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Journal of Computational and Applied Mathematics 71 (1996) 147–162

JOURNAL OF  
COMPUTATIONAL AND  
APPLIED MATHEMATICS

# On the asymptotic behavior of solutions of impulsively damped nonlinear oscillator equations

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Received 17 May 1995

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## Abstract

Intermittently damped oscillator equations are important both in practice and attractivity investigations. The problem of attractivity appears clearly if the damping is concentrated into discrete points.

In this paper we investigate the asymptotic behavior of the impulsive equation

$$\ddot{x} + f(x) = 0 \quad (t \neq t_n)$$

$$\dot{x}(t_n + 0) = b_n \dot{x}(t_n) \quad (t = t_n)$$

( $n = 1, 2, \dots$ ). We find an analogy, but not strict correspondence, to the attractivity results for distributed damping. The attractivity properties mainly depend on the properties of  $f(x)$ .

**Keywords:** Asymptotic behavior; Damped equation; Impulses; Nonlinear equation; Oscillator equation

**AMS classification:** 34D05; 34D20; 34C15

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## 1. Introduction

In this paper we consider the impulsively damped equation

$$\ddot{x} + f(x) = 0 \quad (t \neq t_n), \quad x(t_n + 0) = x(t_n), \quad \dot{x}(t_n + 0) = b_n \dot{x}(t_n), \quad (1)$$

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<sup>1</sup> Research supported by the Mississippi State University Biological and Physical Sciences Research Institute.

<sup>2</sup> Research supported by Hungarian National Foundation for Scientific Research grant no. 6032/6319, and by the Foundation for the Hungarian Higher Education and Research grant no. 776/94.

where  $t_n \rightarrow \infty$  ( $n \rightarrow \infty$ ),  $xf(x) > 0$ ,  $x \neq 0$ ,  $f$  is continuous ( $x \in \mathbb{R}$ ), and we investigate the asymptotic behavior of the solutions. Eq. (1) is the impulsive analogue of the damped oscillator equation

$$\ddot{x} + g(t)\dot{x} + f(x) = 0. \quad (2)$$

This is very obvious if  $g(t)$  is an on–off function and the lengths of the “on” intervals are very short. The literature on the asymptotic behavior of the solutions of (2) is very large, see for example [1, 3, 5–11, 13]. However, there are no known necessary and sufficient conditions for asymptotic stability of the zero solution. Special equations with on–off damping are of importance when discussing necessary conditions, as the following examples show (for details see [6, 7, 9, 13]).

**Example 1.** Let  $p > 0$  be given and let  $g(t) = 1$  if  $t \in [np, np + 1/n]$ ,  $g(t) = 0$  otherwise, and let  $f(x) = |x|^\alpha \operatorname{sgn} x$ .

Case  $\alpha = 1$ : The zero solution is globally asymptotically stable (g.a.s.) if  $p \neq \pi$ , but not asymptotically stable (a.s.) if  $p = \pi$  [4].

Case  $\alpha > 1$ : The zero solution is a.s. for every  $p > 0$ , but g.a.s. is not guaranteed (an example is given in [9]).

Case  $0 < \alpha < 1$ : The question of asymptotic stability is open.

The next example applies our results from Section 3 to the analogous impulsive equation. In the following sections we will explain the analogy.

**Example 2.** Consider the impulsive equation

$$\ddot{x} + |x|^\alpha \operatorname{sgn} x = 0 \quad (t \neq pn), \quad \dot{x}(pm + 0) = \frac{n-2}{n} \dot{x}(pm).$$

Case  $\alpha = 1$ : The zero solution is g.a.s. if  $p \neq k\pi$  ( $k = 1, 2, \dots$ ), and not a.s. if  $p = k\pi$  ( $k = 1, 2, \dots$ ).

Case  $\alpha > 1$ : The zero solution is a.s. but not g.a.s.

Case  $0 < \alpha < 1$ : The zero solution is not a.s.

Notice that the properties are different in the linear, sublinear and superlinear cases.

In this paper we give necessary as well as sufficient conditions for asymptotic stability, and we investigate the structure of the set of solutions. In the next section, we state preliminary lemmas and investigate the analogy to distributed damping. In the third section, we present our asymptotic stability theorems, and the last section contains the details of the proofs. As a special case, we obtain the example given above.

## 2. Definitions, assumptions, preliminaries

We say that the zero solution of (1) or (2) is stable if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|x(0)| + |\dot{x}(0)| < \delta$  implies  $|x(t)| + |\dot{x}(t)| < \varepsilon$  ( $t \geq 0$ ). The zero solution is a.s. if there exists  $\delta > 0$  such that  $|x(0)| + |\dot{x}(0)| < \delta$  implies  $\lim_{t \rightarrow \infty} (x(t), \dot{x}(t)) = (0, 0)$ . The asymptotic stability is global (g.a.s.) if  $\delta = \infty$ .

We will use the following notation. Let  $t_{n(t)}$  denote the largest  $k$  smaller than the given time  $t$ ; formally,  $t_{n(t)} := \max(t_n: t_n < t)$ . For example,  $t_{n(t_{k+1})} = t_k$ . Since  $\lim_{n \rightarrow \infty} t_n = \infty$ , every solution can be continued to  $\infty$ . The solutions are differentiable and  $\dot{x}(t)$  is piecewise continuous and continuous from the left-hand side at every  $t > 0$ . We investigate the solutions with the help of the energy function

$$V(x, \dot{x}) = \dot{x}^2 + 2 \int_0^x f(u) du =: \dot{x}^2 + F(x), \quad (3)$$

and we introduce the notation

$$F_1(x) = 2 \int_0^{-x} f(u) du \quad \text{and} \quad F_2(x) = 2 \int_0^x f(u) du \quad \text{for } x \geq 0,$$

$$\gamma := \min \left( \sup_{x \geq 0} F_1(x), \sup_{x \geq 0} F_2(x) \right).$$

Obviously,  $F(x) = F_1(-x)$  for  $x \leq 0$ .

If it does not result in a misunderstanding, we use the simple notation  $V(t)$  instead of  $V(x(t), \dot{x}(t))$ . If necessary, we will use the number of the equation as a subscript.

Let us calculate the change of the energy along the solutions of (1):

$$\begin{aligned} V(t_{n+1}) - V(t_n) &= V(t_n + 0) - V(t_n) \\ &= \dot{x}^2(t_n + 0) + F(x(t_n + 0)) - \dot{x}^2(t_n) - F(x(t_n)) \\ &= b_n^2 \dot{x}^2(t_n) - \dot{x}^2(t_n) = -\dot{x}^2(t_n)(1 - b_n^2) = -a_n \dot{x}^2(t_n), \end{aligned} \quad (4)$$

where  $a_n = 1 - b_n^2$  is the  $n$ th energy-quantum.

The energy is nonincreasing if  $b_n^2 \leq 1$  independently of the sign of  $b_n$  and is constant between any  $t_n$  and  $t_{n+1}$ . Since we want to avoid “beating” at  $t_n$  and to keep the initial monotonicity of the solutions beyond  $t_n$  (as in the case of distributed damping), we assume  $b_n \geq 0$ . This assumption guarantees that the solutions are differentiable at the extremal points.

We note that if  $|b_n| > 1$ , then the energy increases at  $t_n$ . On the other hand, if we start from  $a_n$  (we know the “energy-quantums”) with  $a_n > 1$ , this decrease in the energy would require an impulse effect for  $x(t)$  as well. In this case,  $x(t)$  is not continuous. In this paper we do not deal with these cases.

There is a critical situation when  $b_n = 0$  ( $a_n = 1$ ). In this case, we lose the uniqueness of the solutions at  $t_n$ . Moreover, there exist solutions that are identically zero for  $t > t_n$ .

In the following, we assume that

$$(H1) \quad 0 \leq b_n \leq 1 \quad (1 \geq a_n \geq 0), \quad n = 1, 2, \dots$$

To guarantee uniqueness, we use the more restrictive condition

$$(H1') \quad 0 < b_n \leq 1 \quad (1 > a_n \geq 0), \quad n = 1, 2, \dots,$$

or even

$$(H2) \quad \liminf_{n \rightarrow \infty} b_n > 0 \quad \left( \limsup_{n \rightarrow \infty} a_n < 1 \right).$$

Now compare the change of the energy along the solutions of (2) and (1). It is known that if  $x(t)$  is a solution of (2), then  $V(\cdot)$  takes the form

$$V_{(2)}(t) = V_{(2)}(x(t), \dot{x}(t)) = V_{(2)}(0) - 2 \int_0^t g(s) \dot{x}^2(s) ds.$$

If  $g(t)$  is on–off, that is,

$$g(t) = \begin{cases} g_n > 0, & t \in [t_n, t_n + i_n] \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

then we have

$$V_{(2)}(t_{n+1}) - V_{(2)}(t_n) = -2 \int_{t_n}^{t_n + i_n} g_n \dot{x}^2(s) ds \approx -2g_n i_n \dot{x}^2(\eta_n),$$

where  $\eta_n \in [t_n, t_n + i_n]$ .

We can see that  $2g_n i_n \approx a_n = 1 - b_n^2$  and  $\int_{t_n}^{t_{n+1}} g(s) ds \approx a_n$  in general. This relation makes it possible to create impulsive damping from distributed ones, and distributed dampings from impulsive ones. But this correspondence is not one-to-one, and the asymptotic behavior can be different for the created equations. As illustrations, let us consider some examples.

**Example 3.** Let  $g(t) = 1$  in Eq. (2) and let  $\{t_n\}$  any monotone sequence tending to infinity. Then

$$2 \int_{t_n}^{t_{n+1}} g(s) ds = 2(t_{n+1} - t_n).$$

If  $t_n = n$ , then  $a_n = 2$  can be the impulse that results in discontinuous solutions. If  $t_n = n/3$ , then  $a_n = \frac{2}{3}$ .

**Example 4.** Let  $g(t) = t$  and let  $t_n = \sqrt{n/2}$ . Then

$$2 \int_{t_n}^{t_{n+1}} g(s) ds = \frac{n+1}{2} - \frac{n}{2} = \frac{1}{2}.$$

But with the choice of  $t_n = \sqrt{n}$ , we obtain  $a_n = 1$ , which is the critical case for uniqueness of solutions.

**Example 5.** Similarly, if  $g(t) = t^2$  and  $t_n = \sqrt[3]{n}$ , then  $a_n = \frac{2}{3}$ .

**Example 6.** For  $g(t) = 1/(t+1)$ , let  $t_n = n$  and

$$2 \int_n^{n+1} \frac{1}{t+1} dt = \log\left(1 + \frac{1}{n+1}\right) \approx \frac{1}{n+1} =: a_n.$$

If  $g(t)$  is an on–off function defined by (5) and  $\lim_{n \rightarrow \infty} i_n = 0$ , then the correspondence is quite definite, i.e.,  $a_n := 2g_n i_n$ . Finally, we consider some examples of this type.

**Example 7.** Let  $f(x) = x$ ,  $g_n = 1/n$ ,  $t_n = n\pi$ , and  $i_n = \frac{1}{12}\pi$  in Eq. (2). Then by the result in [9], the zero solution is g.a.s. But taking  $b_n = \sqrt{1 - \pi/12n}$  and  $n\pi$  for the instants, we can easily verify that the energy is constant along the solutions of (1). On the other hand, if we take  $(n + \frac{1}{12})\pi$  for the instants of impulses, we find that the zero solution is g.a.s. (see [10]).

The analogy is sharper if  $\lim_{n \rightarrow \infty} i_n = 0$ . Then it does not seem to matter how the points of  $[t_n, t_n + i_n]$  are chosen. The above examples illustrate the problem of asymptotic equivalency, which is as yet unsolved.

Let us return to the behavior of the energy along the solutions of (1). Using the equality (4) repeatedly, we obtain

$$V(t) = V(0) - \sum_{t_n < t} a_n \dot{x}^2(t_n) \quad (6)$$

along the solutions of (1). Similarly to (6),  $V(t)$  can be expressed as follows: If  $t_n < t \leq t_{n+1}$ , we have

$$V(t) = V(t_{n+0}) = V(t_n) \left( 1 - a_n \frac{\dot{x}^2(t_n)}{V(t_n)} \right) = V(0) \prod_{t_i < t} \left( 1 - a_i \frac{\dot{x}^2(t_i)}{V(t_i)} \right). \quad (7)$$

A lower estimate is

$$V(t) \geq V(t_n)(1 - a_n) = V(0) \prod_{t_i < t} (1 - a_i) = V(0) \left( \prod_{t_i < t} b_i \right)^2. \quad (8)$$

Using the equality (7), it is easy to prove the following basic lemma.

**Lemma 8.** Under the assumption (H1),  $V(t)$  is nonincreasing for every solution. Consequently, the zero solution of (1) is stable. Moreover, the solutions with initial conditions satisfying  $\dot{x}^2(T) + F(x(T)) < \gamma$  ( $T \geq 0$ ) are bounded. For these solutions,  $\liminf_{t \rightarrow \infty} \dot{x}^2(t) = 0$ .

If (H1') holds, we obtain a Gronwall–Bellman type inequality for  $V(t)$  from (7). Since  $\log(1 - y) \leq -y$  for  $y < 1$ , we have

$$\begin{aligned} \log V(t) &= \log V(0) + \sum_{t_i < t} \log \left( 1 - a_i \frac{\dot{x}^2(t_i)}{V(t_i)} \right) \\ &\leq \log V(0) - \sum_{t_i < t} a_i \frac{\dot{x}^2(t_i)}{V(t_i)}. \end{aligned}$$

We have thus proved the following lemma.

**Lemma 9.** Suppose that (H1') holds. If  $x(t)$  is a solution of (1), then

$$V(t) \leq V(0) \exp \left\{ - \sum_{t_n < t} a_n \frac{\dot{x}^2(t_n)}{V(t_n)} \right\}. \quad (9)$$

From (8) and the fact that  $\log(1 - y) \geq -Cy$  if  $1 > y \geq K$  for some  $K > 0$ , we obtain the following lower estimate for  $V(t)$ .

**Lemma 10.** *Suppose that (H1)–(H2) hold. Then*

$$V(t) \geq V(0)C \exp \left\{ \sum_{t_n < t} a_n \right\}. \quad (10)$$

Using this lemma, we obtain a necessary condition that is analogous to the case of distributed damping.

**Theorem 11.** *Suppose that (H1') holds. If either*

$$(i) \quad \prod_{n=1}^{\infty} b_n > 0,$$

*or (H2) and*

$$(i') \quad \sum_{n=1}^{\infty} a_n = \infty$$

*is satisfied, then for every solution of (1),  $\lim_{t \rightarrow \infty} V(t) > 0$ .*

The proof is a direct application of Lemma 10. We have only to observe that (H2) follows from (i). The theorem is not true if  $b_n = 0$  for some  $n$  since then there are solutions that are identically zero for  $t > t_n$ .

We also need estimates for  $\dot{x}(t)$ . The following lemma is fundamental in the estimation of the energy; it can be checked immediately or noted that it follows from a more general result [2, Corollary 2.4].

**Lemma 12.** *Let  $x(t)$  be a nonzero solution of (1) and  $T \geq 0$ . Then*

$$\dot{x}(t) = \dot{x}(T) \prod_{T \leq t_i < t} b_i - \int_T^t \left( \prod_{s \leq t_i < t} b_i \right) f(x(s)) ds. \quad (11)$$

*Obviously, in the integral, the product can be taken for  $s < t_i < t$ .*

Let us introduce the notation

$$\Phi(s, t) := \prod_{s \leq t_i < t} b_i.$$

If (H1') holds,  $\Phi(s, t)$  can be expressed in the form  $\varphi(t)/\varphi(s)$ , where  $\varphi(t) = \prod_{t_n < t} b_n$ . The equality (11) can then be written in the form

$$\dot{x}(t) = \dot{x}(T) \Phi(T, t) - \int_T^t \Phi(s, t) f(x(s)) ds. \quad (12)$$

The next lemma is also fundamental in attractivity investigations.

**Lemma 13.** Suppose that (H1) holds. Let  $x(t)$  be a solution of (1) that is nonzero on any interval  $[T, \infty)$ . Let  $t'$  and  $t''$  be consecutive zeros of  $\dot{x}(t)$ . Then there exists  $\tilde{t} \in (t', t'')$  for which

$$x(\tilde{t}) = 0.$$

Consequently, the solutions of (1) are either oscillatory or monotonic on some interval  $[T, \infty)$ .

**Proof.** The proof is the same as for Eq. (2) in [5]. Since  $x(t)$  non-trivial,  $f(x(t')), f(x(t'')) \neq 0$ . If  $x(t)$  does not have a zero in  $(t', t'')$ , then  $\inf_{t \in (t', t'')} |f(x(t))| = \min(f(x(t')), f(x(t''))) = \delta > 0$ . We can suppose that  $x(t) > 0$  on  $(t', t'')$ . Using Lemma 12, we obtain

$$\begin{aligned} 0 = \dot{x}(t'') &= \dot{x}(t') \prod_{t' \leq t_i < t''} b_i + \int_{t'}^{t''} \prod_{s < t_i < t''} b_i f(x(s)) ds \\ &> \delta \int_{t'}^{t''} \left( \prod_{s < t_i < t''} b_i \right) ds > \delta(t'' - t_{n(t'')}) > 0, \end{aligned}$$

which is a contradiction.  $\square$

If  $b_n = 0$ , then the solution can become zero from  $t_n$  on. We note that this lemma is not true if  $b_n < 0$  for some  $n$ , since we can obtain nonoscillatory “saw-tooth solutions”.

### 3. Asymptotic stability

On the basis of Theorem 11, the condition  $\prod_{n=1}^{\infty} b_n = \infty$  is necessary for the asymptotic stability of the zero solution. From the inequality (9), we can see that the variation of the energy at  $t_n$  highly depends on  $\dot{x}(t_n)$ .

To estimate the energy, our first task is to localize the sets where  $\dot{x}(t)$  is close to zero. In our criteria, we will somehow avoid these sets. For the analogous investigations on distributed damping, see, e.g., [6, 7, 9, 11].

Consider the undamped equation

$$\ddot{x} + f(x) = 0. \quad (13)$$

The behavior of the solutions of this equation was investigated, e.g., in [12]. We introduce some notation and cite some necessary results from [12].

The functions  $F(x)$ ,  $F_1(x)$ ,  $F_2(x)$  and the number  $\gamma$  are already introduced in Section 2. Let

$$\Delta(r) = \int_{-F_1^{-1}(r)}^{F_2^{-1}(r)} \frac{dx}{\sqrt{r - F(x)}} = \int_0^{F_1^{-1}(r)} \frac{dx}{\sqrt{r - F_1(-x)}} + \int_0^{F_2^{-1}(r)} \frac{dx}{\sqrt{r - F_2(x)}}$$

$$\Delta_0(r_0) = \inf_{0 < r < r_0} \Delta(r) \quad \text{if } r, r_0 < \gamma,$$

$$\Delta_{\infty} = \inf_{0 < r < \infty} \Delta_0(r) \quad \text{if } \gamma = \infty.$$

On the basis of investigations in [12], the following lemma holds.

**Lemma 14** (Reissig et al. [12]). Let  $x(t)$  be a nontrivial oscillatory solution of Eq. (13). Then  $V(x(t), \dot{x}(t)) = r = \text{const.} > 0$  and the distance between any two consecutive zeros of  $\dot{x}(t)$  is equal to

$$\int_{-F_1^{-1}(r)}^{F_2^{-1}(r)} \frac{dx}{\sqrt{r - F(x)}}.$$

Solutions with the property  $V(x(0), \dot{x}(0)) < \gamma$  are oscillatory.

The first statement of this lemma can be extended to Eq. (1) as follows.

**Lemma 15.** Let  $x(t)$  be a solution of (1) such that  $\lim_{t \rightarrow \infty} V(t) = r > 0$  ( $r < \gamma$ ). Then for any  $\varepsilon, \delta > 0$  ( $\delta < r$ ) there exists  $T > 0$  such that if  $T < s_1 < s'_1 < s'_2 < s_2$ ,  $F(x(s'_1)) = F(x(s'_2)) = r - \delta$  and  $\dot{x}(s'_1) = \dot{x}(s'_2) = 0$ , then

$$s'_2 - s'_1 \geq \int_{-F_1^{-1}(r-\delta)}^{F_2^{-1}(r-\delta)} \frac{dr}{\sqrt{(1+\varepsilon)r - F(x)}}.$$

Consequently, if  $x(t)$  is oscillatory and  $\{s_n\}$  is a sequence of zeros of  $\dot{x}(t)$ , then

$$\liminf_{n \rightarrow \infty} (s_{n+1} - s_n) \geq \Delta(r).$$

For an analogous lemma for Eq. (2), see [9]. A proof of Lemma 15 and some of the remainder of our results in this section are given in Section 4.

In [6], Hatvani and Totik introduced the following definition.

**Definition 16.** The sequence  $\{s_n\}_{n=1}^{\infty}$  is  $L$ -discrete if  $\liminf_{n \rightarrow \infty} (s_{n+1} - s_n) \geq L$ .

Using this terminology, the above lemma says that the sequence of the zeros of  $\dot{x}(t)$  is  $\Delta(r)$ -discrete.

Properties of  $\Delta(r)$  are investigated in detail in [12, Chapter 3.1]. For some functions  $f(x)$ ,  $\Delta(r)$  can be calculated precisely, while for others, numeric integration or comparison of results can help in obtaining an estimation. We cite the following result.

**Lemma 17** (Reissig et al. [12, Theorem 3.1.3]). Let  $r_0 > 0$  be given. If  $|\tilde{f}(x)| \geq |f(x)|$  for every  $x$  for which  $F(x) \leq r_0$ , then for these values of  $x$

$$\Delta(\tilde{F}(x)) = \int_0^x \frac{ds}{\sqrt{\tilde{F}(x) - \tilde{F}(s)}} \leq \int_0^x \frac{ds}{\sqrt{F(x) - F(s)}} = \Delta(F(x)).$$

In the special case  $f(x) = N|x|^\alpha \operatorname{sgn} x$ , calculations yield the following expression for  $\Delta(r)$ :

$$(a) \quad \alpha = 1, \quad \Delta(r) = \pi/\sqrt{N};$$

$$(b) \quad \alpha \neq 1, \quad \Delta(r) = Ar^\beta \quad \text{with } \beta = \frac{1-\alpha}{2(\alpha+1)} \quad \text{and}$$

$$A = 2 \left( \frac{\alpha+1}{2N} \right)^{1/(\alpha+1)} \frac{\sqrt{\pi} \Gamma(1/(\alpha+1))}{(\alpha+1) \Gamma((3+\alpha)/2(1+\alpha))}, \quad (14)$$

where  $\Gamma(\cdot)$  denotes the  $\Gamma$  function.



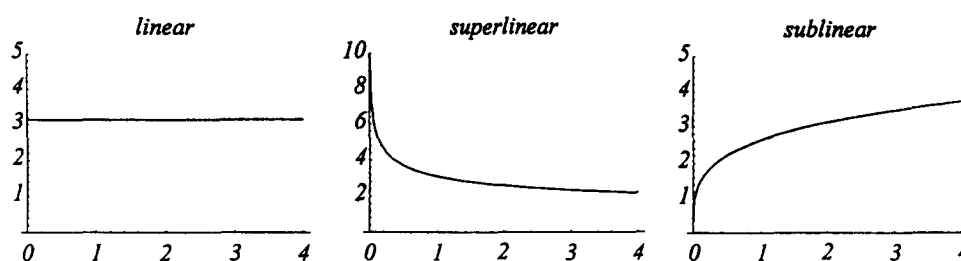


Fig. 1

In the case  $f(x) = x$ , we obtain the known value. In nonlinear cases, the period depends on the amplitude of the solution of the undamped equation, increasing to infinity in the superlinear and decreasing to zero in the sublinear cases. Fig. 1 shows the typical graphs for  $\Delta(r)$  (cases  $r = \frac{1}{3}, 1, 3$ ).

The above lemmas give us a very simple oscillation criterion. If for some  $r_0 > 0$  we have  $\Delta_0(r_0) > 0$  and  $\limsup_{n \rightarrow \infty} (t_{n+1} - t_n) > \Delta_0(r_0)$ , then the every solution of (1) with initial conditions satisfying  $V(x(0), \dot{x}(0)) < r_0$  is oscillatory. But we will see that exactly the other direction of the inequality is required for attractivity.

As was the case for distributed damping [9], to prove asymptotic stability we have to give lower estimates for  $\dot{x}^2(t)$ . Using these estimations for the expression (6)

$$V(t) = V(0) - \sum_{t_n < t} a_n \dot{x}^2(t_n),$$

we obtain asymptotic stability criteria. The following lemma is based on Lemma 12; here we give estimates for  $\dot{x}(t)$ .

**Lemma 18.** Let  $x(t)$  be a solution of (1) for which  $\lim_{t \rightarrow \infty} V(t) = r$  ( $0 < r < \gamma$ ). Let  $r > \delta > 0$  be given. There exists  $T(\delta) > 0$  such that if  $s_2 > s_1 > T$ , then the following statements hold on the interval  $[s_1, s_2]$ .

- (a) If  $F(x(t)) \leq r - \delta$  for  $t \in [s_1, s_2]$ , then  $|\dot{x}(t)| \geq \delta^{1/2}$ .
- (b) If  $F(x(t)) > r - \delta$  and  $x(t)\dot{x}(t) \geq 0$  for  $t \in [s_1, s_2]$ , then  $|\dot{x}(t)| \geq \delta_1 |t - s_2|$ .
- (c) If  $F(x(t)) > r - \delta$ ,  $x(t)\dot{x}(t) \leq 0$  for  $t \in [s_1, s_2]$ , then

$$|\dot{x}(t)| \geq \delta_1 \int_{s_1}^t \Phi(s, t) ds,$$

where  $\delta_1 = \inf \{ |f(u)| : F(u) > r - \delta, |u| < \sup_{t \geq 0} (x(t)) \} > 0$ .

If  $\Phi(s, t) \geq Le^{-K(t-s)}$  for  $t > s$ ,  $t, s \in (s_1, s_2)$ , then  $|\dot{x}(t)| \geq \varepsilon \delta_1 \min(1, t - s_1)$  for some  $0 < \varepsilon < 1$ .

We note that the constants in the above estimates depend only on  $\delta$  and  $r$ . Now we are ready to state our main attractivity theorem in two independent forms. The attractivity conditions are as follows.

**Condition Attr(L).** Suppose that there exists a sequence of intervals  $\{I_n\} = \{[s_n, s_n + i_n]\}$  such that  $s_n \rightarrow \infty$  as  $n \rightarrow \infty$ ,  $i_n > 0$ ,  $s_{n+1} \geq s_n + i_n$ , and

$$\limsup_{n \rightarrow \infty} i_n \leq L \quad (0 \leq L \leq \infty).$$

Let the impulse damping satisfy the following properties:

- (i)  $\liminf_{n \rightarrow \infty} \prod_{s_n < t_j < s_n + i_n} b_j > 0$ ;  
 (ii) for every sequence  $\{u_n\}$  with  $s_n \leq u_n \leq s_n + i_n$

$$\sum_{n=1}^{\infty} \left( \sum_{s_n \leq t_k \leq s_n + i_n} a_k \mu_n^2(t_k, u_n) \right) = \infty,$$

where  $\mu_n(t, u_n) = \min(1, |t - u_n|)$ .

We note that the difference between the subscripts in the sum and the product is essential as we will see in the corollaries. The condition  $\text{Attr}(L)$  can be formalized without (i). In this case, the definition of  $\mu_n$  is more complicated.

We have a more general form with the concept of  $L$ -discrete sets if (ii) is satisfied for every  $t > s$ .

**Condition  $\text{Attr}'(L)$ .** Let  $L > 0$ , and

$$(i') \quad \Phi(s, t) \geq L e^{-K(t-s)}$$

for every  $t > s$  large enough;

- (ii') for every  $L$ -discrete sequence  $\{s_n\}_{n=1}^{\infty}$ , we have

$$\sum_{k=1}^{\infty} a_k v_k^2 = \infty,$$

where  $v_k = \min(1, \text{dist}(\{s_n\}, t_k))$ .

Our main results are as follows.

**Theorem 19.** Suppose that conditions (H1) and  $\text{Attr}(L)$  hold for some  $L \geq 0$ . Let  $x(t)$  be a solution of Eq. (1) for which  $\lim_{t \rightarrow \infty} V(t) = r$  ( $r < \gamma$ ). Then either  $r = 0$ , or  $\Delta(r) \leq L$ .

**Theorem 19'.** Suppose that conditions (H1) and  $\text{Attr}'(L)$  hold for some  $L > 0$ . Let  $x(t)$  be a solution of Eq. (1) for which  $\lim_{t \rightarrow \infty} V(t) = r$  ( $r < \gamma$ ). Then either  $r = 0$ , or  $\Delta(r) < L$ .

It is easy to see that the assumptions (i'),  $\text{Attr}'(L')$  imply  $\text{Attr}(L)$  for every  $L' > L$ .

Our theorems are sharp, as the following result shows.

**Theorem 20.** Let  $t_n = np$  in Eq. (1), (H1) be satisfied, and let  $D_0 = \{r : \Delta(r) = p/k, k = 1, 2, \dots\}$ . The solutions of (1) with the initial conditions  $F(x(t_1)) = r \in D_0$ ,  $\dot{x}(t_1) = 0$  do not tend to zero.

The proof of Theorem 20 is obvious. The following corollary is a very important consequence.

**Corollary 21.** Let  $t_n = np$  ( $p > 0$ ).

- (a) If  $\lim_{r \rightarrow 0} \Delta(r) = 0$ , then the zero solution is not asymptotically stable.  
 (b) If  $\lim_{r \rightarrow \infty} \Delta(r) = 0$ , then the asymptotic stability is not global.

**Proof.** In the first case, for an arbitrarily small positive  $\varepsilon$  there exists a natural number  $K$  such that the equation  $\Delta(r) = p/k$ ,  $k > K$ , has a solution in  $(0, \varepsilon)$ . In the second case, for any natural number  $k$  there exists a solution of the equation  $\Delta(r) = p/k$  on the interval  $(0, \infty)$ .  $\square$

To apply Theorems 19 and 19' we can take several different sequences  $\{s_n\}$ . We obtain corollaries in the simplest form if we use the times  $t_n$  for  $s_n$ . Then  $i_n = t_{n+1} - t_n$ . With this choice, (i) is automatically satisfied. To examine condition (ii), let  $u_n$  be any number in  $[t_n, t_{n+1}]$ . Then

$$\begin{aligned} & \min((t_n - u_n)^2, 1)a_n + \min((t_{n+1} - u_n)^2, 1)a_{n+1} \\ & \leq (\min((t_n - u_n)^2, 1) + \min((t_{n+1} - u_n)^2, 1)) \min(a_n, a_{n+1}) \\ & \leq \min((\tfrac{1}{2}(t_{n+1} - t_n))^2, 1) \min(a_n, a_{n+1}), \end{aligned}$$

since one of the distances  $(t_{n+1} - u_n)$  or  $(u_n - t_n)$  is greater than  $\frac{1}{2}(t_{n+1} - t_n)$ .

**Corollary 22.** Suppose that (H1) holds and there exists a subsequence  $\{t_{k_l}\}$  such that  $\limsup_{l \rightarrow \infty} (t_{k_l+1} - t_{k_l}) = L \geq 0$  ( $< \infty$ ) and

$$\sum_{l=1}^{\infty} (t_{k_l+1} - t_{k_l})^2 \min(a_{k_l+1}, a_{k_l}) = \infty.$$

If  $x(t)$  is a solution of (1) with  $\lim_{t \rightarrow \infty} V(t) = r > 0$ , then  $\Delta(r) \leq L$ .

The conditions of Corollary 22 are satisfied for the impulse damping  $b_n = \sqrt{(n-1)/n}$ ,  $t_n = n$  with  $L = 1$  and  $k_l = l$  ( $l = 1, 2, \dots$ ). But they are not satisfied for  $b_{2n} = \sqrt{(n-1)/n}$ ,  $b_{2n+1} = 1 - 1/n^2$  with any subsequence  $t_{k_l}$ .

The following more general corollary can be applied for such a pulsative damping if we take  $s_n = t_{k_l}$  and  $i_n = t_{k_l+j_l} - t_{k_l}$ .

**Corollary 23.** Suppose that (H1) holds and there exists a subsequence  $\{t_{k_l}\}$  and a sequence  $\{j_l\}$  of positive integers such that

$$t_{k_l+1} \geq t_{k_l+j_l}, \quad \limsup_{l \rightarrow \infty} (t_{k_l+j_l} - t_{k_l}) \leq L$$

for some  $L \geq 0$ ,

$$\liminf_{l \rightarrow \infty} \prod_{t_{k_l} < t_i < t_{k_l+j_l}} b_i > 0,$$

and

$$\sum_{l=1}^{\infty} (t_{k_l+j_l} - t_{k_l})^2 \min(a_{k_l+j_l}, a_{k_l}) = \infty.$$

If  $x(t)$  is a solution of (1) with the property  $\lim_{t \rightarrow \infty} V(t) = r > 0$ , then  $\Delta(r) \leq L$ .

For the proof, we have only to apply the argument appearing before Corollary 22 to the interval  $[t_{k_l}, t_{k_l+j_l}]$ .

The condition on the impulses between  $t_{k_i}$  and  $t_{k_i+j_i}$  is unexpected at first sight, but this is not a real restriction since they are really small. That is why these impulses are not taken into account in estimating the energy.

To obtain attractivity criteria, we need to recall that the energy is decreasing along the solutions, so if  $V(0) = r_0$ , then  $\lim_{t \rightarrow \infty} V(t) < r_0$ . Hence,  $r_0$  can be compared to a number  $L$  for which  $\text{Attr}(L)$  or  $\text{Attr}'(L)$  or their consequence in Corollaries 22 and 23 are satisfied. In this way, we can estimate the attractivity region.

For asymptotic stability, we can easily prove the following theorems. The first one concerns the “nonsublinear” cases, while the second one gives criteria for sublinear systems.

**Theorem 24.** Let  $\Delta_0(r_0) > 0$  for some  $\gamma > r_0 > 0$ . Suppose that one of the conditions  $\text{Attr}(L)$ ,  $\text{Attr}'(L)$ , or the hypotheses of Corollary 22 or 23 are satisfied for some  $0 \leq L \leq \Delta_0(r_0)$ . Then the zero solution is a.s., and every solution with the initial conditions satisfying  $V(0) < r_0$  tends to zero as  $t \rightarrow \infty$ . If, in addition,  $0 \leq L \leq \Delta_\infty$  and  $\gamma = \infty$ , then the asymptotic stability is global.

**Theorem 25.** Let  $\Delta_0(r_0) = 0$  for every  $\gamma > r_0 > 0$ . Suppose that one of the conditions  $\text{Attr}(L)$ ,  $\text{Attr}'(L)$ , or the hypotheses of Corollary 22 or 23 are satisfied for some  $0 < L$ . Then for every solution  $x(t)$ ,  $\lim_{t \rightarrow \infty} V(t) \leq \Delta^{-1}(L)$ . If  $L = 0$ , the zero solution is asymptotically stable. If, in addition,  $\gamma = \infty$ , then the asymptotic stability is global.

**Remark 26.** From Corollary 21 we observe that we can expect asymptotic stability in the case  $\liminf_{r \rightarrow 0} \Delta(r) = 0$  and global asymptotic stability in the case  $\lim_{r \rightarrow \infty} \Delta(r) = 0$  only if  $L = 0$ .

We can use the above results for the special case  $f(x) = |x|^\alpha \operatorname{sgn} x$  ( $\alpha > 0$ ). Let us recall that  $\Delta(r) = A(\alpha)r^{(1-\alpha)/(2(1+\alpha))}$ .

**Corollary 27.** Let the conditions of Corollary 22 be satisfied,  $f(x) = |x|^\alpha \operatorname{sgn} x$ , and  $\limsup_{n \rightarrow \infty} (t_{n+1} - t_n) \leq L$  ( $0 \leq L < \infty$ ).

- (a) Case  $\alpha = 1$ : If  $L < \pi$ , the zero solution is g.a.s.
- (b) Case  $\alpha > 1$ : The zero solution is a.s. A region of attractivity is  $\{F(x_0) + \dot{x}_0^2 < (L/A(\alpha))^{(2(1+\alpha))/(1-\alpha)}\}$ .
- (c) Case  $0 < \alpha < 1$ : If  $L = 0$ , the zero solution is g.a.s. If  $L > 0$ ,  $\lim_{t \rightarrow \infty} V(t) < (L/A(\alpha))^{(2(1+\alpha))/(1-\alpha)}$  for every solution.

Taking the special  $t_n = np$ ,  $p > 0$ , we have the following.

**Corollary 28.** Let  $t_n = np$ , (H1) hold, and  $f(x) = |x|^\alpha \operatorname{sgn} x$ . Assume that

$$\sum \min(a_n, a_{n+1}) = \infty.$$

- (a) Case  $\alpha = 1$ : If  $p < \pi$ , the zero solution is g.a.s. If  $p = \pi$ , the zero solution not is a.s.
- (b) Case  $\alpha > 1$ : The zero solution is asymptotically (but not globally) stable. A region of attractivity is  $\{F(x_0) + \dot{x}_0^2 < (p/A(\alpha))^{(2(1+\alpha))/(1-\alpha)}\}$ .

(c) Case  $0 < \alpha < 1$ : The zero solution is not a.s., and  $\lim_{t \rightarrow \infty} V(t) < (p/A(\alpha))^{(2(1+\alpha))/(1-\alpha)}$  for every solution.

This corollary is used in Example 2 in the introduction.

**Remark 29.** In the linear case, if  $p > \pi$ , we can cut out any interval of length  $\pi$  from the intervals  $(t_n, t_{n+1})$ . In this case, we cannot separate the lengths of the remaining intervals from  $\pi$ . But if the sequence  $\{a_n\}$  has a positive lower bound, the conditions are satisfied for some intervals. Without this restriction, the theorem is true if

$$\limsup_{n \rightarrow \infty} ((t_{n+1} - t_n) \bmod \pi) < \pi.$$

This condition is surely satisfied if  $p/\pi$  is rational.

This cutting method cannot be applied in the nonlinear cases, since the half-period of the solutions of (2) depends on the current value of energy.

Finally we note that other attractivity criteria can be obtained by using Lemma 17 for comparing  $\Delta(r)$  to that of known functions like the above considered odd-powers.

#### 4. Proofs of results in Section 3

**Proof of Lemma 15.** Let  $x(t)$  be a solution of (1) for which  $\lim_{t \rightarrow \infty} V(t) = r > 0$  ( $r < \gamma$ ). Let  $\varepsilon > 0$  be given and let  $T$  be chosen such that

$$r < \dot{x}^2(t) + F(x(t)) < r(t + \varepsilon)$$

for  $t > T$ . Then we have

$$\sqrt{r - F(x(t))} < |\dot{x}(t)| < \sqrt{r(1 + \varepsilon) - F(x(t))},$$

so

$$\frac{|\dot{x}(t)|}{\sqrt{r(1 + \varepsilon) - F(x(t))}} < 1 < \frac{|\dot{x}(t)|}{\sqrt{r - F(x(t))}}.$$

Let  $0 < \delta < r$  and let  $T < s'_1 < s'_2$  have the property that  $F(x(s'_1)) = F(x(s'_2)) = r - \delta$  and  $F(x(t)) < r - \delta$  for  $t \in (s'_1, s'_2)$ . From the above inequality, we obtain

$$s'_2 - s'_1 \geq \int_{-F_1^{-1}(r-\delta)}^{F_2^{-1}(r-\delta)} \frac{dx}{\sqrt{r(1 + \varepsilon) - F(x)}}.$$

If  $s_1, s_2$  are zeros of  $\dot{x}(t)$  such that  $s_1 < s'_1 < s'_2 < s_2$ , then

$$s_2 - s_1 \geq \int_{-F_1^{-1}(r)}^{F_2^{-1}(r)} \frac{dx}{\sqrt{r(1 + \varepsilon) - F(x)}},$$

since  $s_1, s_2$  are independent of  $\delta$ .

Now, let  $x(t)$  be oscillatory and let  $\{s_n\}$  be a sequence of the zeros of  $\dot{x}(t)$ . Then letting  $n \rightarrow \infty$ , we have

$$\liminf_{n \rightarrow \infty} (s_{n+1} - s_n) \geq \int_{-F_1^{-1}(r)}^{F_2^{-1}(r)} \frac{dx}{\sqrt{r(1+\varepsilon) - F(x)}} = \Delta(r).$$

In addition, we can obtain an upper estimate for  $(s'_2 - s'_1)$ ,

$$(s'_2 - s'_1) < \int_{-F_1^{-1}(r-\delta)}^{F_2^{-1}(r-\delta)} \frac{dx}{\sqrt{r - F(x)}} < \Delta(r). \quad \square$$

**Proof of Lemma 18.** Let  $x(t)$  be a solution for which  $\lim_{t \rightarrow \infty} V(t) = r > 0$  ( $r < \gamma$ ). This solution is bounded. Let  $r > \delta > 0$  and the interval  $[s_1, s_2]$  be given.

Case (a): If  $F(x(t)) \leq r - \delta$  ( $t \in [s_1, s_2]$ ), then, since  $F(x(t)) + \dot{x}^2(t) > r$ , we have  $|\dot{x}| \geq \delta^{1/2}$ .

Case (b): Let  $F(x(t)) \geq r - \delta$  and  $\dot{x}(t)x(t) \geq 0$  for  $t \in [s_1, s_2]$ . Using Lemma 12 we obtain

$$0 \leq \dot{x}(s_2) = \Phi(t, s_2)\dot{x}(t) - \int_t^{s_2} f(x(s))\Phi(s, s_2) ds$$

and

$$\dot{x}(t) \geq \int_t^{s_2} f(x(s))\Phi(s, s_2) ds \geq \delta_1(s_2 - t),$$

where  $\delta_1 = \inf\{|f(u)| : F(u) > r - \delta, u < \sup_{t \geq 0} |x(t)|\} > 0$ .

Case (c): Let  $F(x(t)) \geq r - \delta$  and  $x(t)\dot{x}(t) \leq 0$  for  $t \in [s_1, s_2]$ . We can assume that  $\dot{x}(t) \geq 0$ ,  $x(t) < 0$ .

Using again Lemma 12, we have

$$\dot{x}(t) = \Phi(s_1, t)\dot{x}(s_1) - \int_{s_1}^t f(x(s))\Phi(s, t) ds \geq \delta_1 \int_{s_1}^t \Phi(s, t) ds = \delta_1 \int_{s_1}^t \left( \prod_{s < t_j < t} b_j \right) ds.$$

We have already noted that replacing  $\prod_{s \leq t_j < t}$  by  $\prod_{s < t_j < t}$  does not modify the integral.

If  $\Phi(s, t) \geq Le^{-K(t-s)}$  for  $t \geq s$ ,  $t, s \in (s_1, s_2)$ , then

$$\dot{x}(t) \geq \delta_1(1 - e^{K(s_1-t)}) \geq \varepsilon\delta_1 \min(1, t - s_1)$$

for some  $0 < \varepsilon < 1$ .  $\square$

**Proof of Theorem 19.** Suppose that  $\text{Attr}(L)$  is satisfied with  $L \geq 0$ . Let  $x(t)$  be a solution of (1) such that  $\lim_{t \rightarrow \infty} V(x(t), \dot{x}(t)) = r > 0$  ( $r < \gamma$ ). First of all, we can assume that for any  $n > 0$  the intervals  $[s_n, s_n + i_n]$  and  $[s_{n+1}, s_{n+1} + i_{n+1}]$  are disjoint. If this is not the case, but there exists  $k_0$  such that for every  $n$  the intervals  $[s_n, s_n + i_n]$  and  $[s_{n+k_0}, s_{n+k_0} + i_{n+k_0}]$  are disjoint, then in the equality (6)

$$V(t) = V(0) - \sum_{t_n < t} a_n \dot{x}^2(t_n) = \sum_{t_n < t} \sum_{j=1}^{k_0} \frac{a_n \dot{x}^2(t_n)}{k_0},$$

we can estimate the expressions  $a_n \dot{x}^2(t_n)/k_0$  on any  $[s_n, s_n + i_n]$  independently of the others.

Next, we show that  $x(t)$  cannot be monotonic on some  $[T, \infty)$ . Since  $x(t)$  and  $\dot{x}(t)$  are bounded,  $\lim_{t \rightarrow \infty} \dot{x}(t) = 0$ . We can apply Lemma 18 on each  $[s_n, s_n + i_n]$  to estimate  $\dot{x}(t)$  for sufficiently large

$n > N$ . Since  $\{i_n\}$  is bounded,  $|\dot{x}(t)| \geq \varepsilon_1 \min(1, (t - s_n))$  if  $\dot{x}(t) \leq 0$ , and  $|\dot{x}(t)| \geq \varepsilon_2 |t - s_n - i_n|$  if  $\dot{x}(t) \geq 0$ . Then,

$$V(t) \leq V(s_n) - \sum_{n > N, s_n + i_n \leq t} \left( \sum_{t_k \in [s_n, s_n + i_n]} a_k \varepsilon_1^2 \min(1, |t_k - s_n|^2) \right)$$

and

$$V(t) \leq V(s_n) - \sum_{n > N, s_n + i_n \leq t} \left( \sum_{t_k \in [s_n, s_n + i_n]} a_k \varepsilon_2^2 |t_k - s_n - i_n|^2 \right),$$

respectively. The right-hand side tends to  $-\infty$  as  $t \rightarrow \infty$  by (ii) in condition Attr(L), and that is a contradiction.

Now, let  $x(t)$  be oscillatory on some half-ray  $[t_0, \infty)$ . Let us assume that  $0 \leq L < \Delta(r)$ . Then there exist  $\delta, \varepsilon, N$  such that

$$i_n < \int_{-F_1^{-1}(r-\delta)}^{F_2^{-1}(r-\delta)} \frac{dx}{\sqrt{(1+\varepsilon)r - F(x)}} = \Delta_1(r, \delta, \varepsilon)$$

if  $n > N$ , and  $V(t) < r(1 + \varepsilon)$  if  $t > s_N$ . We can apply Lemma 18 with this  $\delta > 0$ .

From Lemma 15, we know that the lengths of the intervals, where  $F(x(t)) \leq r - \delta$ , are not smaller than  $\Delta_1(r, \delta, \varepsilon)$ . So on each  $[s_n, s_n + i_n]$  there are three possibilities.

Case 1:  $(s_n, s_n + i_n)$  does not contain a zero of  $\dot{x}(t)$ ,  $F(x(t)) \leq r - \delta$  or  $\dot{x}(t)x(t) > 0$  if  $t \in (s_n, s_n + i_n)$ . Then

$$\dot{x}(t) \geq \min(\delta^{1/2}, \varepsilon |t - s_n - i_n|).$$

Case 2:  $(s_n, s_n + i_n)$  does not contain any zero of  $\dot{x}(t)$ ,  $F(x(t)) \leq r - \delta$  or  $x(t)\dot{x}(t) < 0$  if  $t \in (s_n, s_n + i_n)$ . Then

$$|\dot{x}(t)| \geq \min(\delta^{1/2}, \varepsilon_3 \min(1, t - s_n)),$$

where  $\varepsilon_3$  is independent of  $n$ .

Case 3: There is a zero  $u_n$  of  $\dot{x}(t)$  in  $(s_n, s_n + i_n)$ . Then the intervals  $(s_n, u_n)$  and  $(u_n, s_n + i_n)$  belong to either of the Cases 1 or 2, respectively.

We note that since  $\dot{x}(t)$  is continuous from the left-hand side,  $\dot{x}(t) = 0$  at an extremum of  $x(t)$ , while  $\dot{x}(t + 0) = 0$  can occur without there being an extremum, since  $b_n = 0$  is not excluded for any  $t_n$ .

Summarizing the above cases, we can state that there is a sequence  $\{u_n\}$  with  $s_n \leq u_n \leq s_n + i_n$  such that one of the properties  $u_n = s_n$ ,  $u_n = s_n + i_n$ ,  $\dot{x}(u_n) = 0$  holds. For  $\dot{x}(t)$ , we have

$$|\dot{x}(t)| \geq \min(\delta^{1/2}, \varepsilon |t - u_n|, \varepsilon_2 \delta^{1/2}, \varepsilon_3 \delta^{1/2}) \geq c_1 \min(1, |t - u_n|).$$

Now, the following estimation for  $V(t)$  is similar to that for monotonic solutions:

$$V(t) \leq V(s_N) - c_1 \sum_{n > N, s_n + i_n < t} \left( \sum_{t_k \in [s_n, s_n + i_n]} a_k (\min(1, |t_k - u_n|))^2 \right).$$

The right-hand side tends to  $-\infty$  as  $t \rightarrow \infty$  because of (ii) in condition Attr(L).  $\square$

**Proof of Theorem 19':** The proof is a slight modification of the above proof. Let  $x(t)$  be a solution for which  $\lim_{t \rightarrow \infty} V(t) = r > 0$  ( $r > \gamma$ ). Because of (i') in  $\text{Attr}'(L)$ ,  $x(t)$  must be oscillatory. Its proof is the same as in the above proof; any  $L$ -discrete sequence can be taken.

Let  $\{s_n\}$  be the sequence of zeros of  $\dot{x}(t)$  ( $\lim_{n \rightarrow \infty} s_n = \infty$ ). By Lemma 15,  $\{s_n\}$  is  $\Delta(r)$ -discrete. If  $n$  is large enough, we can estimate  $\dot{x}(t)$  over each  $[s_n, s_{n+1}]$  as in the proof of Theorem 19. Then  $V(t)$  is majorized with this estimation.  $\square$

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