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Distance for degree raising and reduction of triangular Bézier surfaces

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Abstract

The problem of degree reduction and degree raising of triangular Bézier surfaces is considered. The L_2 and l_2 measures of distance combined with the least-squares method are used to get a formula for the Bézier points. The methods use the matrix representations of the degree reduction and degree raising.

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1. Introduction

In the applications of CAGD, we may have control points which do not fit in a rectangular domain, and it is natural for these points to fit in a triangular domain.

Barycentric coordinates: Let p_1, p_2, p_3 be the vertices of a reference triangle T and p be a point in T then it is always possible to write p as a barycentric combination of p_1, p_2, p_3 as follows:

$$p = up_1 + vp_2 + wp_3,$$

where (u, v, w) are the barycentric coordinates of p with respect to T and are given by the area ratios

$$u = \frac{\text{area}(p, p_2, p_3)}{\text{area}(p_1, p_2, p_3)}, \quad v = \frac{\text{area}(p_1, p, p_3)}{\text{area}(p_1, p_2, p_3)}, \quad w = \frac{\text{area}(p_1, p_2, p)}{\text{area}(p_1, p_2, p_3)},$$

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where $\text{area}(p_1, p_2, p_3)$ is the area of the triangle with vertices p_1, p_2, p_3 . It is clear that $u, v, w \geq 0$ and $u + v + w = 1$.

The generalized Bernstein polynomials: Using the notation $I = (i, j, k)^t$, $U = (u, v, w)^t$ and $|I| = i + j + k$, $|U| = u + v + w$ then the generalized Bernstein polynomials of degree n over a triangle T are defined by

$$B_I^n(U) = \binom{n}{I} U^I = \frac{n!}{i!j!k!} u^i v^j w^k, \quad |I| = n, \quad |U| = 1.$$

The generalized Bernstein polynomials of degree n over a triangle make a partition of unity

$$\sum_{|I|=n} B_I^n(U) = 1.$$

The last sum involves $(n+1)(n+2)/2$ terms, and

$$B_I^n(U) \geq 0 \quad \text{whenever } U = (u, v, w)^t \geq 0.$$

The product of two generalized Bernstein polynomials is also a generalized Bernstein polynomial and given by

$$B_I^n(U) B_J^m(U) = \frac{\binom{I+J}{I}}{\binom{n+m}{n}} B_{I+J}^{n+m}(U), \quad (1)$$

where the binomial coefficient of two vectors $I = (i_1, i_2, i_3)$ and $J = (j_1, j_2, j_3)$ is defined by

$$\binom{I}{J} = \binom{i_1}{j_1} \binom{i_2}{j_2} \binom{i_3}{j_3}.$$

The generalized Bernstein polynomials also satisfy a recurrence relation.

The set $\{B_I^n(U)\}_{|I|=n}$ of Bernstein polynomials of degree n form a basis for the space of polynomials of total degree n over the reference triangle T . Consider the parametric representation

$$X(U) = \sum_{|I|=n} b_I B_I^n(U)$$

of a triangular Bézier surface of degree n with the Bézier points (control points) $\{b_I\}_{|I|=n}$. Applying the de Casteljau algorithm shows that the triangular Bézier patch $X(U)$ satisfies the convex hull property over T in the sense that $\min_{|I|=n} b_I \leq X(U) \leq \max_{|I|=n} b_I$. The problem of degree reduction is concerned with finding another set of Bézier points (control points) $\{c_I\}_{|I|=n-1}$ defining the approximative triangular Bézier surface of degree $n-1$,

$$Y(U) = \sum_{|I|=n-1} c_I B_I^{n-1}(U),$$

so that the least-squares distance $d(X, Y)$ between X and Y is a minimum. The problem of degree reduction for Bézier curves has been studied in [5]. It is proved in [6,7] that the best L_2 -approximation is equivalent to the problem of finding the best Euclidean approximation of the Bernstein–Bézier coefficients. For more on triangular Bézier surfaces, see [1,2,4].

2. Preliminaries

First, the elements of the set of Bézier points $\{b_I\}_{|I|=n}$ are ordered in the following form: $\{b_{n,0,0}, b_{n-1,1,0}, \dots, b_{0,n,0}, b_{0,n-1,1}, \dots, b_{0,0,n}, b_{1,0,n-1}, \dots, b_{n-1,0,1}, b_{n-2,1,1}, \dots, b_{1,n-2,1}, b_{1,n-3,2}, \dots, b_{1,1,n-2}, \dots\}$.

Example. The elements of $\{b_I\}_{|I|=7}$ are ordered in the following form: $\{b_{7,0,0}, b_{6,1,0}, b_{5,2,0}, b_{4,3,0}, b_{3,4,0}, b_{2,5,0}, b_{1,6,0}, b_{0,7,0}, b_{0,6,1}, b_{0,5,2}, b_{0,4,3}, b_{0,3,4}, b_{0,2,5}, b_{0,1,6}, b_{0,0,7}, b_{1,0,6}, b_{2,0,5}, b_{3,0,4}, b_{4,0,3}, b_{5,0,2}, b_{6,0,1}, b_{5,1,1}, b_{4,2,1}, b_{3,3,1}, b_{2,4,1}, b_{1,5,1}, b_{1,4,2}, b_{1,3,3}, b_{1,2,4}, b_{1,1,5}, b_{2,1,4}, b_{3,1,3}, b_{4,1,2}, b_{3,2,2}, b_{2,3,2}, b_{2,2,3}\}$.

The following result gives the integral of the generalized Bernstein polynomials.

Lemma 1. *The integral of the generalized Bernstein polynomials over a triangular region T is given by*

$$\int_T \int B_I^n(U) \, dA = \frac{2A}{(n+1)(n+2)}, \quad (2)$$

where A is the area of the triangular region T .

Proof. The proof is analogous to the formula in [3, Section 5.7] by first showing that

$$\int_T \int \left(\sum_{|I|=n} b_I B_I^n(U) \right) \, dA = \frac{2A}{(n+1)(n+2)} \sum_{|I|=n} b_I$$

and now the lemma follows. \square

Lemma 2. *The L_2 norm of the triangular Bézier surface $X(U) = \sum_{|I|=n} b_I B_I^n(U)$ is given by*

$$\|X(U)\|_2^2 = \frac{2A}{(2n+1)(2n+2)} \binom{2n}{n} \sum_{|I|=n} \sum_{|J|=n} b_I b_J \binom{I+J}{I}. \quad (3)$$

Proof.

$$\begin{aligned} \|X(U)\|_2^2 &= \int_T \int \left| \sum_{|I|=n} b_I B_I^n(U) \right|^2 \, dA \\ &= \int_T \int \sum_{|I|=n} \sum_{|J|=n} b_I b_J B_I^n(U) B_J^n(U) \, dA \\ &= \sum_{|I|=n} \sum_{|J|=n} b_I b_J \int_T \int B_I^n(U) B_J^n(U) \, dA \end{aligned}$$

$$= \sum_{|I|=n} \sum_{|J|=n} b_I b_J \frac{\binom{I+J}{I}}{\binom{2n}{n}} \int_T \int B_{I+J}^{2n}(U) dA.$$

Invoking Lemma 1 completes the proof. \square

Let $r = (n+1)(n+2)/2$ and the $r \times r$ matrix Q_n be given by

$$Q_n = \frac{2A}{(2n+1)(2n+2) \binom{2n}{n}} \left[\begin{pmatrix} I+J \\ I \end{pmatrix} \right]. \quad (4)$$

It is clear from the definition that the matrix Q_n is a real symmetric matrix. Using the mathematical induction, all the upper left submatrices of Q_n have positive determinants and, thus, the matrix Q_n is a symmetric positive definite matrix, see [9]. The sum of each column and each row of the real symmetric matrix Q_n equals to $2A/(n+1)(n+2)$.

Thus, the L_2 norm of $X(U)$ is given in matrix form by

$$\|X(U)\|_2^2 = b^t Q_n b, \quad (5)$$

where the vector b contains the elements of the set of Bézier points $\{b_I\}_{|I|=n}$ ordered in the form described before. It is also clear from Eq. (5) that the matrix Q_n is positive definite.

Example. For the case of unit triangular region with $n=2$ the L_2 norm of $X(U)$ is given by $\|X(U)\|_2^2 = b^t Q_2 b$ where

$$Q_2 = \frac{1}{90} \begin{pmatrix} 6 & 3 & 1 & 1 & 1 & 3 \\ 3 & 4 & 3 & 2 & 1 & 2 \\ 1 & 3 & 6 & 3 & 1 & 1 \\ 1 & 2 & 3 & 4 & 3 & 2 \\ 1 & 1 & 1 & 3 & 6 & 3 \\ 3 & 2 & 1 & 2 & 3 & 4 \end{pmatrix}$$

and $b^t = (b_{2,0,0}, b_{1,1,0}, b_{0,2,0}, b_{0,1,1}, b_{0,0,2}, b_{1,0,1})$.

3. Degree raising

Given a triangular Bézier surface $X(U)$ of degree n with Bézier points $\{b_I\}_{|I|=n}$, we want to write the same triangular Bézier surface $X(U)$ using a basis of degree $n+1$ with Bézier points $\{b_I^*\}_{|I|=n+1}$;

$$\sum_{|I|=n} b_I B_I^n(U) = \sum_{|I|=n+1} b_I^* B_I^{n+1}(U). \quad (6)$$

4. The L_2 norm

Let $X(U)$ and $Y(U)$ be triangular Bézier surfaces of degree n and $n + 1$, respectively, with

$$X(U) = \sum_{|I|=n} b_I B_I^n(U),$$

$$Y(U) = \sum_{|I|=n+1} c_I B_I^{n+1}(U),$$

where $\{b_I\}_{|I|=n}$ and $\{c_I\}_{|I|=n+1}$ are the corresponding Bézier points. We consider the L_2 measure of distance between $X(U)$ and $Y(U)$ as follows:

$$d_2^2(X, Y) = \int_T \int |X(U) - Y(U)|^2 dA$$

$$= \int_T \int \left| \sum_{|I|=n} b_I B_I^n(U) - \sum_{|I|=n+1} c_I B_I^{n+1}(U) \right|^2 dA.$$

Using the degree raising in (7) we get

$$d_2^2(X, Y) = \int_T \int \left| \sum_{|I|=n+1} b_I^* B_I^{n+1}(U) - \sum_{|I|=n+1} c_I B_I^{n+1}(U) \right|^2 dA$$

$$= \int_T \int \left| \sum_{|I|=n+1} (b_I^* - c_I) B_I^{n+1}(U) \right|^2 dA.$$

Letting $d_I = b_I^* - c_I$, and using Lemma 2 gives

$$d_2^2(X, Y) = \int_T \int \sum_{|I|=n+1} \sum_{|J|=n+1} d_I d_J B_I^{n+1}(U) B_J^{n+1}(U) dA$$

$$= \frac{1}{\binom{2n+2}{n+1}} \frac{2A}{(2n+3)(2n+4)} \sum_{|I|=n+1} \sum_{|J|=n+1} d_I d_J \binom{I+J}{I}$$

$$= d^t Q_{n+1} d,$$

where the $((n+2)(n+3)/2) \times ((n+2)(n+3)/2)$ matrix Q_{n+1} is defined in (4).

Thus, the L_2 measure of distance between the triangular Bézier surfaces $X(U)$ and $Y(U)$ of degree n and $n + 1$, respectively, is given in the following theorem.

Theorem 3. The L_2 measure of distance between the triangular Bézier surfaces $X(U)$ and $Y(U)$ of degree n and $n + 1$, respectively, is given by

$$d_2(X, Y) = \sqrt{d^t Q_{n+1} d},$$

where $d = b^* - c$.

5. Degree reduction

In this section, we use the L_2 norm and the discrete l_2 norm to measure the distance of degree reduction between the triangular Bézier surfaces X and Y .

Given a set of Bézier points $\{b_I\}_{|I|=n}$ which defines the triangular Bézier surface of degree n ,

$$X(U) = \sum_{|I|=n} b_I B_I^n(U),$$

we want to find another set of Bézier points $\{c_I\}_{|I|=n-1}$ defining the approximative triangular Bézier surface of degree $n - 1$,

$$Y(U) = \sum_{|I|=n-1} c_I B_I^{n-1}(U),$$

so that the least-squares distance

$$d_2(X, Y) = \sqrt{d^t Q_n d} \quad (8)$$

between $\{b_I\}_{|I|=n}$ and $\{c_I\}_{|I|=n-1}$ is minimized. Substituting $d = b - c^*$, where $c^* = T_n c$, and doing some simplifications in $d_2^2(X, Y)$ we get

$$\begin{aligned} d^t Q_n d &= (b - c^*)^t Q_n (b - c^*) \\ &= (b - T_{n-1} c)^t Q_n (b - T_{n-1} c) \\ &= b^t Q_n b - b^t Q_n T_{n-1} c - c^t T_{n-1}^t Q_n b + c^t T_{n-1}^t Q_n T_{n-1} c \\ &= b^t Q_n b - 2c^t T_{n-1}^t Q_n b + c^t T_{n-1}^t Q_n T_{n-1} c. \end{aligned}$$

We use the least-squares method to find c , see [8]. For a minimum of $d^t Q_n d$ to occur, it is necessary that the derivative of $d^t Q_n d$ with respect to the elements of the vector c is zero. Solving the normal equations

$$0 = \frac{\partial}{\partial c} (d^t Q_n d) = 2(-T_{n-1}^t Q_n b + T_{n-1}^t Q_n T_{n-1} c)$$

gives

$$T_{n-1}^t Q_n T_{n-1} c = T_{n-1}^t Q_n b.$$

Since $T_{n-1}^t Q_n T_{n-1} = Q_{n-1}$, and the matrix Q_{n-1} is a symmetric positive definite matrix, the matrix $T_{n-1}^t Q_n T_{n-1}$ is invertible. Hence $(T_{n-1}^t Q_n T_{n-1})^{-1}$ exists, and the least-squares distance is

minimized by choosing

$$c = (T_{n-1}^t Q_n T_{n-1})^{-1} T_{n-1}^t Q_n b. \quad (9)$$

The matrix $(T_{n-1}^t Q_n T_{n-1})^{-1} T_{n-1}^t Q_n$ is the pseudo-inverse (generalized inverse) of the matrix T_{n-1} , see [9, Section 13.7]. Substituting the last formula for c in the least-squares distance $d_2^2(X, Y)$ in (8) gives the error in the following theorem.

Theorem 4. *The error of the L_2 measure of distance for the degree reduction satisfies*

$$\varepsilon^2 = b^t Q_n b - b^t Q_n T_{n-1} (T_{n-1}^t Q_n T_{n-1})^{-1} T_{n-1}^t Q_n b.$$

For the discrete l_2 measure of distance of degree reduction we want the least-squares distance

$$d_D(X, Y) = \sqrt{d^t d} \quad (10)$$

between $\{b_I\}_{|I|=n}$ and $\{c_I\}_{|I|=n-1}$ to be minimized. After some calculations, similar to the last case, we get

$$d^t d = b^t b - 2c^t T_{n-1}^t b + c^t T_{n-1}^t T_{n-1} c.$$

Finding and solving the normal equations gives the solution

$$c_D = (T_{n-1}^t T_{n-1})^{-1} T_{n-1}^t b. \quad (11)$$

The matrix $(T_{n-1}^t T_{n-1})^{-1} T_{n-1}^t$ is the pseudo-inverse (generalized inverse) of the matrix T_{n-1} . Substituting the last formula for c in the least-squares distance $d_D^2(X, Y)$ in (10) gives the error in the following theorem.

Theorem 5. *The error of the l_2 measure of distance for the degree reduction satisfies*

$$\varepsilon^2 = b^t b - b^t T_{n-1} (T_{n-1}^t T_{n-1})^{-1} T_{n-1}^t b.$$

Eq. (9), which illustrates degree reduction with Bézier surfaces, is the Moore–Penrose inverse of the degree elevation matrix T_{n-1} . Eq. (11), which illustrates degree reduction with Bézier control points, is the Moore–Penrose inverse of the degree elevation matrix T_{n-1} . So c and c_D are the same solution.

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References

- [1] C. de Boor, B-form basics, in: G. Farin (Ed.), *Geometric Modeling: Algorithms and New Trends*, SIAM, Philadelphia, 1987, pp. 131–148.
- [2] G. Farin, Triangular Bernstein–Bézier patches, *Comput. Aided Geom. Des.* 3 (1986) 83–127.

- [3] G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, Boston, 1996.
- [4] J. Hoschek, D. Lasser, *Fundamentals of Computer Aided Geometric Design*, A.K. Peters, Wellesley, MA, 1993.
- [5] B.G. Lee, Y. Park, The distance for the Bézier curves and degree reduction, *Bull. Austral. Math. Soc.* 59 (1997) 507–515.
- [6] D. Lutterkort, J. Peters, U. Reif, Polynomial degree reduction in the L_2 -norm equals best Euclidean approximation of Bézier coefficients, *Comput. Aided Geom. Des.* 166 (1999) 607–612.
- [7] J. Peters, U. Reif, Least squares approximation of Bézier coefficients provides best degree reduction in the L_2 -norm, *J. Approx. Theory* 104 (2000) 90–97.
- [8] J. Rice, *The Approximation of Functions*, Vol. 1, Linear Theory, Addison-Wesley, Reading, MA, 1964.
- [9] C. Ueberhuber, *Numerical Computation 2, Methods, Software, and Analysis*, Springer, Berlin, 1997.