

Radiation boundary conditions for time-dependent waves based on complete plane wave expansions

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ABSTRACT

We develop complete plane wave expansions for time-dependent waves in a half-space and use them to construct arbitrary order local radiation boundary conditions for the scalar wave equation and equivalent first order systems. We demonstrate that, unlike other local methods, boundary conditions based on complete plane wave expansions provide nearly uniform accuracy over long time intervals. This is due to their explicit treatment of evanescent modes. Exploiting the close connection between the boundary condition formulations and discretized absorbing layers, corner compatibility conditions are constructed which allow the use of polygonal artificial boundaries. Theoretical arguments and simple numerical experiments are given to establish the accuracy and efficiency of the proposed methods.

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1. Introduction

Accurate and efficient domain truncation is crucial to the development of comprehensive computational tools for simulating wave propagation in the time domain. The ideal method would provide arbitrary accuracy at low cost on computational domains which are tightly fit around scatterers, sources, and other inhomogeneities of arbitrary support. Although there has been great progress on these issues in recent years, no existing method meets all three goals [1,2]. On the one hand there are fast, low-memory implementations of exact nonlocal conditions [3–5]. These provide excellent long time accuracy at low direct computational cost, but the geometry of the artificial boundary is inflexible. For a three-dimensional scattering problem one must use a sphere, which can be wasteful computational volume, and one must also implement a spherical harmonic transform, which is still not so efficient. As an alternative one can use a perfectly matched layer (PML) [6,7] or arbitrary order local radiation boundary condition sequences [8–12] in polygonal computational domains. In their standard implementations, however, each of these geometrically flexible methods suffers severe accuracy degradation over long times. See, for example, the exact reflection analyses based on the Cagniard-de Hoop method given in [13–15].

In this paper we present a method which combines the excellent long time accuracy of the nonlocal conditions with the geometric flexibility of the local methods, providing, in our view, a satisfactory solution to the time-domain domain truncation problem for isotropic systems. Our fundamental premise is that the poor long time performance of standard methods is a result of insufficient accuracy in the evanescent mode regime. Thus we supplement them with terms designed to absorb evanescent modes. Although evanescent mode corrections have been considered before (e.g. [16]), we believe ours is the first comprehensive development.

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The approximations developed here are based on representations of the wave field as a sum of purely propagative and purely evanescent modes. Some complementary experiments with this representation in waveguides are given in [17], and alternative methods for computing the parameters are described in [18]. The numerical results presented below extend those given in [17] by incorporating corner compatibility conditions to solve scattering problems, while the theoretical analysis directly accounts for a finite separation between sources and the boundary as well as finite time effects. Distinct representations can be derived in terms of waves which both propagate and decay. Boundary conditions based on these mixed representations are derived and tested for first order systems in [19,20], but they have not been extended to second order problems.

2. Complete plane wave representations

We begin by considering a field, $u(x, y, t)$, which satisfies the scalar wave equation in the half-space $x > -\delta$ for some $\delta > 0$. That is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \quad x > -\delta, \quad y \in \mathbb{R}^{d-1}, \quad t > 0, \quad (1)$$

with $u = 0$ at $t = 0$. Of course we suppose the field is produced by sources, scatterers, and other inhomogeneities located in the half-space $x < -\delta$. These effects are all accounted for by Dirichlet data at $x = -\delta$. The evolution in x of the transverse-Fourier-Laplace transform of the solution of (1) then is:

$$\hat{u}(x, k, s) = \hat{u}(-\delta, k, s) e^{-(s^2 + |k|^2)^{1/2}(x+\delta)}. \quad (2)$$

Here $k \in \mathbb{R}^{d-1}$ are the dual Fourier variables to the transverse spatial coordinates y , $|k|$ is the Euclidean norm, and $s = c\bar{s}$ is the dual Laplace variable to time. The branch of the square root is chosen to have positive real part for $\Re s > 0$.

Now invert the transform, integrating with respect to k over the real line and with respect to s over the imaginary axis, setting $s = i\omega$. Going over to polar coordinates in the dual spatial variables, $\rho \geq 0$, $\theta \in \mathbb{S}^{d-2}$ with area element dA , we obtain:

$$u(x, y, t) = \int_0^\infty \int_{\mathbb{S}^{d-2}} \int_{-\infty}^\infty e^{i\omega t + i\rho(\theta \cdot y) - \frac{x+\delta}{c}(c^2\rho^2 - \omega^2)^{1/2}} \bar{u} \rho^{d-2} d\omega dA d\rho, \quad (3)$$

$$\bar{u} = \hat{u}(-\delta, \rho\theta, i\omega). \quad (4)$$

We now separate this multiple integral into two pieces according to the square root, $\gamma = (c^2\rho^2 - \omega^2)^{1/2}$. In the first region:

$$\rho = \frac{|\omega|}{c} \sin \phi, \quad \gamma = i\omega \cos \phi, \quad 0 \leq \phi < \frac{\pi}{2}. \quad (5)$$

In the second region:

$$\rho = \frac{\sqrt{c^2\sigma^2 + \omega^2}}{c}, \quad \gamma = c\sigma, \quad 0 < \sigma < \infty. \quad (6)$$

Introducing the functions

$$\Phi(\tau, y, \phi) = \int_{\mathbb{S}^{d-2}} \int_{-\infty}^\infty e^{i\omega\tau + i\frac{|\omega|}{c} \sin \phi(\theta \cdot y)} \tilde{u}_p \cdot \left(\frac{|\omega|}{c}\right)^{d-1} \sin^{d-2} \phi \cos \phi d\omega dA, \quad (7)$$

$$\Upsilon(t, y, \sigma) = \int_{\mathbb{S}^{d-2}} \int_{-\infty}^\infty e^{i\omega t + i\frac{\sqrt{c^2\sigma^2 + \omega^2}}{c}(\theta \cdot y)} \tilde{u}_e \cdot \frac{(c^2\sigma^2 + \omega^2)^{\frac{d-3}{2}}}{c^{d-3}} \sigma d\omega dA, \quad (8)$$

$$\tilde{u}_p = \hat{u}\left(-\delta, \frac{|\omega|}{c} \sin \phi \theta, i\omega\right), \quad \tilde{u}_e = \hat{u}\left(-\delta, \frac{\sqrt{c^2\sigma^2 + \omega^2}}{c} \theta, i\omega\right) \quad (9)$$

we derive our complete plane wave representation of the solution:

$$u(x, y, t) = \int_0^{\frac{\pi}{2}} \Phi\left(t - \frac{\cos \phi}{c}(x + \delta), y, \phi\right) d\phi + \int_0^\infty e^{-\sigma(x+\delta)} \Upsilon(t, y, \sigma) d\sigma. \quad (10)$$

Clearly, (10) expresses the solution as a superposition of propagating plane waves at all possible angles (the ϕ integral) and evanescent waves at all possible decay rates (the σ integral).

Note that the representation holds for any field component of a system which satisfies the scalar wave equation. Thus our construction is applicable to problems in acoustics, electromagnetics, and elasticity.

We remark that alternative representations could be derived by choosing different inversion contours. In [19] such a representation is considered where the contour is given by $\Re(s) = T^{-1}$ and T represents the solution time. Then one obtains

a single integral involving waves which both propagate and decay. For first order systems this alternative representation yields approximate radiation conditions which are somewhat more efficient than the ones presented here. However, for second order systems the representation used here leads to more direct implementations, though we will make some use of the alternative formula in our error analysis.

3. Discrete approximations and local boundary condition sequences

The straightforward way to produce finite approximate plane wave representations is to replace the integrals in (10) by quadrature rules. In general pwd suppose the propagating mode integral is replaced by a quadrature rule with n_p nodes, ϕ_j and corresponding weights, w_j , and the evanescent mode integral by a rule with n_e nodes, σ_j , and weights, γ_j . We then have:

$$u(x, y, t) \approx \sum_{j=1}^{n_p} w_j \Phi \left(t - \frac{\cos \phi_j}{c} (x + \delta), y, \phi_j \right) + \sum_{j=1}^{n_e} \gamma_j e^{-\sigma_j(x+\delta)} \Upsilon(t, y, \sigma_j). \quad (11)$$

Note that the approximate representation involves a sum over plane waves propagating in a finite set of directions and evanescent modes decaying at a finite number of rates.

3.1. Definition of auxiliary functions

Our construction of the boundary conditions does not rely on the direct construction of the amplitude functions appearing in (11). Instead we begin by introducing a sequence of auxiliary functions according to the recipe given in [9], generalized to take account of the evanescent modes. Precisely, for $x > -\delta$ we set:

$$\cos \phi_1 \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \cos \phi_1 \frac{\partial u_1}{\partial t}, \quad (12)$$

and recursively define $u_j, j = 2, \dots$ by the equations:

$$\left(\cos \phi_j \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u_{j-1} = \left(\cos \phi_j \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u_j, \quad j = 2, \dots, n_p, \quad (13)$$

$$\left(\sigma_j + \frac{\partial}{\partial x} \right) u_{n_p+j-1} = \left(\sigma_j - \frac{\partial}{\partial x} \right) u_{n_p+j}, \quad j = 1, \dots, n_e \quad (14)$$

supplemented by null initial data,

$$u_j(x, y, 0) = 0, \quad j > 1. \quad (15)$$

These equations can be integrated along characteristics in the negative x direction so that u_j is indeed determined by u_{j-1} and thus ultimately by u . The motivation for the form of this recursion is as follows. Suppose that the finite representation (11) is exact. Noting that each term in the sum is in the kernel of one of the commuting differential operators appearing on the left-hand side of (13)–(14) we conclude that the recursion terminates, $u_{n_p+n_e} = 0$.

We now use (13)–(14) to construct boundary conditions on the plane $x = 0$. An easy induction argument shows that the auxiliary functions themselves are solutions of the scalar wave equation [9]. We can use this fact to eliminate the x -derivatives in favor of derivatives which are tangent to the boundary. Before doing so we note that (13)–(14) admits an entirely different interpretation as a nonstandard semidiscretization of a perfectly matched layer. In particular if we solve for the x -derivatives we have:

$$\begin{aligned} \frac{1}{2} \left(\frac{\partial u_{j-1}}{\partial x} + \frac{\partial u_j}{\partial x} \right) &= \frac{\cos \phi_j}{2c} \left(\frac{\partial u_j}{\partial t} - \frac{\partial u_{j-1}}{\partial t} \right), \quad j = 2, \dots, n_p, \\ \frac{1}{2} \left(\frac{\partial u_{n_p+j-1}}{\partial x} + \frac{\partial u_{n_p+j}}{\partial x} \right) &= \frac{\sigma_j}{2} (u_{n_p+j} - u_{n_p+j-1}), \quad j = 1, \dots, n_e. \end{aligned} \quad (16)$$

We observe that this is formally equivalent to a second order difference approximation to the x -derivatives with a combination of the derivative-dependent and real grid spacings:

$$\Delta x_{j-1/2} = \frac{2c}{\cos \phi_j \frac{\partial}{\partial t}}, \quad \Delta x_{n_p+j-1/2} = \frac{2}{\sigma_j}. \quad (17)$$

A standard discrete PML would involve a grid spacing which is a combination of both derivatives and real numbers. We note that PMLs with purely derivative-dependent grid spacings were introduced in [21]. There the idea was to reproduce optimal rational approximations to exact boundary condition in the propagating mode regime. Formulations very similar to ours were introduced in [12].

3.2. Structure of the boundary conditions for the second order wave equation

For the scalar wave equation the procedure is to apply first order differential operators in $\frac{\partial}{\partial x}$ to the recursion relations and then to eliminate all x derivatives. The detailed calculations are given in [17]. These generalize the constructions given in [9,11] to allow for the evanescent mode corrections. The final result is a coupled system of wave equations solved along the boundaries. The variables are the u_j defined above and a seam variable, ψ :

$$\cos \phi_1 \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \cos \phi_1 \frac{\partial u_1}{\partial t}, \quad (18)$$

$$L \frac{\partial^2 \mathbf{U}}{\partial t^2} + l_b \mathbf{e}_{n_p} \frac{\partial^2 \psi}{\partial t^2} - c \mathbf{e}_{n_p+1} \frac{\partial \psi}{\partial t} = c^2 (M \nabla_{\tan}^2 + c^2 S) \mathbf{U}. \quad (19)$$

Here $\mathbf{U} = (u_1 \dots u_{n_p+n_e+1})^T$ and the matrices L , M and S are tightly banded.

We emphasize that the construction of (19) relies on the particular algebraic structure of the recursions (13)–(14) and the scalar wave equation. We have been unable to repeat it for the mixed recursions used for first order systems in [19,20], where both $\cos \phi_j$ and σ_j appear in the same equations. Thus we advocate the use of (13)–(14) for second order formulations. We have also been unable to directly combine (19) at neighboring edges into a consistent corner treatment. Thus we introduce extra variables to produce a first order formulation at the corners. We do not know if these limitations are fundamental or if they can be removed by some systematic algebraic treatment. In any case, as we will see below, the construction of the boundary conditions and corner closures is far more straightforward in the first order case.

3.3. Boundary conditions for first order systems and corner closures

Consider a system of the form:

$$\frac{\partial w}{\partial t} + cA \frac{\partial w}{\partial x} + c \sum_k B_k \frac{\partial w}{\partial y_k} = 0. \quad (20)$$

First of all we associate a sequence of auxiliary variables with each field component by solving Eqs. (13)–(14) with u_j replaced by a vector field w_j . We then multiply the recursion relations by A and use (20) to eliminate normal derivatives. Replacing (12) by a recursion of the form (13) and identifying w with w_0 , we obtain for $j = 1, \dots, n_p + n_e$:

$$\left(AS_j - \frac{\partial}{\partial t} \right) w_{j-1} - c \sum_k B_k \frac{\partial w_{j-1}}{\partial y_k} = \left(AS_j + \frac{\partial}{\partial t} \right) w_j + c \sum_k B_k \frac{\partial w_j}{\partial y_k}, \quad (21)$$

where

$$S_j = \begin{cases} \cos \phi_j \frac{\partial}{\partial t}, & j \leq n_p, \\ c \sigma_{j-n_p}, & j > n_p. \end{cases} \quad (22)$$

To close these relations we consider normal characteristic variables. In particular denote by r_+ a variable which is outgoing relative to the computational domain and by r_- a variable which is incoming relative to the computational domain. We set:

$$\frac{\partial r_{+,0}}{\partial t} = \text{interior value}, \quad r_{-,n_p+n_e} = 0. \quad (23)$$

Obviously the conditions developed above for planar boundaries can only be used in restrictive circumstances, namely waveguides or transversely periodic media as considered in [17]. To use them for scattering problems we must develop methods for terminating the boundary conditions at corners where adjacent pieces of the boundary intersect. Here, as first noted in [12], one can exploit the connection with PML by constructing the compatibility relations in a manner consistent with the standard construction of PML corner layers. That is, we simply use the formal frequency-dependent grid spacings (17) in each spatial variable. This point of view clarifies the algebraic structure of the method we introduced in [9].

We consider here the first order system. As mentioned above we do not have a direct construction in the second order case; the numerical experiments reported below for the scalar wave equation use corner variables associated with a related first order equation. For simplicity we will only present results for right angle corners aligned with the coordinate axes in two space dimensions. The technique works for general angles, though we have observed a loss of accuracy as the angle becomes acute. The corner variables are $w_{j,k}(t)$ for $j, k = 0, \dots, n_p + n_e$. If $k = 0$ these correspond to the corner values of the auxiliary variables associated with the edge parallel to the y -axis and if $j = 0$ they correspond to the auxiliary variables associated with the edge parallel to the x -axis. We formally discretize the equation on a box with grid spacings (17) and their analogues in y :

$$\begin{aligned} \frac{\partial w_{j-1,k}}{\partial t} + \frac{\partial w_{j-1,k-1}}{\partial t} + \frac{\partial w_{j,k}}{\partial t} + \frac{\partial w_{j,k-1}}{\partial t} &= AS_j(w_{j-1,k-1} - w_{j,k-1} + w_{j-1,k} - w_{j,k}) \\ &+ BS_k(w_{j-1,k-1} - w_{j-1,k} + w_{j,k-1} - w_{j,k}). \end{aligned} \quad (24)$$

To complete the description of the corner system we supplement these with data taken from the edges plus termination conditions. Precisely we impose the time derivatives of outgoing characteristic variables from each edge when either $j = 0$ or $k = 0$ and we impose termination conditions on outgoing characteristic variables for $j = n_p + n_e$ or $k = n_p + n_e$, combining the equations if $j = k = n_p + n_e$. These can be shown to provide a closed system of ordinary differential equations. See [19] for details.

4. Analysis and parameter selection

To complete the definition of the method we must choose the nodes ϕ_j and σ_j appearing in (11). There are many ways this could be done, and here we base their choice on the approximation of the integrals (11). A viable alternative for the second order case is to choose both ϕ and σ following Zolotarev's optimal approximations to square root functions [22, Ch. 4]. Such approximations have been used in the purely evanescent case in [23] and for the purely propagative case in [21]. Their combined use will be considered in [18].

4.1. Quadrature formulas

As the first integral is defined on a bounded domain it is possible to apply standard Gaussian quadrature. Precisely we take:

$$\phi_j = (z_j + 1) \cdot \frac{\pi}{4}, \quad -1 = z_1 < z_2 < \dots < z_{n_p} < 1, \quad (25)$$

where the z_j 's are the left endpoint Gauss–Radau nodes.

The determination of the second quadrature scheme is more complicated as one must introduce a length scale. This corresponds to the underlying fact that we are approximating an operator with a homogeneous symbol by one with an inhomogeneous symbol. In [24] Yarvin and Rokhlin construct generalized Gaussian quadrature rules of the form:

$$\sum_{j=1}^{n_e} w_j e^{-s_j z} f(s_j) \approx \int_0^\infty e^{-sz} f(s) ds, \quad (26)$$

for $z \in [1, 500]$ and certain functions f . In our case we seek approximations valid for $x \geq \delta$. Making the change of variables $x = \delta z$ the approximation becomes:

$$\sum_{j=1}^{n_e} w_j e^{-\frac{s_j}{\delta} x} f(s_j). \quad (27)$$

Based on this formula we set:

$$\sigma_j = \frac{s_j}{\delta}, \quad (28)$$

where s_j is the j th Yarvin–Rokhlin node. (The nodes themselves are available at the URL <http://www.netlib.org/pdes/multipole/wts500.f>.)

4.2. Accuracy

We seek to obtain error estimates valid for fixed $\delta > 0$ up to some finite time T . Here we follow the straightforward approach based on Fourier–Laplace transformation developed in [25]. Returning to Eq. (2), we write down an expression for the transform of the reflected wave in terms of the Dirichlet data at $x = -\delta$. It takes the form:

$$\hat{e}(x, k, s) = R(k, s) \hat{u}(-\delta, k, s) e^{\gamma(x-\delta)}, \quad x \leq 0, \quad (29)$$

where, as computed in [17], R is given by:

$$R(k, s) = -\frac{\bar{s} \cos \phi_1 - \gamma}{\bar{s} \cos \phi_1 + \gamma} \prod_{j=2}^{n_p} \left(\frac{\bar{s} \cos \phi_j - \gamma}{\bar{s} \cos \phi_j + \gamma} \right)^2 \prod_{j=1}^{n_e} \left(\frac{\sigma_j - \gamma}{\sigma_j + \gamma} \right)^2. \quad (30)$$

As in [25] we set $\bar{s} = \frac{1}{cT}(1 + i\omega)$. Error estimates in standard L_2 -based Sobolev norms can then be derived directly using Parseval's relation. In these estimates the controlling factor is simply:

$$\rho = \max_{\bar{s} = \frac{1}{cT}(1+i\omega), k \in \mathbb{R}} |\operatorname{Re}^{-\gamma\delta}|. \quad (31)$$

Table 1

Error estimates for various parameters.

n_p	n_e	η	$\bar{\rho}$	n_p	n_e	η	$\bar{\rho}$
5	15	10^{-3}	4.9×10^{-4}	3	3	10^{-2}	1.0×10^{-2}
6	14	10^{-3}	1.9×10^{-4}	6	6	10^{-2}	3.3×10^{-4}
7	13	10^{-3}	8.4×10^{-5}	9	9	10^{-2}	3.6×10^{-5}
8	12	10^{-3}	5.5×10^{-5}	12	12	10^{-2}	6.3×10^{-6}
9	11	10^{-3}	3.6×10^{-5}	15	15	10^{-2}	7.4×10^{-7}
10	10	10^{-3}	2.5×10^{-5}	3	3	10^{-3}	1.2×10^{-2}
11	9	10^{-3}	1.8×10^{-5}	6	6	10^{-3}	4.2×10^{-4}
12	8	10^{-3}	1.4×10^{-5}	9	9	10^{-3}	4.3×10^{-5}
13	7	10^{-3}	1.2×10^{-5}	12	12	10^{-3}	7.9×10^{-6}
14	6	10^{-3}	1.2×10^{-5}	15	15	10^{-3}	1.0×10^{-6}
15	5	10^{-3}	1.8×10^{-5}	3	3	10^{-4}	1.3×10^{-2}
16	4	10^{-3}	2.8×10^{-5}	6	6	10^{-4}	6.4×10^{-4}
17	3	10^{-3}	6.0×10^{-5}	9	9	10^{-4}	6.8×10^{-5}
18	2	10^{-3}	1.7×10^{-4}	12	12	10^{-4}	1.5×10^{-5}
19	1	10^{-3}	1.9×10^{-4}	15	15	10^{-4}	2.9×10^{-6}

It is shown in [19] that for (s, k) restricted to this region γ takes on the values:

$$\gamma = i\omega \frac{\cos \theta}{cT} + \frac{1}{cT \cos \theta}, \quad \theta \in [0, \pi/2). \quad (32)$$

Introducing

$$\eta = \frac{\delta}{cT} \quad (33)$$

we find

$$\begin{aligned} \rho = e^{-\frac{\eta}{\cos \theta}} & \left(\frac{(\cos \phi_1 - \cos \theta)^2 \omega^2 + (\cos \phi_1 - \sec \theta)^2}{(\cos \phi_1 + \cos \theta)^2 \omega^2 + (\cos \phi_1 + \sec \theta)^2} \right)^{1/2} \prod_{j=2}^{n_p} \left(\frac{(\cos \phi_j - \cos \theta)^2 \omega^2 + (\cos \phi_j - \sec \theta)^2}{(\cos \phi_j + \cos \theta)^2 \omega^2 + (\cos \phi_j + \sec \theta)^2} \right) \\ & \times \prod_{j=1}^{n_e} \left(\frac{\eta^2 \cos^2 \theta \omega^2 + (s_j - \eta \sec \theta)^2}{\eta^2 \cos^2 \theta \omega^2 + (s_j + \eta \sec \theta)^2} \right). \end{aligned} \quad (34)$$

Clearly, the accuracy for any fixed choice of n_p and n_e will be a function of η . We estimate the average value of ρ for $\eta = 10^{-2}$, $\eta = 10^{-3}$, and $\eta = 10^{-4}$, by sampling over 5×10^7 points for $(\omega, \theta) \in [0, 10^4] \times [0, \frac{\pi}{2})$. We first consider the determination of n_p and n_e fixing their sum to be 20 and studying the variation of $\bar{\rho}$ for $\eta = 10^{-3}$. The results are shown in Table 1.

We now fix $n_p = n_e$ and consider $\bar{\rho}$ as a function of $n_p + n_e$ and η . The results are also shown in Table 1. They demonstrate spectral convergence in boundary condition order and weak dependence on η .

We note that error bounds follow from estimating the maximum of ρ , and these are not so impressive for the parameters chosen here. In [19] we choose optimal parameters based on the direct minimization of the $\rho_{\max}$. Then we prove that the number of terms, P , required to guarantee an error ϵ satisfies:

$$P \leq C \ln \frac{1}{\epsilon} \cdot \ln \frac{1}{\eta}. \quad (35)$$

However, these optimal approximations involve recursions which mix propagating and evanescent terms. Although this leads to no special difficulties for applications to first order systems, we have not been able to extend our derivation of second order equations for the auxiliary variables.

5. Numerical experiments

We now present some numerical experiments illustrating the accuracy and stability of the proposed conditions. Additional experiments for the scalar wave equation in waveguides are described in [17], and for first order systems in [19,20]. Here we consider both second and first order formulations and solve using high order difference approximations on structured grids and discontinuous Galerkin methods on unstructured grids. In each case we compare with an exact solution evaluated using quadratures. (These will be made available by the first author to anyone wishing to repeat the experiments.)

We consider first a solution of the scalar wave equation produced by point sources approximately located at the following coordinates: $(-2.998717, -.00189)$, $(-1.86719, .001304)$, $(.0012307, .0012819)$, $(2.98397, -.001099)$. The time signal of each source is a Gaussian centered at $t = -.5$ with amplitude 1 and variance 95. As the amplitude is effectively zero for $t \geq 0$ we solve it as a Cauchy problem.

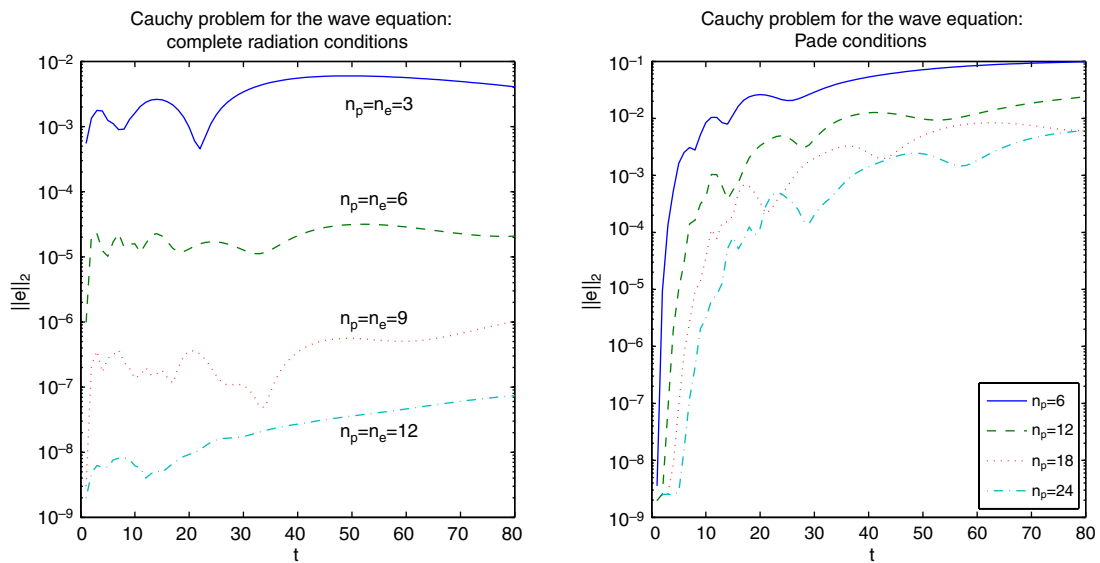


Fig. 1. Errors in solutions of the Cauchy problem for the scalar wave equation as the boundary condition order is increased. Complete radiation conditions are compared with the Padé (Engquist–Majda) conditions.

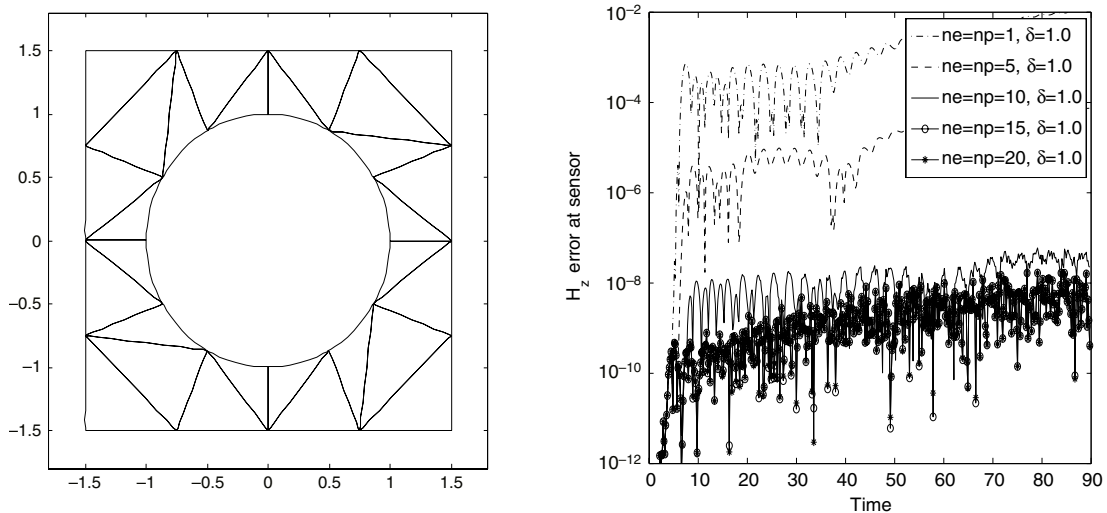


Fig. 2. Errors in solutions of the perfectly conducting cylinder scattering problem for the TE Maxwell system as the boundary condition order is increased. The mesh is also shown.

The computational domain was taken to be the rectangle $[-4, 4] \times [-1, 1]$. We used a uniform grid with $\Delta x = \Delta y = \frac{1}{150}$ and solved using the 8th order space–time difference scheme described in [17]. We implemented the complete radiation conditions with $n_p = n_e$ and the parameters described above. (The length scale parameter for the Yarvin–Rokhlin nodes was $\delta = 1$.) In Fig. 1 we plot the approximate L_2 errors over time for various choices of boundary condition order. The excellent long time accuracy and spectral convergence of the method is clear, as is the good correlation between the observed errors and the predictions in Table 1. (Here $\eta = 1/80$.) For purposes of comparison we repeat the experiment using the well-known Engquist–Majda sequence. Here the importance of the evanescent mode corrections is obvious as the long time accuracy of the solutions without them is severely degraded.

In our second example we solve the TE Maxwell system exterior to a perfectly conducting cylinder of unit radius. The incoming field was produced by a point source located at $(2.25, 0)$ with signal $\sin^6 t$, $t \geq 0$. We solve using the nodal discontinuous Galerkin method described in [26] with degree 10 polynomials in each element. In Fig. 2 we plot the triangulation used as well as the error at a point in the shadow zone. Here again we chose $\delta = 1$ and we observe excellent long time accuracy and spectral convergence. Note that as we increase the number of terms in the boundary condition beyond $n_p = n_e = 10$ we can no longer achieve significant improvements due to the discretization error.

6. Conclusions

In conclusion we have presented strong evidence that local radiation boundary condition sequences based on complete plane wave representations provide near optimal efficiency with geometric flexibility. We plan in the future to demonstrate three-dimensional implementations as well as additional applications.

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