



On the orthogonality and convolution orthogonality via the Kontorovich–Lebedev transform

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ABSTRACT

Notions of orthogonality and convolution orthogonality are explored with the use of the Kontorovich–Lebedev transform and its convolution. New classes of the corresponding orthogonal polynomials and functions are investigated. Integral representations, orthogonality relations and explicit expressions are established.

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1. Introduction and preliminary results

It is known that a relationship between different systems of orthogonal polynomials can be realized, using Fourier transform. For instance, Fourier transform of Jacobi polynomials yields Continuous Hahn polynomials (see [1]). In the literature other Fourier-type transforms of classical orthogonal polynomials have been considered. In particular, the action of the Fourier–Jacobi transform (or the Oleviskii transform [2]) on classical orthogonal polynomials have been considered in [3].

The aim of the present contribution is to deal with the Kontorovich–Lebedev transform and its modifications in order to establish new biorthogonal sequences of functions and polynomials, and to emphasize the role of Continuous Dual Hahn and Wilson polynomials as in [1] the role of Jacobi polynomials is shown in the framework of Fourier transformation.

In fact, let f be a complex-valued function defined on $\mathbb{R}_+ \equiv (0, \infty)$. The Kontorovich–Lebedev transform [2,4] is defined by the following integral

$$(Ff)(\tau) = \int_0^\infty K_{i\tau}(x)f(x)dx, \quad \tau \in \mathbb{R}_+. \quad (1.1)$$

Here $K_{i\tau}(x)$ is the modified Bessel function of the second kind, or the Macdonald function of argument $x > 0$ and pure imaginary subscript $i\tau$ (see [5], Vol. II). It can be defined by the Fourier cosine transform

$$K_{i\tau}(x) = \int_0^\infty e^{-x \cosh u} \cos(\tau u) du, \quad x > 0, \quad (1.2)$$

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and, conversely, by the inversion formula we immediately obtain

$$e^{-x \cosh u} = \frac{2}{\pi} \int_0^\infty K_{i\tau}(x) \cos(\tau u) d\tau. \quad (1.3)$$

Moreover, it is an eigenfunction for the differential operator

$$\mathcal{A} \equiv x^2 - x \frac{d}{dx} x \frac{d}{dx}, \quad (1.4)$$

i.e. we have

$$\mathcal{A}K_{i\tau}(x) = \tau^2 K_{i\tau}(x). \quad (1.5)$$

The modified Bessel function has the following asymptotic behavior with respect to x [5], Vol. II

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} [1 + O(1/x)], \quad x \rightarrow +\infty, \quad (1.6)$$

$$K_\nu(x) = O(x^{-|\operatorname{Re} \nu|}), \quad x \rightarrow 0, \quad (1.7)$$

$$K_0(x) = O(-\log x), \quad x \rightarrow 0 \quad (1.8)$$

and with respect to the index $\nu = i\tau$

$$K_{i\tau}(x) = O\left(\frac{e^{-\pi\tau/2}}{\sqrt{\tau}}\right), \quad \tau \rightarrow +\infty. \quad (1.9)$$

The following uniform inequality will be useful in the sequel (see [4], formula (1.100))

$$|K_{i\tau}(x)| \leq e^{-\delta\tau} K_0(x \cos(\delta)), \quad \delta \in \left[0, \frac{\pi}{2}\right]. \quad (1.10)$$

As is known [4], the Kontorovich–Lebedev transform (1.1) extends to an isometric map

$$F : L_2(\mathbb{R}_+; xdx) \rightarrow L_2\left(\mathbb{R}_+; \frac{2}{\pi^2} \tau \sinh(\pi\tau) d\tau\right)$$

and the inversion formula holds

$$xf(x) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) K_{i\tau}(x) (Ff)(\tau) d\tau, \quad x > 0, \quad (1.11)$$

where integrals (1.1), (1.11) converge with respect to norms of the image spaces. Moreover, the Parseval equality takes place

$$\int_0^\infty |f(x)|^2 x dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) |(Ff)(\tau)|^2 d\tau \quad (1.12)$$

and, via the parallelogram identity, we have the generalized Parseval equality

$$\int_0^\infty f(x) \overline{g(x)} x dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) (Ff)(\tau) \overline{(Fg)(\tau)} d\tau, \quad (1.13)$$

where Fg is the Kontorovich–Lebedev transform (1.1) of a function $g \in L_2(\mathbb{R}_+; xdx)$.

According to [2,4], the convolution $f * g$ of two functions f, g from the space $L_1(\mathbb{R}_+; K_0(px)dx)$, $0 < p \leq 1$, related to the Kontorovich–Lebedev transform is given by the formula

$$(f * g)(x) = \frac{1}{2x} \int_0^\infty \int_0^\infty e^{-\frac{y^2+t^2}{2yt}x - \frac{yt}{2x}} f(y)g(t) dy dt, \quad x > 0. \quad (1.14)$$

We have

Theorem 1. Let $f, g \in L_1(\mathbb{R}_+; K_0(px)dx)$, $0 < p \leq 1$. Then the convolution $f * g \in L_1(\mathbb{R}_+; K_0(p^2x)dx)$ and satisfies the Young-type inequality

$$\|f * g\|_{L_1(\mathbb{R}_+; K_0(p^2x)dx)} \leq \|f\|_{L_1(\mathbb{R}_+; K_0(px)dx)} \|g\|_{L_1(\mathbb{R}_+; K_0(px)dx)}. \quad (1.15)$$

Moreover, the Kontorovich–Lebedev transform (1.1) of the convolution (1.14) is the product of the Kontorovich–Lebedev transforms, i.e.

$$F(f * g)(\tau) = (Ff)(\tau)(Fg)(\tau), \quad (1.16)$$

and when $p \in (0, 1/2)$ the Parseval-type equality holds for all $x > 0$,

$$(f * g)(x) = \frac{2}{x\pi^2} \int_0^\infty \tau \sinh(\pi\tau) K_{i\tau}(x) (Ff)(\tau) (Fg)(\tau) d\tau. \quad (1.17)$$

Proof. In fact, the existence of the convolution (1.14) as a function of the space $L_1(\mathbb{R}_+; K_0(p^2x)dx)$ and inequality (1.15) follow from the following estimate

$$\begin{aligned} \|f * g\|_{L_1(\mathbb{R}_+; K_0(p^2x)dx)} &= \int_0^\infty \frac{K_0(p^2x)}{2x} \left| \int_0^\infty \int_0^\infty e^{-\frac{y^2+t^2}{2yt}x - \frac{yt}{2x}} f(y)g(t)dydt \right| dx \\ &\leq \int_0^\infty \frac{K_0(p^2x)}{2x} \int_0^\infty \int_0^\infty e^{-\frac{y^2+t^2}{2yt}x - \frac{yt}{2x}} |f(y)g(t)| dydt dx \\ &= \int_0^\infty \frac{K_0(x)}{2x} \int_0^\infty \int_0^\infty e^{-\frac{y^2+t^2}{2ytp^2}x - \frac{ytp^2}{2x}} |f(y)g(t)| dydt dx \\ &\leq \int_0^\infty \frac{K_0(x)}{2x} \int_0^\infty \int_0^\infty e^{-\frac{y^2+t^2}{2yt}x - \frac{ytp^2}{2x}} |f(y)g(t)| dydt dx \\ &= \int_0^\infty K_0(py)|f(y)|dy \int_0^\infty K_0(pt)|g(t)|dt = \|f\|_{L_1(\mathbb{R}_+; K_0(px)dx)} \|g\|_{L_1(\mathbb{R}_+; K_0(px)dx)}, \end{aligned}$$

where the interchange of the order of integration is permitted via Fubini's theorem and the integral with respect to x is calculated by the Macdonald product formula for the modified Bessel functions (see [4], formula (1.103)). In order to prove formula (1.16) we apply the Kontorovich–Lebedev transform to the convolution, change the order of integration by Fubini's theorem and use the Macdonald formula, taking into account the embeddings

$$L_1(\mathbb{R}_+; K_0(p^2x)dx) \subseteq L_1(\mathbb{R}_+; K_0(px)dx) \subseteq L_1(\mathbb{R}_+; K_0(x)dx).$$

Finally, we establish representation (1.17) for convolution (1.14). To do this, we appeal to the following index integral for the convolution kernel (see [4], formula (4.36))

$$\exp\left(-\frac{1}{2}\left(\frac{y^2+t^2}{yt}x + \frac{yt}{x}\right)\right) = \frac{4}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y) K_{i\tau}(t) d\tau. \quad (1.18)$$

Hence, employing inequality (1.10), we deduce

$$\begin{aligned} \exp\left(-\frac{1}{2}\left(\frac{y^2+t^2}{yt}x + \frac{yt}{x}\right)\right) &\leq \frac{4}{\pi^2} K_0(x \cos(\delta)) K_0(y \cos(\delta)) K_0(t \cos(\delta)) \\ &\times \int_0^\infty \tau \sinh(\pi\tau) e^{-3\delta\tau} d\tau = \frac{24\delta}{\pi(9\delta^2 - \pi^2)^2} K_0(x \cos(\delta)) K_0(y \cos(\delta)) K_0(t \cos(\delta)), \end{aligned}$$

which gives the inequality for all $(x, y, t) \in \mathbb{R}_+^3$, $\delta \in (\frac{\pi}{3}, \frac{\pi}{2})$

$$\exp\left(-\frac{1}{2}\left(\frac{y^2+t^2}{yt}x + \frac{yt}{x}\right)\right) \leq \frac{24\delta}{\pi(9\delta^2 - \pi^2)^2} K_0(x \cos(\delta)) K_0(y \cos(\delta)) K_0(t \cos(\delta)). \quad (1.19)$$

Hence, plugging the right-hand side of (1.18) in (1.14) and changing the order of integration owing to inequality (1.19) and conditions of the theorem for $p = \cos(\delta) \in (0, 1/2)$, we get representation (1.17) of the convolution $f * g$. \square

Lemma 1. Let $f \in L_1(\mathbb{R}_+; K_0(px)x^{-1}dx)$, $0 < p \leq 1$. Then the Kontorovich–Lebedev transform (1.1) $(Fg)(\tau)$ of the function $g(x) = f(x)/x$ is the composition of the Fourier cosine and Laplace transforms, i.e.

$$(Fg)(\tau) = \int_0^\infty \cos(\tau u) \int_0^\infty e^{-x \cosh u} f(x) \frac{dx du}{x}, \quad \tau > 0. \quad (1.20)$$

Moreover, if, in addition, f belongs to the space $L_r(\mathbb{R}_+; K_0(px)x^{-a}dx)$, $r, a > 1$, $p \in (0, 1)$, then it can be written in the form

$$(Fg)(\tau) = \frac{1}{\tau} \int_0^\infty \sin(\tau u) \sinh(u) \int_0^\infty e^{-x \cosh u} f(x) dx du \quad (1.21)$$

and

$$\lim_{u \rightarrow \infty} \sinh(u) \int_0^\infty e^{-x \cosh u} f(x) dx = 0. \quad (1.22)$$

Proof. Since $f \in L_1(\mathbb{R}_+; K_0(px)x^{-1}dx)$ we have the estimate

$$\begin{aligned} \int_0^\infty \left| \cos(\tau u) \int_0^\infty e^{-x \cosh u} f(x) \frac{dx}{x} \right| &\leq \int_0^\infty du \int_0^\infty e^{-x \cosh u} |f(x)| \frac{dx}{x} \\ &= \int_0^\infty K_0(x) |f(x)| \frac{dx}{x} \leq \|f\|_{L_1(\mathbb{R}_+; K_0(px)x^{-1}dx)} < \infty. \end{aligned}$$

Therefore formula (1.20) follows immediately via Fubini's theorem and integral representation (1.2). Then integrating by parts in the integral with respect to u and eliminating the integrated terms due to the absolute and uniform convergence by $u \in \mathbb{R}_+$ of the integral with respect to x , we obtain (1.21), differentiating by u under the integral sign. Indeed, this is allowed due to the estimate for some $N > 0$ large enough

$$\begin{aligned} \sinh(u) \int_N^\infty e^{-x \cosh u} |f(x)| dx &= \sinh(u) \int_N^\infty e^{-(1-p)x \cosh u} e^{-px \cosh u} |f(x)| dx \\ &\leq \sinh(u) \left(\int_N^\infty e^{-(1-p)r'x \cosh u} x^{ar'/r} dx \right)^{1/r'} \left(\int_N^\infty e^{-prx \cosh u} \frac{|f(x)|^r}{x^a} dx \right)^{1/r} \\ &\leq \Gamma^{1/r'} \left(\frac{ar'}{r} + 1 \right) \frac{\sinh(u)}{[(1-p)r' \cosh(u)]^{1+(a-1)/r}} \left(\int_N^\infty e^{-prx \cosh u} \frac{|f(x)|^r}{x^a} dx \right)^{1/r} \\ &\leq \Gamma^{1/r'} \left(\frac{ar'}{r} + 1 \right) [(1-p)r']^{-1-(a-1)/r} \\ &\quad \times \sup_{x \geq N} \left[\frac{e^{-prx}}{K_0(px)} \right]^{1/r} \left(\int_N^\infty K_0(px) \frac{|f(x)|^r}{x^a} dx \right)^{1/r} \rightarrow 0, \quad N \rightarrow \infty, \end{aligned}$$

where $r' = r/(r-1)$ and $\Gamma(z)$ is the Euler gamma function [5], Vol. I. Finally, condition (1.22) is an immediate consequence of the inequality

$$\begin{aligned} \sinh(u) \int_0^\infty e^{-x \cosh u} |f(x)| dx &\leq \Gamma^{1/r'} \left(\frac{ar'}{r} + 1 \right) \frac{[\cosh(u)]^{(1-a)/r}}{[(1-p)r']^{1+(a-1)/r}} \\ &\quad \times \sup_{x>0} \left[\frac{e^{-prx}}{K_0(px)} \right]^{1/r} \|f\|_{L_r(\mathbb{R}_+; K_0(px)x^{-a} dx)} \rightarrow 0, \quad u \rightarrow \infty. \quad \square \end{aligned}$$

Corollary 1. Let $f, g \in L_1(\mathbb{R}_+; K_0(px)dx)$, $0 < p < 1/2$, and $\omega \in L_1(\mathbb{R}_+; K_0(px)x^{-1}dx)$. Then the following equality takes place

$$\int_0^\infty (f * g)(x) \omega(x) dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) (Ff)(\tau) (Fg)(\tau) q(\tau) d\tau, \quad (1.23)$$

where

$$q(\tau) = \int_0^\infty K_{i\tau}(x) \omega(x) \frac{dx}{x}. \quad (1.24)$$

Proof. The proof follows immediately, multiplying both sides of (1.17) by ω and integrating over \mathbb{R}_+ . The interchange of the order of integration on the right-hand side of the obtained equality is allowed via Fubini's theorem and inequality (1.19). \square

Let ω be a positive function such that $\omega \in L_1((0, 1); x^{-2}dx) \cap L_1((1, \infty); x^{-1}dx)$. Making use of integral representations for the modified Bessel function via integration by parts [5], Vol. II

$$\begin{aligned} K_{i\tau}(x) &= \frac{1}{\sinh(\pi \tau/2)} \int_0^\infty \sin(\tau u) \sin(x \sinh(u)) du \\ &= \frac{1}{x \sinh(\pi \tau/2)} \int_0^\infty \frac{\cos(x \sinh(u))}{\cosh^2(u)} [\tau \cos(\tau u) \cosh(u) - \sin(\tau u) \sinh(u)] du, \end{aligned} \quad (1.25)$$

we substitute the first integral in (1.25) into (1.24), having the equality

$$q(\tau) = \frac{1}{\sinh(\pi \tau/2)} \int_0^\infty \int_0^\infty \sin(\tau u) \sin(x \sinh(u)) \omega(x) \frac{dx du}{x}. \quad (1.26)$$

Our goal is to justify the interchange of the order of integration in (1.26), proving the formula

$$q(\tau) = \frac{1}{\sinh(\pi \tau/2)} \int_0^\infty \sin(\tau u) du \int_0^\infty \sin(x \sinh(u)) \omega(x) \frac{dx}{x}. \quad (1.27)$$

Indeed, we write the integral in (1.27) in the form

$$\begin{aligned} &\lim_{N \rightarrow \infty} \int_0^N \sin(\tau u) du \int_0^\infty \sin(x \sinh(u)) \omega(x) \frac{dx}{x} \\ &= \lim_{N \rightarrow \infty} \int_0^\infty \int_0^N \sin(\tau u) \sin(x \sinh(u)) \omega(x) \frac{dudx}{x}, \end{aligned} \quad (1.28)$$

where the interchange of the order of integration is due to the dominated convergence theorem by virtue of the estimate

$$\int_0^N |\sin(\tau u)| du \int_0^\infty |\sin(x \sinh(u))| \omega(x) \frac{dx}{x} \leq N \int_0^\infty \omega(x) \frac{dx}{x} < \infty.$$

Now, in order to pass to the limit under the integral sign on the right-hand side of (1.28), we appeal to the same justification as above, employing the following estimate via integration by parts (cf. (1.25))

$$\begin{aligned} \left| \int_0^N \sin(\tau u) \sin(x \sinh(u)) du \right| &= \frac{1}{x} \left| -\frac{\sin(N\tau) \cos(x \sinh(N))}{\cosh(N)} \right. \\ &+ \int_0^N \frac{\cos(x \sinh(u))}{\cosh^2(u)} [\tau \cos(\tau u) \cosh(u) - \sin(\tau u) \sinh(u)] du \left| \right. \\ &\leq \frac{1}{x} \left[1 + \int_0^\infty \frac{1}{\cosh^2(u)} [\tau \cosh(u) + \sinh(u)] du \right] \end{aligned}$$

and the condition $\omega \in L_1(\mathbb{R}_+; x^{-2}dx)$. This proves (1.27). Further, appealing to Theorem 123 in [6] about the non-negativeness of the Fourier sine transform, we are ready, as a direct consequence, to formulate the following result.

Lemma 2. Let ω be a positive non-increasing function over \mathbb{R}_+ such that $\omega(x) = o(x)$, $x \rightarrow \infty$, $\omega \in L_1((0, 1); x^{-2}dx) \cap L_1((1, \infty); x^{-1}dx)$ and let the function

$$\varphi(u) = \int_0^\infty \sin(xu) \omega(x) \frac{dx}{x} \quad (1.29)$$

be non-increasing on \mathbb{R}_+ . Then the weight function (1.24) $q(\tau) \geq 0$, $\tau \in \mathbb{R}_+$.

Proof. It is easily seen from the conditions of the lemma and Theorem 123 in [6] that $\varphi(u)$ as the Fourier sine transform (1.29) is non-negative. Then since $\varphi(\sinh(u))$ is non-increasing on \mathbb{R}_+ , integrable over $(0, 1)$ owing to the estimate

$$\int_0^1 |\varphi(\sinh(u))| du \leq \|\omega\|_{L_1(\mathbb{R}_+; x^{-1}dx)}$$

and tends to zero at infinity due to the Riemann–Lebesgue lemma, its Fourier sine transform (1.27) is non-negative as well as via Theorem 123 in [6]. Hence $q(\tau) \geq 0$, $\tau \in \mathbb{R}_+$. \square

Returning to equality (1.23), we observe that for positive ω and non-negative q its left-hand side satisfies properties of the inner product and forms the so-called convolution Hilbert spaces studied in [4, Chapter 4]. This means that one can consider the corresponding convolution orthogonality of functions from the associated Lebesgue spaces. The notion of the convolution orthogonality was introduced for the first time in [7] for the Laplace convolution, and the discrete case was investigated in [8]. Our goal is to explore in the sequel concrete orthogonal sequences of functions and polynomials with respect to the convolution (1.14) and Parseval-type equalities (1.13), (1.23). To do this we will employ, in particular, Wilson's and Continuous Dual Hahn polynomials [9]. We note that in [3, 10] similar problems were examined, involving Fourier and Fourier–Jacobi transforms. A recent overview of hypergeometric polynomials and their q -analogues is given in [11]. Therein the reader can find the Askey tableau and q -Askey tableau. Continuous Dual Hahn and Wilson polynomials appear in the Askey tableau in the ${}_3F_2$ and ${}_4F_3$ levels, respectively.

Finally, in this section, let $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R} \setminus \{0\}$, and consider the modified Kontorovich–Lebedev transform by the formula

$$(F_{\alpha, \beta} f)(\tau) = \int_0^\infty K_{i\tau}(\alpha x^\beta) f(x) dx. \quad (1.30)$$

We have (see (1.1)) $(F_{1,1} f)(\tau) \equiv (Ff)(\tau)$. Furthermore, in the same manner as above, changing variables and functions, one can show that the modified Kontorovich–Lebedev transform (1.30) extends to an isometric map

$$F_{\alpha, \beta} : L_2(\mathbb{R}_+; x dx) \rightarrow L_2\left(\mathbb{R}_+; \frac{2|\beta|}{\pi^2} \tau \sinh(\pi \tau) d\tau\right), \quad (1.31)$$

the inversion formula holds

$$xf(x) = \frac{2|\beta|}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) K_{i\tau}(\alpha x^\beta) (F_{\alpha, \beta} f)(\tau) d\tau, \quad x > 0, \quad (1.32)$$

where integrals (1.30), (1.31) converge with respect to norms of the image spaces, and the Parseval equality is valid

$$\int_0^\infty f(x) \overline{g(x)} x dx = \frac{2|\beta|}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) (F_{\alpha, \beta} f)(\tau) \overline{(F_{\alpha, \beta} g)(\tau)} d\tau. \quad (1.33)$$

2. The use of the Wilson and Continuous Dual Hahn polynomials

With a slight modification of the Kontorovich–Lebedev transform (1.1) it becomes an automorphism on the vector space of polynomials \mathcal{P} (see [12,13]). Moreover, it involves orthogonal and multiple orthogonal polynomials or d -orthogonal ones. For instance, the Continuous Dual Hahn polynomials appear as the Kontorovich–Lebedev transform of a 2-orthogonal sequence of Laguerre type. We note that [14] is the starting point of d -orthogonality that constitutes a particular case of multiple orthogonality that appears in an elegant way in [11,15]. Our goal here is to extend Parseval-type equalities (1.13), (1.23) to different orthogonal polynomials and functions, generating new orthogonal and convolution orthogonal systems related to the Kontorovich–Lebedev transform and its modifications.

2.1. Kontorovich–Lebedev transformations and Laguerre polynomials

It is straightforward to verify, appealing to the asymptotic behavior (1.6), (1.7), (1.8) of the modified Bessel function, that $\mathcal{P} \subset L_1(\mathbb{R}_+; K_0(px)dx)$, $0 < p \leq 1$. This means that the Parseval-type equality (1.23) holds for polynomial functions f, g . However, the Parseval equality (1.13) to be valid for a polynomial f should be reestablished under some sufficient conditions on a function g . To do this, we shall prove

Theorem 2. Let $f \in \mathcal{P}$ and g be such that its Kontorovich–Lebedev transform $Fg \in L_2(\mathbb{R}_+; \tau \sinh((\pi + \mu)\tau)d\tau)$, for some $\mu > 0$. Then equality (1.13) holds.

Proof. Since evidently, $L_2(\mathbb{R}_+; \tau \sinh((\pi + \mu)\tau)d\tau) \subset L_2(\mathbb{R}_+; \tau \sinh(\pi\tau)d\tau)$, $\mu > 0$, we get from (1.12) that $g \in L_2(\mathbb{R}_+; xdx)$ and integral (1.11) for g

$$xg(x) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) K_{i\tau}(x)(Fg)(\tau) d\tau, \quad (2.1)$$

converges in the mean square sense with respect to the norm in $L_2(\mathbb{R}_+; xdx)$. But the estimate (see (1.10))

$$\begin{aligned} \int_0^\infty \tau \sinh(\pi\tau) |K_{i\tau}(x)(Fg)(\tau)| d\tau &\leq K_0(x \cos(\delta)) \int_0^\infty \tau \sinh(\pi\tau) e^{-\delta\tau} |(Fg)(\tau)| d\tau \\ &\leq K_0(x \cos(\delta)) \left(\int_0^\infty \frac{\tau \sinh^2(\pi\tau) e^{-2\delta\tau}}{\sinh((\pi + \mu)\tau)} d\tau \right)^{1/2} \|Fg\|_{L_2(\mathbb{R}_+; \tau \sinh((\pi + \mu)\tau)d\tau)} \\ &= C_{\delta, \mu} K_0(x \cos(\delta)) \|Fg\|_{L_2(\mathbb{R}_+; \tau \sinh((\pi + \mu)\tau)d\tau)}, \end{aligned}$$

where $C_{\delta, \mu} > 0$ is the constant

$$C_{\delta, \mu} = \left(\int_0^\infty \frac{\tau \sinh^2(\pi\tau) e^{-2\delta\tau}}{\sinh((\pi + \mu)\tau)} d\tau \right)^{1/2}, \quad \delta \in \left(\max \left(0, \frac{\pi - \mu}{2} \right), \frac{\pi}{2} \right),$$

guarantees the existence of (2.1) as a Lebesgue integral for all $x > 0$. Therefore, substituting it in the left-hand side of (1.13), we change the order of integration by Fubini's theorem owing to the estimate

$$\begin{aligned} \int_0^\infty |f(x)| \int_0^\infty \tau \sinh(\pi\tau) |K_{i\tau}(x)(Fg)(\tau)| d\tau dx \\ \leq C_{\delta, \mu} \|Fg\|_{L_2(\mathbb{R}_+; \tau \sinh((\pi + \mu)\tau)d\tau)} \int_0^\infty |f(x)| K_0(x \cos(\delta)) dx < \infty, \end{aligned}$$

to complete the proof of Theorem 2. \square

We will show below that the Parseval equality (1.13) generates various systems of orthogonal polynomials and functions, and it is closely related, in particular, to the Wilson and Continuous Dual Hahn orthogonalities. Precisely, following [9], we define the sequence of Wilson orthogonal polynomials $\{W_n(t)\}_{n \geq 0}$ such that

$$\begin{aligned} \int_0^\infty \left| \frac{\Gamma(a + it)\Gamma(b + it)\Gamma(c + it)\Gamma(d + it)}{\Gamma(2it)} \right|^2 W_n(t^2) W_m(t^2) dt \\ = 2\pi \delta_{n,m} \frac{n! \Gamma(n + a + b) \Gamma(n + a + c) \Gamma(n + a + d) \Gamma(n + b + c) \Gamma(n + b + d) \Gamma(n + c + d)}{\Gamma(2n + a + b + c + d) (n + a + b + c + d - 1)_n}, \end{aligned}$$

where $\delta_{n,m}$ is the Kronecker delta, $(z)_n$, $n \in \mathbb{N}_0$, is the Pochhammer symbol, $a, b, c, d > 0$, i is the imaginary unit and

$$\begin{aligned} W_n(t^2) \equiv W_n(t^2; a, b, c, d) &= (a + b)_n (a + c)_n (a + d)_n \\ &\times {}_4F_3(-n, n + a + b + c + d - 1, a + it, a - it; a + b, a + c, a + d; 1) \end{aligned} \quad (2.2)$$

are Wilson's polynomials being expressed in terms of the generalized hypergeometric function ${}_4F_3$ (cf. [5], Vol. I). It is shown that W_n is symmetric in all four parameters a, b, c, d .

The sequence of Continuous Dual Hahn orthogonal polynomials $\{S_n(t)\}_{n \geq 0}$ such that (see [9])

$$\int_0^\infty \left| \frac{\Gamma(a+it)\Gamma(b+it)\Gamma(c+it)}{\Gamma(2it)} \right|^2 S_n(t^2)S_m(t^2)dt \\ = 2\pi\delta_{n,m} n! \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c),$$

where

$$S_n(t^2) \equiv S_n(t^2; a, b, c) = (a+b)_n(a+c)_n \\ \times {}_3F_2(-n, a+it, a-it; a+b, a+c; 1) \quad (2.3)$$

are Continuous Dual Hahn polynomials in terms of the ${}_3F_2$ generalized hypergeometric function. These polynomials are symmetric in all three parameters.

Let us consider sequence of Laguerre orthogonal polynomials $\{L_n^\alpha(t)\}_{n \geq 0}$ (see [5], Vol. II)

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}, \quad \alpha > -1. \quad (2.4)$$

We will see as this orthogonality generates other orthogonalities of polynomials and functions, making use of the Parseval equality (1.13) and mapping properties of the Kontorovich–Lebedev transform. In fact, let us introduce the sequence of functions $\{F_n(\tau; \alpha, \beta, \mu, \eta)\}_{n \geq 0}$, which are defined by the following Kontorovich–Lebedev integral

$$F_n(\tau; \alpha, \beta, \mu, \eta) = \int_0^\infty x^\beta e^{-\mu x} L_n^\alpha(x) K_{i\tau}(\eta x) dx, \quad \alpha, \beta > -1, \quad \mu \geq 0, \quad \eta > 0. \quad (2.5)$$

Proposition 1. Let $\alpha > \beta > -1$. Then for $0 < \mu \leq 1$, $\gamma = \alpha - \beta - 1$ the sequences $\{F_n(\tau; \alpha, \beta, \mu, \mu)\}_{n \geq 0}$ and $\{F_n(\tau; \alpha, \gamma, 1 - \mu, \mu)\}_{n \geq 0}$ are biorthogonal, i.e.

$$\int_0^\infty \tau \sinh(\pi \tau) F_n(\tau; \alpha, \beta, \mu, \mu) F_m(\tau; \alpha, \gamma, 1 - \mu, \mu) d\tau = \frac{\pi^2}{2n!} \Gamma(n+\alpha+1) \delta_{n,m}. \quad (2.6)$$

Besides, the general term of the sequence $\{F_n(\tau; \alpha, \beta, \mu, \mu)\}_{n \geq 0}$ can be represented in terms of a ${}_3F_2$ hypergeometric function, that is a polynomial generalizing the Continuous Dual Hahn polynomials, namely,

$$F_n(\tau; \alpha, \beta, \mu, \mu) = \frac{\sqrt{\pi}}{(2\mu)^{\beta+1} n!} \frac{|\Gamma(\beta+1-i\tau)|^2 (\alpha+1)_n}{\Gamma(\beta+3/2)} \\ \times {}_3F_2\left(-n, \beta+1+i\tau, \beta+1-i\tau; \alpha+1, \beta+\frac{3}{2}; \frac{1}{2\mu}\right). \quad (2.7)$$

The general term of the sequence $\{F_n(\tau; \alpha, \gamma, 1 - \mu, \mu)\}_{n \geq 0}$ can be represented, in turn, in terms of a finite sum of the Gauss ${}_2F_1$ hypergeometric function, i.e.

$$F_n(\tau; \alpha, \gamma, 1 - \mu, \mu) = \sqrt{\pi} \frac{|\Gamma(\gamma+1+i\tau)|^2 (1+\alpha)_n}{(2\mu)^{\gamma+1} n! \Gamma(\gamma+3/2)} \sum_{k=0}^n \frac{(-n)_k (\gamma+1-i\tau)_k (\gamma+1+i\tau)_k}{(2\mu)^k k! (1+\alpha)_k (\gamma+3/2)_k} \\ \times {}_2F_1\left(\frac{k+\gamma+1+i\tau}{2}, \frac{k+\gamma+1-i\tau}{2}; k+\gamma+\frac{3}{2}; \frac{2\mu-1}{\mu^2}\right). \quad (2.8)$$

Proof. Indeed, biorthogonality (2.6) follows immediately from the mapping L_2 -properties of the Kontorovich–Lebedev transform (1.1) (see (1.12) and Theorem 2), if we let $\gamma = \alpha - \beta - 1 > -1$ and then for $0 < \mu \leq 1$ apply the Parseval identity (1.13) in the orthogonality relation (2.4) that Laguerre polynomials satisfy. Furthermore, the function $F_n(\tau; \alpha, \beta, \mu, \mu)$ can be calculated explicitly, employing Entry 3.14.3.1 in [16]

$$\int_0^\infty x^{s-1} e^{-\mu x} K_{i\tau}(\mu x) dx = \frac{\sqrt{\pi}}{(2\mu)^s} \frac{\Gamma(s-i\tau)\Gamma(s+i\tau)}{\Gamma(s+1/2)}, \quad \operatorname{Re}(s) > 0 \quad (2.9)$$

and the closed form expression of Laguerre polynomials as hypergeometric functions

$$L_n^\alpha(x) = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}. \quad (2.10)$$

Hence we derive from (2.5)

$$\begin{aligned} F_n(\tau; \alpha, \beta, \mu, \mu) &= \frac{\sqrt{\pi}}{(2\mu)^{\beta+1}} \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} (2\mu)^{-k} \frac{\Gamma(\beta+k+1-i\tau)\Gamma(\beta+k+1+i\tau)}{\Gamma(\beta+k+3/2)} \\ &= \frac{\sqrt{\pi}}{(2\mu)^{\beta+1}n!} \frac{|\Gamma(\beta+1-i\tau)|^2 (\alpha+1)_n}{\Gamma(\beta+3/2)} \\ &\quad \times {}_3F_2\left(-n, \beta+1+i\tau, \beta+1-i\tau; \alpha+1, \beta+\frac{3}{2}; \frac{1}{2\mu}\right), \end{aligned}$$

where the latter ${}_3F_2$ hypergeometric function is a polynomial in τ , slightly generalizing the Continuous Dual Hahn polynomial (2.3). This proves (2.7). On the other hand, we calculate $F_n(\tau; \alpha, \gamma, 1-\mu, \mu)$ in terms of the Gauss hypergeometric function ${}_2F_1$ [5], Vol. I. Precisely, appealing to (2.10), Entry 3.14.3.3 in [16] and the Boltz formula for the Gauss hypergeometric function (see [4], formula (1.53)), we have

$$\begin{aligned} \int_0^\infty x^{s-1} e^{-(1-\mu)x} K_{i\tau}(\mu x) dx &= \sqrt{\pi} \frac{\Gamma(s-i\tau)\Gamma(s+i\tau)}{(2\mu)^s \Gamma(s+1/2)} \\ &\quad \times {}_2F_1\left(\frac{s-i\tau}{2}, \frac{s+i\tau}{2}; s+\frac{1}{2}; \frac{2\mu-1}{\mu^2}\right). \end{aligned} \quad (2.11)$$

Then we find, finally, the following expressions

$$\begin{aligned} F_n(\tau; \alpha, \gamma, 1-\mu, \mu) &= \int_0^\infty x^\gamma e^{-(1-\mu)x} L_n^\alpha(x) K_{i\tau}(\mu x) dx \\ &= \sqrt{\pi} \frac{|\Gamma(\gamma+1+i\tau)|^2 (1+\alpha)_n}{(2\mu)^{\gamma+1} n! \Gamma(\gamma+3/2)} \sum_{k=0}^n \frac{(-n)_k (\gamma+1-i\tau)_k (\gamma+1+i\tau)_k}{(2\mu)^k k! (1+\alpha)_k (\gamma+3/2)_k} \\ &\quad \times {}_2F_1\left(\frac{k+\gamma+1+i\tau}{2}, \frac{k+\gamma+1-i\tau}{2}; k+\gamma+\frac{3}{2}; \frac{2\mu-1}{\mu^2}\right). \end{aligned}$$

Hence we get (2.8) and complete the proof of Proposition 1. \square

Corollary 2. Let $\mu = 1/2$, $\gamma = \beta$, $\alpha = 2\beta+1$ in (2.6). Then it represents the orthogonality for a particular case of Continuous Dual Hahn polynomials (2.3) with $a = b = \beta+1$, $c = 1/2$.

Proof. In fact, letting $\mu = 1/2$ in (2.6), (2.7), we employ the reflection and duplication formulas for gamma function [5], Vol. I, we end up with a particular case of the Continuous Dual Hahn orthogonality, namely,

$$\begin{aligned} \int_0^\infty \left| \frac{\Gamma(\beta+1+it)\Gamma(\gamma+1+it)}{\Gamma(it)} \right|^2 S_n\left(t^2; \beta+1, \gamma+1, \frac{1}{2}\right) S_m\left(t^2; \beta+1, \gamma+1, \frac{1}{2}\right) dt \\ = \frac{n!}{2} \delta_{n,m} \Gamma(n+\beta+\gamma+2) \Gamma\left(n+\beta+\frac{3}{2}\right) \Gamma\left(n+\gamma+\frac{3}{2}\right). \end{aligned} \quad (2.12)$$

Hence $\gamma = \beta$ yields Continuous Dual Hahn polynomials (2.3) with $a = b = \beta+1$, $c = 1/2$. \square

Corollary 3. Let $\mu = 1$ in (2.8). Then the general term of the sequence $\{F_n(\tau; \alpha, \gamma, 0, 1)\}_{n \geq 0}$ can be expressed in terms of the ${}_4F_3$ hypergeometric functions:

$$\begin{aligned} F_{2m}(\tau; \alpha, \gamma, 0, 1) &= \frac{(1+\alpha)_{2m} 2^\gamma}{(2m)!} \left[\frac{1}{2} \left| \Gamma\left(\frac{\gamma+1+i\tau}{2}\right) \right|^2 \right. \\ &\quad \times {}_4F_3\left(-m, \frac{1}{2}-m, \frac{\gamma+1+i\tau}{2}, \frac{\gamma+1-i\tau}{2}; \frac{1}{2}, \frac{1+\alpha}{2}, 1+\frac{\alpha}{2}; 1\right) \\ &\quad \left. - \frac{2m}{1+\alpha} \left| \Gamma\left(\frac{\gamma+2+i\tau}{2}\right) \right|^2 \right. \\ &\quad \left. \times {}_4F_3\left(1-m, \frac{1}{2}-m, \frac{\gamma+2+i\tau}{2}, \frac{\gamma+2-i\tau}{2}; \frac{3}{2}, 1+\frac{\alpha}{2}, \frac{3+\alpha}{2}; 1\right) \right], \quad m \in \mathbb{N}, \\ F_{2m+1}(\tau; \alpha, \gamma, 0, 1) &= \frac{(1+\alpha)_{2m+1} 2^\gamma}{(2m+1)!} \left[\frac{1}{2} \left| \Gamma\left(\frac{\gamma+1+i\tau}{2}\right) \right|^2 \right. \end{aligned} \quad (2.13)$$

$$\begin{aligned}
& \times {}_4F_3 \left(-m, -\frac{1}{2} - m, \frac{\gamma + 1 + i\tau}{2}, \frac{\gamma + 1 - i\tau}{2}; \frac{1}{2}, \frac{1 + \alpha}{2}, 1 + \frac{\alpha}{2}; 1 \right) \\
& - \frac{2m + 1}{1 + \alpha} \left| \Gamma \left(\frac{\gamma + 2 + i\tau}{2} \right) \right|^2 \\
& \times {}_4F_3 \left(-m, \frac{1}{2} - m, \frac{\gamma + 2 + i\tau}{2}, \frac{\gamma + 2 - i\tau}{2}; \frac{3}{2}, 1 + \frac{\alpha}{2}, \frac{3 + \alpha}{2}; 1 \right) \Big], \quad m \in \mathbb{N}.
\end{aligned} \tag{2.14}$$

Proof. The proof follows immediately from (2.5), (2.10), Entry 3.14.1.3 in [16] and properties of the Pochhammer symbol for the even and odd subscripts. Precisely, we derive

$$\begin{aligned}
F_{2m}(\tau; \alpha, \gamma, 0, 1) &= \int_0^\infty x^\gamma L_{2m}^\alpha(x) K_{i\tau}(x) dx = \frac{(1 + \alpha)_{2m}}{(2m)!} \sum_{k=0}^{2m} \frac{2^{k+\gamma-1} (-2m)_k}{k! (1 + \alpha)_k} \\
&\times \Gamma \left(\frac{k + \gamma + 1 + i\tau}{2} \right) \Gamma \left(\frac{k + \gamma + 1 - i\tau}{2} \right) = \frac{(1 + \alpha)_{2m}}{(2m)!} \left[\left| \Gamma \left(\frac{\gamma + 1 + i\tau}{2} \right) \right|^2 \right. \\
&\times \sum_{k=0}^m \frac{2^{2k+\gamma-1} (-2m)_{2k}}{(2k)! (1 + \alpha)_{2k}} \left(\frac{\gamma + 1 + i\tau}{2} \right)_k \left(\frac{\gamma + 1 - i\tau}{2} \right)_k \\
&+ \left| \Gamma \left(\frac{\gamma + 2 + i\tau}{2} \right) \right|^2 \sum_{k=0}^{m-1} \frac{2^{2k+\gamma} (-2m)_{2k+1}}{(2k+1)! (1 + \alpha)_{2k+1}} \left(\frac{\gamma + 2 + i\tau}{2} \right)_k \left(\frac{\gamma + 2 - i\tau}{2} \right)_k \Big] \\
&= \frac{(1 + \alpha)_{2m} 2^\gamma}{(2m)!} \left[\frac{1}{2} \left| \Gamma \left(\frac{\gamma + 1 + i\tau}{2} \right) \right|^2 \right. \\
&\times {}_4F_3 \left(-m, \frac{1}{2} - m, \frac{\gamma + 1 + i\tau}{2}, \frac{\gamma + 1 - i\tau}{2}; \frac{1}{2}, \frac{1 + \alpha}{2}, 1 + \frac{\alpha}{2}; 1 \right) \\
&- \frac{2m}{1 + \alpha} \left| \Gamma \left(\frac{\gamma + 2 + i\tau}{2} \right) \right|^2 \\
&\times {}_4F_3 \left(1 - m, \frac{1}{2} - m, \frac{\gamma + 2 + i\tau}{2}, \frac{\gamma + 2 - i\tau}{2}; \frac{3}{2}, 1 + \frac{\alpha}{2}, \frac{3 + \alpha}{2}; 1 \right) \Big], \\
F_{2m+1}(\tau; \alpha, \gamma, 0, 1) &= \int_0^\infty x^\gamma L_{2m+1}^\alpha(x) K_{i\tau}(x) dx = \frac{(1 + \alpha)_{2m+1}}{(2m+1)!} \sum_{k=0}^{2m+1} \frac{2^{k+\gamma-1} (-2m-1)_k}{k! (1 + \alpha)_k} \\
&\times \Gamma \left(\frac{k + \gamma + 1 + i\tau}{2} \right) \Gamma \left(\frac{k + \gamma + 1 - i\tau}{2} \right) = \frac{(1 + \alpha)_{2m+1}}{(2m+1)!} \left[\left| \Gamma \left(\frac{\gamma + 1 + i\tau}{2} \right) \right|^2 \right. \\
&\times \sum_{k=0}^m \frac{2^{2k+\gamma-1} (-2m-1)_{2k}}{(2k)! (1 + \alpha)_{2k}} \left(\frac{\gamma + 1 + i\tau}{2} \right)_k \left(\frac{\gamma + 1 - i\tau}{2} \right)_k \\
&+ \left| \Gamma \left(\frac{\gamma + 2 + i\tau}{2} \right) \right|^2 \sum_{k=0}^m \frac{2^{2k+\gamma} (-2m-1)_{2k+1}}{(2k+1)! (1 + \alpha)_{2k+1}} \left(\frac{\gamma + 2 + i\tau}{2} \right)_k \left(\frac{\gamma + 2 - i\tau}{2} \right)_k \Big] \\
&= \frac{(1 + \alpha)_{2m+1} 2^\gamma}{(2m+1)!} \left[\frac{1}{2} \left| \Gamma \left(\frac{\gamma + 1 + i\tau}{2} \right) \right|^2 \right.
\end{aligned}$$

$$\begin{aligned} & \times {}_4F_3 \left(-m, -\frac{1}{2} - m, \frac{\gamma + 1 + i\tau}{2}, \frac{\gamma + 1 - i\tau}{2}; \frac{1}{2}, \frac{1 + \alpha}{2}, 1 + \frac{\alpha}{2}; 1 \right) \\ & - \frac{2m + 1}{1 + \alpha} \left| \Gamma \left(\frac{\gamma + 2 + i\tau}{2} \right) \right|^2 \\ & \times {}_4F_3 \left(-m, \frac{1}{2} - m, \frac{\gamma + 2 + i\tau}{2}, \frac{\gamma + 2 - i\tau}{2}; \frac{3}{2}, 1 + \frac{\alpha}{2}, \frac{3 + \alpha}{2}; 1 \right) \Big]. \quad \square \end{aligned}$$

A similar analysis can be done, considering the modified Kontorovich–Lebedev transform $F_{\eta, 1/2} f$ (cf. (1.30)). Indeed, introducing the sequence of functions $\{G_n(\tau; \alpha, \beta, \mu, \eta)\}_{n \geq 0}$ by the formula

$$G_n(\tau; \alpha, \beta, \mu, \eta) = \int_0^\infty x^\beta e^{-\mu x} L_n^\alpha(x) K_{i\tau}(\eta\sqrt{x}) dx, \quad (2.15)$$

we prove the following result.

Proposition 2. Let $\alpha > \beta > -1$, $\eta > 0$. Then for $0 < \mu \leq 1$, $\gamma = \alpha - \beta - 1$ the sequences $\{G_n(\tau; \alpha, \beta, \mu, \eta)\}_{n \geq 0}$ and $\{G_n(\tau; \alpha, \gamma, 1 - \mu, \eta)\}_{n \geq 0}$ are biorthogonal, i.e.

$$\int_0^\infty \tau \sinh(\pi\tau) G_n(\tau; \alpha, \beta, \mu, \eta) G_m(\tau; \alpha, \gamma, 1 - \mu, \eta) d\tau = \frac{\pi^2}{n!} \Gamma(n + \alpha + 1) \delta_{n,m}. \quad (2.16)$$

Moreover, the general term $G_n(\tau; \alpha, \beta, \mu, \eta)$ (2.15) is represented as the finite sum of the Whittaker functions $W_{\rho, \sigma}(z)$:

$$\begin{aligned} G_n(\tau; \alpha, \beta, \mu, \eta) &= \frac{(1 + \alpha)_n}{\eta n!} \mu^{-(1/2 + \beta)} e^{\eta^2/(8\mu)} \left| \Gamma \left(\beta + 1 + \frac{i\tau}{2} \right) \right|^2 \sum_{k=0}^n \frac{(-n)_k}{k!} \\ &\times \frac{(\beta + 1 - \frac{i\tau}{2})_k (\beta + 1 - \frac{i\tau}{2})_k}{(1 + \alpha)_k} W_{-(1/2 + \beta + k), i\tau/2} \left(\frac{\eta^2}{4\mu} \right). \end{aligned} \quad (2.17)$$

Proof. Biorthogonality (2.16) holds owing to isometry (1.31) for the modified Kontorovich–Lebedev transform (2.15), the Parseval identity (1.33), the Laguerre orthogonality (2.4) and Theorem 2. Then (2.10) and Entry 3.14.3.10 in [16] suggest the equalities

$$\begin{aligned} G_n(\tau; \alpha, \beta, \mu, \eta) &= \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n + \alpha}{n - k} \int_0^\infty x^{\beta + k} e^{-\mu x} K_{i\tau}(\eta\sqrt{x}) dx \\ &= \frac{(1 + \alpha)_n}{\eta n!} \mu^{-(1/2 + \beta)} e^{\eta^2/(8\mu)} \left| \Gamma \left(\beta + 1 + \frac{i\tau}{2} \right) \right|^2 \sum_{k=0}^n \frac{(-n)_k}{k!} \\ &\times \frac{(\beta + 1 - \frac{i\tau}{2})_k (\beta + 1 - \frac{i\tau}{2})_k}{(1 + \alpha)_k} W_{-(1/2 + \beta + k), i\tau/2} \left(\frac{\eta^2}{4\mu} \right), \end{aligned}$$

where $W_{\nu, \mu}(z)$ is the Whittaker function [5], Vol. II. This gives (2.17) and completes the proof. \square

Corollary 4. Let $\mu = 0$ in (2.15). Then the general term of the sequence $\{G_n(\tau; \alpha, \gamma, 0, \eta)\}_{n \geq 0}$ can be expressed in terms of the ${}_3F_1$ hypergeometric functions:

$$\begin{aligned} G_n(\tau; \alpha, \gamma, 0, \eta) &= \frac{(1 + \alpha)_n}{2n!} \left(\frac{4}{\eta^2} \right)^{\gamma + 1} \left| \Gamma \left(\gamma + 1 + \frac{i\tau}{2} \right) \right|^2 \\ &\times {}_3F_1 \left(-n, \gamma + 1 + \frac{i\tau}{2}, \gamma + 1 - \frac{i\tau}{2}; 1 + \alpha; \frac{4}{\eta^2} \right). \end{aligned} \quad (2.18)$$

Proof. The limit case $\mu = 0$ is treated, employing the equality (see, for instance, in [2])

$$\int_0^\infty x^{s-1} K_{i\tau}(\eta\sqrt{x}) dx = \eta^{-2s} 2^{2s-1} \Gamma \left(s + \frac{i\tau}{2} \right) \Gamma \left(s - \frac{i\tau}{2} \right), \quad \operatorname{Re}(s) > 0. \quad (2.19)$$

Hence, recalling (2.10), we have, accordingly,

$$\begin{aligned} G_n(\tau; \alpha, \gamma, 0, \eta) &= \frac{1}{2} \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} \left(\frac{4}{\eta^2}\right)^{\gamma+k+1} \Gamma\left(k+\gamma+1+\frac{i\tau}{2}\right) \\ &\times \Gamma\left(k+\gamma+1-\frac{i\tau}{2}\right) = \frac{(1+\alpha)_n}{2n!} \left(\frac{4}{\eta^2}\right)^{\gamma+1} \left|\Gamma\left(\gamma+1+\frac{i\tau}{2}\right)\right|^2 \\ &\times {}_3F_1\left(-n, \gamma+1+\frac{i\tau}{2}, \gamma+1-\frac{i\tau}{2}; 1+\alpha; \frac{4}{\eta^2}\right). \quad \square \end{aligned}$$

2.2. Orthogonal sequences of functions and convolution orthogonality

In this subsection we will find new sequences of functions which are biorthogonal with respect to convolution (1.14) for the Kontorovich–Lebedev transform (1.1) and its modification for the modified Kontorovich–Lebedev transform (1.30). To do this, the Continuous Dual Hahn and Wilson polynomials will be involved. In fact, considering the Continuous Dual Hahn polynomials (2.3) and their orthogonality, we appeal to (2.19) in order to write them in the form of the modified Kontorovich–Lebedev transform (1.30) with $\alpha = 2$, $\beta = 1/2$. Consequently, using the definition of the ${}_3F_2$ hypergeometric function, it is not difficult to obtain the formula

$$\left|\Gamma\left(a+\frac{i\tau}{2}\right)\right|^2 S_n\left(\frac{\tau^2}{4}; a, b, c\right) = (F_{2,1/2} f_n)(\tau), \quad (2.20)$$

where

$$f_n(x) = 2(a+b)_n(a+c)_n x^{a-1} {}_1F_2(-n; a+b, a+c; x). \quad (2.21)$$

Hence the Continuous Dual Hahn orthogonality can be rewritten as follows

$$\begin{aligned} \frac{1}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) \left|\Gamma\left(c+\frac{i\tau}{2}\right)\right|^2 (F_{2,1/2} f_n)(\tau) (F_{2,1/2} g_m)(\tau) d\tau \\ = 4\delta_{n,m} n! \Gamma(n+a+b) \Gamma(n+a+c) \Gamma(n+b+c), \end{aligned} \quad (2.22)$$

where

$$g_m(x) = 2(a+b)_m(b+c)_m x^{b-1} {}_1F_2(-m; a+b, b+c; x). \quad (2.23)$$

We will show that (2.22) generates the convolution orthogonality of the sequences $\{f_n\}_{n \geq 0}$, $\{g_n\}_{n \geq 0}$ related to the modified Kontorovich–Lebedev transform $F_{2,1/2}f$. But this will be possible if we modify the convolution (1.14). Indeed, making use of simple substitutions, we define the modified convolution as follows

$$(f \hat{*} g)(x) = \frac{1}{4x} \int_0^\infty \int_0^\infty e^{-(y+t)\sqrt{\frac{x}{yt}} - \sqrt{\frac{yt}{x}}} f(y)g(t) dy dt, \quad x > 0. \quad (2.24)$$

Moreover, formula (1.18) presumes the identity

$$(F_{2,1/2} f \hat{*} g)(\tau) = (F_{2,1/2} f)(\tau) (F_{2,1/2} g)(\tau). \quad (2.25)$$

Therefore the Parseval identity (1.33) implies an analog of the equality (1.23)

$$\int_0^\infty (f \hat{*} g)(x) \omega(x) dx = \frac{1}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) \hat{q}(\tau) (F_{2,1/2} f)(\tau) (F_{2,1/2} g)(\tau) d\tau, \quad (2.26)$$

where

$$\hat{q}(\tau) = \int_0^\infty K_{i\tau}(2\sqrt{x}) \omega(x) \frac{dx}{x}. \quad (2.27)$$

Thus we arrive at

Proposition 3. Let $a, b, c > 0$. The sequences $\{f_n\}_{n \geq 0}$, $\{g_n\}_{n \geq 0}$ given by formulas (2.21), (2.23) are biorthogonal with respect to convolution (2.24), i.e

$$\int_0^\infty (f_n \hat{*} g_m)(x) x^c dx = 2\delta_{n,m} n! \Gamma(n+a+b) \Gamma(n+a+c) \Gamma(n+b+c). \quad (2.28)$$

Proof. The proof of the identity (2.28) is an immediate consequence of (2.19), (2.22), (2.26), (2.27) with $\omega(x) = x^c$. \square

In order to employ the original convolution (1.14) and Parseval-type equality (1.23) we will appeal to (1.1), (2.9) and Entry 3.14.3.2 in [16] ($\mu > 0$)

$$\int_0^\infty x^{s-1} e^{\mu x} K_{it}(\mu x) dx = \frac{\cosh(\pi \tau)}{\sqrt{\pi}(2\mu)^s} \Gamma(s - i\tau) \Gamma(s + i\tau) \Gamma\left(\frac{1}{2} - s\right), \quad 0 < \operatorname{Re}(s) < \frac{1}{2}. \quad (2.29)$$

Then the Continuous Dual Hahn orthogonality can be written accordingly

$$\begin{aligned} & \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) q(\tau) (Ff_n)(\tau) (Fg_m)(\tau) d\tau \\ &= \frac{n!}{2^c \sqrt{\pi}} \frac{\Gamma(n+a+b) \Gamma(n+a+c) \Gamma(n+b+c) \Gamma(1/2-c)}{\Gamma(a+1/2) \Gamma(b+1/2)} \delta_{n,m}, \end{aligned} \quad (2.30)$$

where $a, b, > 0$, $0 < c < 1/2$ and

$$f_n(x) = \frac{2^a}{\sqrt{\pi}} x^{a-1} e^{-x} (a+b)_n (a+c)_n {}_2F_2(-n, a+1/2; a+b, a+c; 2x), \quad (2.31)$$

$$g_m(x) = \frac{2^b}{\sqrt{\pi}} x^{b-1} e^{-x} (a+b)_m (b+c)_m {}_2F_2(-m, b+1/2; a+b, b+c; 2x), \quad (2.32)$$

$$q(\tau) = \frac{\cosh(\pi \tau)}{2^c \sqrt{\pi}} |\Gamma(c+i\tau)|^2 \Gamma\left(\frac{1}{2}-c\right) = \int_0^\infty x^{c-1} e^x K_{it}(x) dx. \quad (2.33)$$

Proposition 4. Let $a, b > 0$, $0 < c < 1/2$. The sequences of functions given by formulas (2.31), (2.32) are biorthogonal with respect to convolution (1.14), i.e

$$\int_0^\infty (f_n * g_m)(x) e^x x^c dx = \frac{n!}{2^c \sqrt{\pi}} \frac{\Gamma(n+a+b) \Gamma(n+a+c) \Gamma(n+b+c) \Gamma(1/2-c)}{\Gamma(a+1/2) \Gamma(b+1/2)} \delta_{n,m}. \quad (2.34)$$

Proof. Since sequences of functions $\{f_n\}_{n \geq 0}$, $\{g_n\}_{n \geq 0}$ given by (2.31), (2.32) evidently satisfy conditions of Theorem 1, the corresponding Parseval-type identity (1.17) for the convolution $f_n * g_m$ holds. Hence, multiplying both sides of (1.17) by $e^x x^c$ and integrating over \mathbb{R}_+ , we obtain

$$\int_0^\infty (f_n * g_m)(x) e^x x^c dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) q(\tau) (Ff_n)(\tau) (Fg_m)(\tau) d\tau,$$

where q is defined by (2.33) and the interchange of the order of integration is ensured by Fubini's theorem owing to the estimate (see (1.12), (2.29))

$$\begin{aligned} & \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) |(Ff_n)(\tau) (Fg_m)(\tau)| \int_0^\infty x^{c-1} e^x |K_{it}(x)| dx d\tau \\ & \leq \frac{2}{\pi^2} \int_0^\infty x^{c-1} e^x K_0(x) dx \int_0^\infty \tau \sinh(\pi \tau) |(Ff_n)(\tau) (Fg_m)(\tau)| d\tau \\ & \leq \frac{|\Gamma(c)|^2}{2^c \sqrt{\pi}} \Gamma\left(\frac{1}{2}-c\right) \|f_n\|_{L_2(\mathbb{R}_+, x dx)} \|g_m\|_{L_2(\mathbb{R}_+, x dx)} < \infty. \end{aligned}$$

Consequently, combining with (2.30), we end up with (2.34), completing the proof of Proposition 4. \square

Let us consider Wilson's polynomials (2.2). We will show that similar to (2.22) it generates the convolution orthogonality by means of the modified Kontorovich-Lebedev transform $F_{2,1/2}f$. The key ingredient will be the formula (see Entry 3.14.18.4 in [16])

$$\begin{aligned} & \int_0^\infty K_{it}(2\sqrt{x}) K_{it}(2\sqrt{x}) x^{s-1} dx = \frac{1}{4\Gamma(2s)} \Gamma\left(s + \frac{\nu+i\tau}{2}\right) \Gamma\left(s - \frac{\nu+i\tau}{2}\right) \\ & \times \Gamma\left(s + \frac{\nu-i\tau}{2}\right) \Gamma\left(s - \frac{\nu-i\tau}{2}\right), \quad \operatorname{Re}(s) > \frac{|\operatorname{Re}(\nu)|}{2}. \end{aligned} \quad (2.35)$$

But first we recall (2.19) to rewrite Wilson's orthogonality in the form

$$\begin{aligned} & \frac{1}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) \left| \Gamma\left(c + \frac{i\tau}{2}\right) \Gamma\left(d + \frac{i\tau}{2}\right) \right|^2 (F_{2,1/2} f_n)(\tau) (F_{2,1/2} g_m)(\tau) d\tau \\ &= 4\delta_{n,m} \frac{n! \Gamma(n+a+b) \Gamma(n+a+c) \Gamma(n+a+d) \Gamma(n+b+c) \Gamma(n+b+d) \Gamma(n+c+d)}{\Gamma(2n+a+b+c+d) (n+a+b+c+d-1)_n}, \end{aligned} \quad (2.36)$$

where

$$f_n(x) = 2x^{a-1}(a+b)_n(a+c)_n(a+d)_n \times {}_2F_3(-n, n+a+b+c+d-1; a+b, a+c, a+d; x), \quad (2.37)$$

$$g_n(x) = 2x^{b-1}(a+b)_n(b+c)_n(b+d)_n \times {}_2F_3(-n, n+a+b+c+d-1; a+b, b+c, b+d; x). \quad (2.38)$$

Proposition 5. Let $a, b, c, d > 0$. The sequences $\{f_n\}_{n \geq 0}, \{g_n\}_{n \geq 0}$ given by formulas (2.37), (2.38) are biorthogonal with respect to convolution (2.24), i.e

$$\int_0^\infty (f_n \hat{*} g_m)(x) K_{c-d}(2\sqrt{x}) x^{(c+d)/2} dx = \delta_{n,m} \frac{n! \Gamma(n+a+b) \Gamma(n+a+c) \Gamma(n+a+d) \Gamma(n+b+c) \Gamma(n+b+d) (c+d)_n}{\Gamma(2n+a+b+c+d) (n+a+b+c+d-1)_n}. \quad (2.39)$$

Proof. Appealing to (2.26) and choosing $\omega(x) = K_{c-d}(2\sqrt{x}) x^{(c+d)/2}$, we employ (2.35) with $s = (c+d)/2$, $v = c-d$ and (2.36) to get (2.39). \square

Concerning the original convolution (1.14), we prove the following

Proposition 6. Let $a, b, d > 0$, $0 < c < 1/2$. The sequences $\{f_n\}_{n \geq 0}, \{g_n\}_{n \geq 0}$ with general terms

$$f_n(x) = \frac{2^a}{\sqrt{\pi}} x^{a-1} e^{-x} (a+b)_n (a+c)_n (a+d)_n \times {}_3F_3(-n, n+a+b+c+d-1, a+1/2; a+b, a+c, a+d; 2x), \quad (2.40)$$

$$g_m(x) = \frac{2^b}{\sqrt{\pi}} x^{b-1} e^{-x} (a+b)_m (b+c)_m (b+d)_m \times {}_3F_3(-m, m+a+b+c+d-1, b+1/2; a+b, b+c, b+d; 2x), \quad (2.41)$$

respectively, are biorthogonal with respect to convolution (2.14) and the orthogonality identity holds

$$\int_0^\infty (f_n * g_m)(x) \omega(x) dx = \frac{\delta_{n,m}}{2^{c+d}} \frac{n! \Gamma(n+a+b) \Gamma(n+a+c) \Gamma(n+a+d) \Gamma(n+b+c) \Gamma(n+b+d) \Gamma(n+c+d) \Gamma(1/2-c)}{\Gamma(1/2+d) \Gamma(2n+a+b+c+d) (n+a+b+c+d-1)_n}, \quad (2.42)$$

where

$$\omega(x) = x^d \int_0^\infty \frac{K_d(x+y)}{(x+y)^d} e^y y^{c+d-1} dy, \quad x > 0. \quad (2.43)$$

Proof. Indeed, analogously to (2.30), we recall equalities (1.16), (2.2), (2.9), (2.29), to expose the right-hand side of (1.23) in the form

$$\frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi \tau) q(\tau) (Ff_n)(\tau) (Fg_m)(\tau) d\tau = \frac{\delta_{n,m}}{2^{c+d}} \frac{n! \Gamma(n+a+b) \Gamma(n+a+c) \Gamma(n+a+d) \Gamma(n+b+c) \Gamma(n+b+d) \Gamma(n+c+d) \Gamma(1/2-c)}{\Gamma(1/2+d) \Gamma(2n+a+b+c+d) (n+a+b+c+d-1)_n}, \quad (2.44)$$

where $0 < c < 1/2$,

$$q(\tau) = \int_0^\infty K_{i\tau}(x) (t^{c-1} e^t * t^{d-1} e^{-t})(x) dx, \quad (2.45)$$

and f_n, g_n are given by formulas (2.40), (2.41), respectively. In the meantime, the convolution (1.14) $(t^{c-1} e^t * t^{d-1} e^{-t})(x)$ in (2.45) can be treated, taking into account the formula (see Entry 2.2.1.8 in [16])

$$\int_0^\infty x^{s-1} e^{-ax-b/x} dx = 2 \left(\frac{b}{a}\right)^{s/2} K_s(2\sqrt{ab}), \quad a, b > 0, s \in \mathbb{C}. \quad (2.46)$$

Then we get from (1.14)

$$(t^{c-1}e^t * t^{d-1}e^{-t})(x) = x^{d-1} \int_0^\infty \frac{K_d(x+y)}{(x+y)^d} e^y y^{c+d-1} dy$$

and, consequently, arrive at the convolution orthogonality (2.42) of functions (2.40), (2.41) with the weight function (2.43), obtaining from (2.44) and (1.23). \square

3. Orthogonal and 2-orthogonal polynomials of the Prudnikov type

In this section we shall show how the Prudnikov-type orthogonal and 2-orthogonal polynomials (see [17–20]), which are associated with the scaled Macdonald functions $\rho_\nu(x) = 2x^{\nu/2}K_\nu(2\sqrt{x})$, generate new biorthogonal systems of functions by virtue of the modified Kontorovich–Lebedev transform (1.30) $F_{2,1/2}f$. Precisely, let us consider the following orthogonalities

$$\int_0^\infty P_n^{\nu,\alpha}(x)P_m^{\nu,\alpha}(x)x^\alpha \rho_\nu(x)dx = \delta_{n,m}, \quad \nu \geq 0, \alpha > 0, \quad (3.1)$$

$$\int_0^\infty Q_m^\nu(x)Q_n^\nu(x)e^{-x}\rho_\nu(x)dx = \delta_{n,m}, \quad \nu > 0, \quad (3.2)$$

$$\int_0^\infty q_m^\nu(x)q_n^\nu(x)e^{-1/x}\rho_\nu(x)\frac{dx}{x} = \delta_{n,m}, \quad \nu > 0, \quad (3.3)$$

and 2-orthogonality conditions ($\alpha > 0$) that is a particular case of the multiple orthogonality with respect to the vector of measures ($x^\alpha \rho_\nu(x)$, $x^\alpha \rho_{\nu+1}(x)$)

$$\int_0^\infty p_{2n}^{\nu,\alpha}(x)\rho_\nu(x)x^{\alpha+m}dx = 0, \quad m = 0, 1, 2, \dots, n-1, \quad (3.4)$$

$$\int_0^\infty p_{2n}^{\nu,\alpha}(x)\rho_{\nu+1}(x)x^{\alpha+m}dx = 0, \quad m = 0, 1, 2, \dots, n-1, \quad (3.5)$$

$$\int_0^\infty p_{2n+1}^{\nu,\alpha}(x)\rho_\nu(x)x^{\alpha+m}dx = 0, \quad m = 0, 1, 2, \dots, n, \quad (3.6)$$

$$\int_0^\infty p_{2n+1}^{\nu,\alpha}(x)\rho_{\nu+1}(x)x^{\alpha+m}dx = 0, \quad m = 0, 1, 2, \dots, n-1. \quad (3.7)$$

Notice that according to the notation of multiple orthogonal polynomials in [15] $p_{2n}^{\nu,\alpha}(x)$, $p_{2n+1}^{\nu,\alpha}(x)$ are the polynomials associated with the indices (n, n) , $(n+1, n)$, respectively. We will emphasize the following results.

Proposition 7. The sequences of polynomials $\{S_n^{\nu,\alpha}(\tau^2)\}_{n \geq 0}$, $\{U_n^{\nu,\alpha}(\tau^2)\}_{n \geq 0}$ whose general terms are given by formulas

$$S_n^{\nu,\alpha}(\tau^2) = \sum_{k=0}^n a_{n,k}(1+i\tau)_k(1-i\tau)_k, \quad (3.8)$$

$$U_n^{\nu,\alpha}(\tau^2) = \sum_{k=0}^n \frac{a_{n,k}}{(2\alpha+\nu)_{2k}}(\alpha+\nu+i\tau)_k(\alpha+\nu-i\tau)_k(\alpha+i\tau)_k(\alpha-i\tau)_k, \quad (3.9)$$

respectively, where coefficients $a_{n,k}$ of the Prudnikov polynomials $P_n^{\nu,\alpha}$ are given explicitly in [19] are biorthogonal, i.e.

$$\int_0^\infty S_n^{\nu,\alpha}(\tau^2)U_m^{\nu,\alpha}(\tau^2) \left| \frac{\tau \Gamma(\nu+\alpha+i\tau) \Gamma(\alpha+i\tau)}{\Gamma(i\tau+1/2)} \right|^2 d\tau = \frac{1}{2} \Gamma(2\alpha+\nu) \delta_{n,m}. \quad (3.10)$$

Proof. In fact, recalling the Parseval identity (1.33) for the modified Kontorovich–Lebedev transform $F_{2,1/2}f$, we obtain from (3.1) the equalities

$$\begin{aligned} \int_0^\infty P_n^{\nu,\alpha}(x)P_m^{\nu,\alpha}(x)x^\alpha \rho_\nu(x)dx &= \frac{1}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) F_{2,1/2}(P_n^{\nu,\alpha}(x))(\tau) \\ &\times F_{2,1/2}(P_m^{\nu,\alpha}(x)x^{\alpha-1}\rho_\nu(x))(\tau) d\tau = \delta_{n,m}. \end{aligned} \quad (3.11)$$

Meanwhile, the use of (2.19) yields (cf. (3.8))

$$\begin{aligned} F_{2,1/2}(P_n^{\nu,\alpha}(x))(\tau) &= \frac{1}{2} \left| \Gamma\left(1 + \frac{i\tau}{2}\right) \right|^2 \sum_{k=0}^n a_{n,k} \left(1 + \frac{i\tau}{2}\right)_k \left(1 - \frac{i\tau}{2}\right)_k \\ &= \frac{1}{2} \left| \Gamma\left(1 + \frac{i\tau}{2}\right) \right|^2 S_n^{\nu,\alpha}\left(\frac{\tau^2}{4}\right), \end{aligned} \quad (3.12)$$

where $a_{n,k}$ are the coefficients of polynomials $P_n^{\nu,\alpha}$ given in [19]. Analogously, equality (2.35) implies (cf. (3.9))

$$\begin{aligned} F_{2,1/2} \left(P_m^{\nu,\alpha}(x) x^{\alpha-1} \rho_\nu(x) \right) (\tau) &= \frac{1}{2\Gamma(2\alpha+\nu)} \left| \Gamma \left(\alpha + \nu + \frac{i\tau}{2} \right) \right|^2 \left| \Gamma \left(\alpha + \frac{i\tau}{2} \right) \right|^2 \\ &\times \sum_{k=0}^m \frac{a_{m,k}}{(2\alpha+\nu)_{2k}} \left(\alpha + \nu + \frac{i\tau}{2} \right)_k \left(\alpha + \nu - \frac{i\tau}{2} \right)_k \left(\alpha + \frac{i\tau}{2} \right)_k \left(\alpha - \frac{i\tau}{2} \right)_k \\ &= \frac{1}{2\Gamma(2\alpha+\nu)} \left| \Gamma \left(\alpha + \nu + \frac{i\tau}{2} \right) \right|^2 \left| \Gamma \left(\alpha + \frac{i\tau}{2} \right) \right|^2 U_m^{\nu,\alpha} \left(\frac{\tau^2}{4} \right). \end{aligned} \quad (3.13)$$

Therefore, substituting right-hand sides of latter equalities in (3.12), (3.13) into the right-hand side of the first equality in (3.11), making a simple substitution in the integral and employing the reflection, addition and duplication formulas for the gamma function, we arrive at (3.10), completing the proof of the proposition. \square

Appealing again to (1.33), (2.35) and Entry 3.14.3.10 in [16] (cf. (2.17))

$$\int_0^\infty x^{s-1} e^{-ax^2} K_\mu(bx) dx = \frac{a^{(1-s)/2}}{2b} e^{b^2/(8a)} \Gamma \left(\frac{s+\mu}{2} \right) \Gamma \left(\frac{s-\mu}{2} \right) W_{(1-s)/2, \mu/2} \left(\frac{b^2}{4a} \right),$$

where $a, b > 0$, $\operatorname{Re}(s) > |\operatorname{Re}(\mu)|$, the orthogonality (3.2) of the Prudnikov-type polynomials Q_n^ν leads us to

Proposition 8. *Let $\nu > 0$. The following functions*

$$\begin{aligned} F_{2,1/2} \left(Q_n^\nu(x) e^{-x} x^{\nu/2-1} \right) (\tau) &= \frac{\sqrt{e}}{2} \left| \Gamma \left(\frac{i\tau + \nu}{2} \right) \right|^2 \sum_{k=0}^n b_{n,k} \left(\frac{\nu + i\tau}{2} \right)_k \left(\frac{\nu - i\tau}{2} \right)_k \\ &\times W_{(1-\nu)/2-k, i\tau/2}(1), \quad \tau > 0, \end{aligned} \quad (3.14)$$

$$\begin{aligned} F_{2,1/2} \left(Q_n^\nu(x) K_\nu(2\sqrt{x}) \right) (\tau) &= \frac{1}{4} \left| \Gamma \left(1 + \frac{\nu + i\tau}{2} \right) \Gamma \left(1 - \frac{\nu + i\tau}{2} \right) \right|^2 \\ &\times \sum_{k=0}^n \frac{b_{n,k}}{(2)_{2k}} \left(1 + \frac{\nu + i\tau}{2} \right)_k \left(1 - \frac{\nu + i\tau}{2} \right)_k \left(1 + \frac{\nu - i\tau}{2} \right)_k \left(1 - \frac{\nu - i\tau}{2} \right)_k, \quad \tau > 0, \end{aligned} \quad (3.15)$$

where $F_{2,1/2}f$ is the modified Kontorovich–Lebedev transform (1.30) and $b_{n,k}$ are coefficients of the polynomials Q_n^ν being calculated explicitly in [20], are biorthogonal in the sense of the identity

$$\int_0^\infty \tau \sinh(\pi\tau) F_{2,1/2} \left(Q_n^\nu(x) e^{-x} x^{\nu/2-1} \right) (\tau) F_{2,1/2} \left(Q_m^\nu(x) K_\nu(2\sqrt{x}) \right) (\tau) d\tau = \frac{\pi^2}{2} \delta_{n,m}. \quad (3.16)$$

Analogously we treat the orthogonality (3.3). However, to do this, we will need the formula (see [16], Entry 3.14.3.13)

$$\begin{aligned} 2 \int_0^\infty e^{-1/x} K_{i\tau}(2\sqrt{x}) x^{s-1} dx &= \Gamma \left(s + \frac{i\tau}{2} \right) \Gamma \left(s - \frac{i\tau}{2} \right) {}_0F_2 \left(1 - s + \frac{i\tau}{2}, 1 - s - \frac{i\tau}{2}; -1 \right) \\ &+ \Gamma \left(-s - \frac{i\tau}{2} \right) \Gamma(-i\tau) {}_0F_2 \left(1 + s + \frac{i\tau}{2}, 1 + i\tau; -1 \right) \\ &+ \Gamma \left(-s + \frac{i\tau}{2} \right) \Gamma(i\tau) {}_0F_2 \left(1 + s - \frac{i\tau}{2}, 1 - i\tau; -1 \right), \quad s \in \mathbb{C}. \end{aligned}$$

Hence, recalling (1.33) and (2.35), it gives

Proposition 9. *Let $\nu > 0$. The following functions*

$$\begin{aligned} F_{2,1/2} \left(q_n^\nu(x) e^{-1/x} x^{-2} \right) (\tau) &= \frac{1}{2} \left| \Gamma \left(\frac{i\tau}{2} - 1 \right) \right|^2 \sum_{k=0}^n c_{n,k} \left(\frac{i\tau}{2} - 1 \right)_k \left(-\frac{i\tau}{2} - 1 \right)_k \\ &\times {}_0F_2 \left(2 - k + \frac{i\tau}{2}, 2 - k - \frac{i\tau}{2}; -1 \right) + \frac{\Gamma(-i\tau)}{2} \sum_{k=0}^n c_{n,k} \Gamma \left(1 - k - \frac{i\tau}{2} \right) \end{aligned}$$

$$\begin{aligned} & \times {}_0F_2\left(k + \frac{i\tau}{2}, 1 + \frac{i\tau}{2}; -1\right) + \frac{\Gamma(i\tau)}{2} \sum_{k=0}^n c_{n,k} \Gamma\left(1 - k + \frac{i\tau}{2}\right) \\ & \times {}_0F_2\left(k - \frac{i\tau}{2}, 1 - \frac{i\tau}{2}; -1\right), \quad \tau > 0, \end{aligned} \quad (3.17)$$

$$\begin{aligned} F_{2,1/2}(q_m^\nu(x)\rho_\nu(x))(\tau) &= \frac{1}{2\Gamma(2+\nu)} \left| \Gamma\left(1 + \nu + \frac{i\tau}{2}\right) \Gamma\left(1 + \frac{i\tau}{2}\right) \right|^2 \\ & \times \sum_{k=0}^m \frac{c_{m,k}}{(2+\nu)_{2k}} \left(1 + \nu + \frac{i\tau}{2}\right)_k \left(1 + \nu - \frac{i\tau}{2}\right)_k \left(1 + \frac{i\tau}{2}\right)_k \left(1 - \frac{i\tau}{2}\right)_k, \quad \tau > 0, \end{aligned} \quad (3.18)$$

where coefficients $c_{m,k}$ of the polynomials q_n^ν are calculated explicitly in [20], are biorthogonal in the sense of the identity

$$\int_0^\infty \tau \sinh(\pi\tau) F_{2,1/2}(q_n^\nu(x)e^{-1/x}x^{-2})(\tau) F_{2,1/2}(q_m^\nu(x)\rho_\nu(x))(\tau) d\tau = \pi^2 \delta_{n,m}. \quad (3.19)$$

Finally, new families of 2-orthogonal polynomials appear in a natural way when we deal with modified Kontorovich-Lebedev transformation (1.30) of 2-orthogonal polynomials defined by (3.4)–(3.7). We will proceed this with the use of the explicit formula for polynomials $p_n^{\nu,\alpha}$ from [17]

$$p_n^{\nu,\alpha}(x) = (-1)^n (1+\alpha)_n (1+\alpha+\nu)_n {}_1F_2(-n; 1+\alpha, 1+\alpha+\nu; x).$$

Then, employing (1.33), (2.19), (2.35) as above, we derive from (3.4)–(3.7), correspondingly, the following orthogonality conditions

$$\int_0^\infty V_{2n}^{\nu,\alpha}(\tau^2) \left| \frac{\tau \Gamma(1+\nu+i\tau) \Gamma(\alpha+i\tau+m)}{\Gamma(i\tau+1/2)} \right|^2 d\tau = 0, \quad m = 0, 1, 2, \dots, n-1, \quad (3.20)$$

$$\int_0^\infty R_{2n}^{\nu,\alpha}(\tau^2) \left| \frac{\tau \Gamma(2+\nu+i\tau) \Gamma(\alpha+i\tau+m)}{\Gamma(i\tau+1/2)} \right|^2 d\tau = 0, \quad m = 0, 1, 2, \dots, n-1, \quad (3.21)$$

$$\int_0^\infty V_{2n+1}^{\nu,\alpha}(\tau^2) \left| \frac{\tau \Gamma(1+\nu+i\tau) \Gamma(\alpha+i\tau+m)}{\Gamma(i\tau+1/2)} \right|^2 d\tau = 0, \quad m = 0, 1, 2, \dots, n, \quad (3.22)$$

$$\int_0^\infty R_{2n+1}^{\nu,\alpha}(\tau^2) \left| \frac{\tau \Gamma(2+\nu+i\tau) \Gamma(\alpha+i\tau+m)}{\Gamma(i\tau+1/2)} \right|^2 d\tau = 0, \quad m = 0, 1, 2, \dots, n-1, \quad (3.23)$$

where general terms of the polynomial sequences $\{V_n^{\nu,\alpha}(\tau^2)\}_{n \geq 0}$, $\{R_n^{\nu,\alpha}(\tau^2)\}_{n \geq 0}$ are defined accordingly

$$\begin{aligned} V_n^{\nu,\alpha}(\tau^2) &= (-1)^n (1+\alpha)_n (1+\alpha+\nu)_n \\ & \times {}_5F_4\left(-n, 1+\nu+i\tau, 1+\nu-i\tau, 1+i\tau, 1-i\tau; \frac{\nu+2}{2}, \frac{\nu+3}{2}, 1+\alpha, 1+\alpha+\nu; \frac{1}{4}\right), \\ R_n^{\nu,\alpha}(\tau^2) &= (-1)^n (1+\alpha)_n (1+\alpha+\nu)_n \\ & \times {}_5F_4\left(-n, 2+\nu+i\tau, 2+\nu-i\tau, 1+i\tau, 1-i\tau; \frac{\nu+3}{2}, \frac{\nu+4}{2}, 1+\alpha, 1+\alpha+\nu; \frac{1}{4}\right). \end{aligned}$$

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