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hp-FEM convergence for unilateral contact problems with Tresca friction in plane linear elastostatics

Joachim Gwinner *

Universität der Bundeswehr München, D-85577 Neubiberg/Munich, Germany

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ABSTRACT

This paper is concerned with the convergence of the *hp*-version of the finite element method (*hp*-FEM) for some nonsmooth unilateral problems in linear elastostatics. We consider in particular the deformation of an elastic body unilaterally supported by a rigid foundation, admitting Tresca friction (given friction) along the rigid foundation, solely subjected to body forces and surface tractions without being fixed along some part of its boundary. For the discretization of the unilateral constraint and the nonsmooth friction functional we employ Gauss–Lobatto quadrature. We show convergence of the *hp*-FEM approximations for mechanically definite problems without imposing any regularity assumption. Moreover we treat the coercive case, when the body is fixed along some part of the boundary. Based on an abstract Céa–Falk estimate and operator interpolation arguments, we establish an a priori error estimate in the energy norm under a reasonable regularity assumption.

1. Introduction

In this paper we investigate the *hp*-version of the finite element method (*hp*-FEM) applied to unilateral contact with Tresca friction in planar linear elastostatics. As already proposed by Panagiotopoulos [1] a fixed point approach to unilateral contact problems obeying the more realistic Coulomb law leads to a sequence of Tresca frictional unilateral contact problems. This approach has been recently substantiated by Dostál, Haslinger and Kučera in [2,3], who implemented the fixed point method by novel splitting techniques. Thus the efficient numerical solution of Tresca unilateral contact problems remains an interesting topic of research.

While the analysis of the standard *h*-version FEM for nonsmooth unilateral contact problems is well documented in the literature (see [4–10] and the references listed therein), the situation is less favorable for the *p*-version and particularly the *hp*-version of the FEM, where the approximation properties of spaces of piecewise polynomials are quantified in terms of both the local mesh size and the local polynomial degree. For the closely related variational inequalities of the first kind arising from non-frictional Signorini and unilateral contact problems, Maischak and Stephan analyzed *hp*-boundary element methods (*hp*-BEM) in [11,12] and obtained convergence rates under certain regularity assumptions on the exact solution. Then Chernov, Maischak and Stephan [13] provided results for the frictional two-body contact problem in the *hp*-BEM; however, the variational crimes associated with approximating the nondifferentiable friction functional j, which is clearly necessary in a high order context, were not addressed. On the other hand, Dörsek and Melenk, based on an analysis in Besov spaces, derived in [14] for the *hp*-FEM approximation of a pure frictional contact problem (without unilateral boundary conditions) an a priori error estimate consisting of a polynomial error term and an additional log error term. An a priori error estimate without such an additional log error term was earlier given by the author in [15], albeit for the *p*-BEM approximation of a simplified scalar model problem extracted from the full friction unilateral contact problem.

* Tel.: +49 8960042533.
E-mail address: joachim.gwinner@unibw-muenchen.de.

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Let us also refer to [16] for a recent numerical study and comparison results for the h-, p-, hp-version of the FEM for the solution of the unilateral frictionless 2D Hertzian contact problem. Moreover, Dörsek and Melenk were able to demonstrate numerically in some friction model problems in [17,14] that the exponential convergence of the hp-FEM, well-known for linear boundary value problems, can also obtained for this class of nonlinear free boundary value problems, if an appropriate adaptive strategy is used.

In the present paper, we focus on two issues. Firstly, we apply the discretization theory of [6] and its extension to semicoercive variational inequalities of the second type in [18]. We combine this theory with Gauss–Lobatto quadrature for the discretization of the unilateral constraint and the nonsmooth friction functional. Thus we can show convergence of the hp-FEM approximations on quadrilaterals for semicoercive unilateral Tresca friction problems without imposing any regularity assumption. Secondly we treat the coercive case, when the body is fixed along some part of its boundary. Here based on an abstract Céa–Falk estimate taken from [15] and operator interpolation arguments, we establish an hp-FEM a priori error estimate on quadrilaterals in the energy norm under a reasonable regularity assumption. This error estimate sharpens the error bound given earlier by [14] for the pure frictional contact problem and gives the same p-FEM error order on quadrilaterals as in [11] for the frictionless unilateral contact problem. Moreover we close here a gap in the proof of the consistency error in [15] which was communicated to the author by J.M. Melenk. Thus we complement and extend the convergence analysis given in [13–15,12] in several respects.

The plan of the paper is as follows. The next Section 2 presents the variational problem, its hp-FEM approximation, and collects preliminary material. The main results are in Sections 3 and 4; in Section 3 we establish hp-norm convergence without a regularity assumption, in Section 4 we prove an a priori error bound under a reasonable regularity assumption. Finally in Section 5 we give some concluding remarks and an outlook.

2. The unilateral frictional contact problem and its *hp*-FEM approximation

Let us consider an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary Γ that splits into three disjoint parts Γ_0, Γ_T, Γ_c such that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_T} \cup \overline{\Gamma_c}$. Zero displacements are prescribed on Γ_0, surface tractions $\underline{T} \in (L^2(\Gamma_T))^2$ act on Γ_T, and on the part Γ_c unilateral contact and Tresca friction conditions between the body and a perfectly rigid foundation hold. Thus Γ_c contains the free boundary of the unilateral contact. In the model of Tresca friction (given friction) one assumes a known slip bound $g \in L^\infty(\Gamma_c),\ g \geq 0$. Moreover, the body is subject to body forces $\underline{F} \in (L^2(\Omega))^2$. To make the contact problem meaningful, we assume meas $(\Gamma_c) > 0$, but we do not require meas $(\Gamma_0) > 0$.

We denote by $H^s()$ the usual Sobolev spaces on Ω or on parts of Γ with norms defined using the Slobodeckij seminorms. We also use the short $\underline{H}^s = (H^s)^2$ for the vectorial Sobolev spaces. In particular, we have the space of *virtual displacements*

$$\mathcal{V} = \{\underline{v} \in \underline{H}^1(\Omega) \mid \gamma_0\, \underline{v} = 0\},$$

where $\gamma_o = \gamma_{\Gamma 0} : \underline{H}^1(\Omega) \to \underline{H}^{\frac{1}{2}}(\Gamma_0)$ is the trace map onto Γ_0, and its convex closed subset of *kinematically admissible* displacements

$$\mathcal{K} = \{\underline{v} \in \mathcal{V} \mid (\gamma_c\, \underline{v})_n \leq d\}.$$

Here, likewise $\gamma_c = \gamma_{\Gamma c} : \underline{H}^1(\Omega) \to \underline{H}^{\frac{1}{2}}(\Gamma_c) \subset (L^2(\Gamma_c))^2$, further $d \in C(\overline{\Gamma_c})$, $d \geq 0$ is the initial gap between the body and the rigid foundation, and with the unit outer normal $\underline{n} \in (L^\infty(\Gamma))^2$ to the boundary, a vector field $\underline{w}$ at the boundary has its normal component $w_n = \underline{w} \cdot \underline{n}$ and its normal component $\underline{w}_t = \underline{w} - w_n\, \underline{n}$.

Adopting standard notations from linear elasticity, $\varepsilon(\underline{v}) = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$ denotes the small strain tensor to the displacement field $\underline{v}$ and $\sigma(\underline{v}) = C : \varepsilon(\underline{v})$ the stress tensor. Here, C is the Hooke tensor, assumed to be uniformly positive definite with L^∞ coefficients. This leads to the bilinear form, linear functional, sublinear functional, and to the *total potential energy* of the body, respectively,

$$a(\underline{u}, \underline{v}) = \int_\Omega \varepsilon(\underline{u}) : C : \varepsilon(\underline{v})\, dx,$$

$$l(\underline{v}) = \int_\Omega \underline{F} \cdot \underline{v}\, dx + \int_{\Gamma_T} \underline{T} \cdot \underline{v}\, ds,$$

$$j(\underline{v}) = \int_{\Gamma_c} g\, |\underline{v}_t|\, ds,$$

$$J(\underline{v}) = \frac{1}{2} a(\underline{v}, \underline{v}) - l(\underline{v}) + j(\underline{v}).$$

In these terms, the *variational formulation* of the unilateral contact problem with Tresca friction reads as follows:
Find a minimizer $\underline{u} \in \mathcal{K}$ of the functional $J(\underline{v})$, $\underline{v} \in \mathcal{K}$!
Another equivalent formulation is the *variational inequality* problem (π) *of second kind*: Find $\underline{u} \in \mathcal{K}$ such that for all $\underline{v} \in \mathcal{K}$,

$$a(\underline{u}, \underline{v} - \underline{u}) + j(\underline{v}) - j(\underline{u}) \geq l(\underline{v} - \underline{u}). \tag{1}$$

There exists a unique solution $\underline{u}$ (see e.g [19,8]), if Γ_0 has positive measure and hence the bilinear form is coercive by Korn's inequality, see e.g. [20,21]. In the semicoercive case, when $\Gamma_0 = \emptyset$, following [22,10], one introduces the subspace of *rigid body motions*:

$$\begin{aligned}\mathcal{R} &= \left\{\underline{r} \in [H^1(\Omega)]^2 : \varepsilon(\underline{r}) = 0\right\} \\ &= \left\{\underline{r} \in [H^1(\Omega)]^2 : r_1 = a_1 - bx_2, r_2 = a_2 + bx_1; a_1, a_2, b \in \mathbb{R}\right\}\end{aligned}$$

and the *recession cone* [23],

$$\mathcal{K}_{\text{rec}} = \{\underline{v} \in \mathcal{V} \mid (\gamma_c\, \underline{v})_n \le 0\}$$

of the convex subset $\mathcal{K}$. Then choose $\underline{r}^* \in \mathcal{R}$ with $\underline{r}^*_n \le \min\{d(\underline{x}) : \underline{x} \in \Gamma_c\}$, insert $\underline{r}^* \in \mathcal{K}$ in (1), obtain since $j(\underline{u}) \ge 0$

$$a(\underline{u}, \underline{u}) - l(\underline{u}) \le j(\underline{r}^*) - l(\underline{r}^*),$$

hence the a priori estimate

$$a(\underline{u}, \underline{u}) \le c_0 + c_1 \|\underline{u}\|_{\underline{H}^1(\Omega)} \tag{2}$$

for some constants $c_0, c_1 \ge 0$. Further inserting $\underline{v} = \underline{r}^* + \rho\underline{r} \in \mathcal{K}$ in (1) with arbitrary $\rho \ge 0, \underline{r} \in \mathcal{R} \cap \mathcal{K}_{\text{rec}}$ and letting $\rho \to \infty$ gives as a necessary condition for the existence of a solution to the recession condition

$$l(\underline{r}) \equiv \int_\Omega \underline{F} \cdot \underline{r}\, dx + \int_{\Gamma_T} \underline{T} \cdot \underline{r}\, ds \le j(\underline{r}) \equiv \int_{\Gamma_c} g\, |\underline{r}_t|\, ds, \quad \forall \underline{r} \in \mathcal{R} \cap \mathcal{K}_{\text{rec}}.$$

Existence of a solution is guaranteed (see [23]) under the strengthened condition

$$l(\underline{r}) < j(\underline{r}), \quad \forall \underline{r} \in \mathcal{R} \cap \mathcal{K}_{\text{rec}} \setminus \{0\},$$

or under the simpler, but stronger condition (see [19])

$$l(\underline{r}) < 0, \quad \forall \underline{r} \in \mathcal{R} \cap \mathcal{K}_{\text{rec}} \setminus \{0\}.$$

Both sufficient conditions clearly only make sense if $\mathcal{R} \cap \mathcal{K}_{\text{rec}}$ does not contain a subspace; for a study of the case of a nontrivial subspace we also refer to [19].

To conclude the preliminaries for our finite element analysis we state an essential hypothesis, namely the density relation

$$\overline{\mathcal{K} \cap [C^\infty(\overline{\Omega})]^2} = \mathcal{K}. \tag{3}$$

We note that (see [8]) (3) holds true in a polygonal domain Ω (what we assume from now on for simplicity), if there is only a finite number of "end points" $\overline{\Gamma}_c \cap \overline{\Gamma}_T, \overline{\Gamma}_0 \cap \overline{\Gamma}_T, \overline{\Gamma}_c \cap \overline{\Gamma}_0$.

For simplicity let Ω be a polygonal, planar domain and let g be a piecewise constant function on Γ_C. These are no restrictions of generality. In fact, the p- and hp-finite element approximation on curvilinear domains is well-understood, see [24]. The analysis to follow can be extended to higher dimensional domains by tensor product approximation.

Let $\mathcal{T}_N$ $(N \in \mathbb{N})$ be a shape regular [25] sequence of meshes consisting of affine quadrilaterals $Q \in \mathcal{T}_N$ with diameter $h_{N,Q}$ such that all corners of Γ and all "end points" $\overline{\Gamma}_c \cap \overline{\Gamma}_T, \overline{\Gamma}_T \cap \overline{\Gamma}_0, \overline{\Gamma}_0 \cap \overline{\Gamma}_c$ are nodes of $\mathcal{T}_N$. Moreover, we introduce the set of edges on the contact boundary,

$$\mathcal{E}_{c,N} = \{E : E \subset \Gamma_c \text{ is an edge of } \mathcal{T}_N\}$$

and assume that g is constant on each edge $E \in \mathcal{E}_{c,N}$. Obviously, for every $E \in \mathcal{E}_{c,N}$ there exists a unique $Q_E \in \mathcal{T}_N$ such that E is an edge of Q_E.

Further we denote by $p_{N,Q} \in \mathbb{N}$ a polynomial degree for each $Q \in \mathcal{T}_N$. We assume that neighboring elements have comparable polynomial degrees, i.e. there exists a constant $c > 0$ such that for elements $Q, Q' \in \mathcal{T}_N$ with $\overline{Q} \cap \overline{Q'} \neq \emptyset$ there holds

$$c^{-1}\, p_{N,Q} \le p_{N,Q'} \le c\, p_{N,Q}.$$

Let $\Pi^p(Q)$ be the tensor product space of polynomials of degree p in each variable. This gives the FE subspace

$$\mathcal{V}_N = \{\underline{v}_N \in \mathcal{V} : \underline{v}_N \mid Q \in (\Pi^{p_{N,Q}}(Q))^2, \forall Q \in \mathcal{T}_N\}.$$

Similar to [17,14,11,12] we employ Gauss–Lobatto quadrature in the discretization procedure. To this end we introduce for $q \ge 1$ on the reference interval $[-1, 1]$ the $q+1$ Gauss–Lobatto points, i.e., the zeros ξ_j^{q+1} $(0 \le j \le q)$ of $(1-\xi^2)L_q'(\xi)$, where L_q denotes the Legendre polynomial of degree q. Note that $\xi_0^{q+1} = -1$ and $\xi_q^{q+1} = 1$ are the end points of the reference interval. It is known (see [26, Chapter I, Section 4]) that there exist positive weights

$$\omega_j^{q+1} := \frac{1}{q(q+1)L_q^2(\xi_j^{q+1})}$$

such that the quadrature formula

$$\int_{-1}^{1} \phi(\xi)\, d\xi = \sum_{j=0}^{q} \omega_j^{q+1} \phi(\xi_j^{q+1})$$

is exact for all polynomials ϕ up to degree $2q-1$.

For any $E \in \mathcal{E}_{c,N}$ we introduce the quadrature order $q_{N,E}$ such that $q_{N,E} = p_{N,Q_E}$. By affine transformation $F_E : [-1,1] \to \overline{E}$ we define the set $G_{E,N}$ of $q_{N,E}+1$ Gauss–Lobatto points for each element E of $\mathcal{E}_{c,N}$ and set $G_{c,N} := \bigcup\{G_{E,N} : E \in \mathcal{E}_{c,N}\}$. Choosing the Gauss–Lobatto points as control points of the unilateral constraint, we define

$$\mathcal{K}_N := \{\underline{v}_N \in \mathcal{V}_N : (\gamma_c\, \underline{v}_N)_n \le d \text{ on } G_{c,N}\}.$$

Clearly, $\mathcal{K}_N$ is a convex closed subset of $\mathcal{V}_N$. Note however, $\mathcal{K}_N$ is generally not contained in $\mathcal{K}$ for polynomial degree ≥ 2 or for a non-concave obstacle d.

We also approximate the nonlinear nonsmooth functional j using the above quadrature rule by

$$j_N(\underline{v}) = j_{c,N}(\gamma_c\, \underline{v})_t,\; j_{c,N}(\psi) = \sum_{E\in\mathcal{E}_{c,N}} g_E \sum_{j=0}^{q_{N,E}} \omega_j^{q_{N,E}+1} \left|\psi \circ F_E(\xi_j^{q_{N,E}+1})\right|,$$

where g_E denotes the constant value of the function g on E. Then $j_N, j_{c,N}$ are sublinear functionals, with $j_{c,N}$ uniformly bounded on $C(\overline{\Gamma_c})$. Note that for the piecewise polygonal boundary Γ, $\underline{v}_t \circ F_E$ is piecewise polynomial of the same degree as $\underline{v}$.

Thus we arrive at the following discrete variational problem (π_N) as an approximation to our variational problem (π): Find $\underline{u}_N \in \mathcal{K}_N$ such that for all $\underline{v}_N \in \mathcal{K}_N$

$$a(\underline{u}_N, \underline{v}_N - \underline{u}_N) + j_N(\underline{v}_N) - j_N(\underline{u}_N) \ge l(\underline{v}_N - \underline{u}_N). \tag{4}$$

Similarly to the above bound (2) we obtain the a priori bound

$$a(\underline{u}_N, \underline{u}_N) \le c_0 + c_1 \|\underline{u}_N\|_{\underline{H}^1(\Omega)} \tag{5}$$

for some constants $c_0, c_1 \ge 0$ independent of N.

Note that we only replaced the nonlinear functional j by its approximate j_N. In most computations, however, also a and l have to be replaced by some approximations that take into account e.g. numerical integration or approximation of a curved boundary. Since such approximations are well documented in the literature of h- and hp-finite element analysis of elliptic boundary value problems (see [24,25]), we omit this aspect here.

Associated to the Gauss–Lobatto points $G_{E,N}$ we have the local interpolation operator $i_{E,q} = i_{E,N} : C^0(\overline{E}) \to \mathcal{P}_q(E)$ with $q = q_{N,E}$ given by

$$(i_{E,N}\eta)(x) = \eta(x), \quad \forall\, x \in G_{E,N},\; \eta \in C^0(\overline{E})$$

and the global interpolation operator $i_{c,N}$ on $C^0(\overline{\Gamma_c})$ defined by

$$i_{c,N}\eta = \sum_{E\in\mathcal{E}_{c,N}} (i_{E,N}\eta)|\overline{E}, \quad \forall\, \eta \in C^0(\overline{\Gamma}).$$

Likewise associated to the Gauss–Lobatto points $G_{Q,N} = F_Q\{(\xi_i^{p+1}, \xi_j^{p+1}) \mid 0 \le i,j \le p\}$ with $p = p_{N,Q}$ and the affine transformation $F_Q : [-1,1]^2 \to \overline{Q}$ we have the local interpolation operator $i_{Q,p} = i_{Q,N} : C^0(\overline{Q}) \to \mathcal{P}_p(Q)$ with $p = p_{N,E}$ given by

$$(i_{Q,N}\psi)(x) = \psi(x), \quad \forall\, x \in G_{Q,N},\; \psi \in C^0(\overline{Q})$$

and the global interpolation operator i_N on $C^0(\overline{\Omega})$ defined by

$$i_N\psi = \sum_{Q\subset\Omega} (i_{Q,N}\psi)|\overline{Q}, \quad \forall\, \psi \in C^0(\overline{\Omega}).$$

For later use we recall from [26, Theorems 13.4, 14.2]; [27, Theorems 4.7, 5.9] the following results on the polynomial interpolation error in the reference interval $\hat{E} = (-1,1)$, respectively in the reference square $\hat{Q} = (-1,1)^2$.

Theorem 2.1. (i) *For any real numbers r and s satisfying $s > (1+r)/2$ and $0 \le r \le 1$, there exists a positive constant c depending only on s such that for any function $\eta \in H^s(\hat{E})$ the following estimate holds*

$$\|\eta - i_{\hat{E},q}\eta\|_{H^r(\hat{E})} \le c\, q^{r-s}\, \|\eta\|_{H^s(\hat{E})}. \tag{6}$$

(ii) *For any real numbers r and s satisfying $s > 1 + r/2$ and $0 \le r \le 1$, there exists a positive constant c depending only on s such that for any function $\psi \in H^s(\hat{Q})$ the following estimate holds*

$$\|\eta - i_{\hat{Q},p}\psi\|_{H^r(\hat{Q})} \le c\, p^{r-s}\, \|\psi\|_{H^s(\hat{Q})}. \tag{7}$$

3. A *hp*-approximation result

Without any regularity assumption for the solution $\underline{u}$ of (π) we can show the following convergence result for the *hp*-FEM solutions $\underline{u}_N$ of (π_N) in the energy norm.

Theorem 3.1. *Let the solution $\underline{u}$ of (π) exist uniquely. Suppose that for the polygonal domain Ω, there are only a finite number of "end points" $\overline{\Gamma}_c \cap \overline{\Gamma}_0, \overline{\Gamma}_c \cap \overline{\Gamma}_T, \overline{\Gamma}_T \cap \overline{\Gamma}_c$ and the gap function d belongs to $H^{1/2+\varepsilon}(\Gamma_c)$ for some $\varepsilon > 0$. Then for $N \to \infty$ with $\min_{Q\in\mathcal{T}_N} h_{N,Q}^{-1}\, p_{N,Q} \to \infty$ there holds $\underline{u}_N \to \underline{u}$ with respect to the $\underline{H}^1(\Omega)$ norm.*

Proof. Here we adapt the discretization theory of Glowinski [6] to more general semicoercive variational inequalities of the second kind over a convex subset instead over the whole space, see also [18, Theorem 2.2]. Thus we have to show the following hypotheses:

H1 If $\underline{v}_N \rightharpoonup \underline{v}$ (weak convergence) in $\mathcal{V}$ for $N \to \infty$ with $\underline{v}_N \in \mathcal{K}_N$, then $\underline{v} \in \mathcal{K}$ and

$$\liminf_{N\to\infty} j_N(\underline{v_N}) \geq j(\underline{v}).$$

H2 There exist a subset $M \subset \mathcal{K}$ dense in $\mathcal{K}$ and mappings $\varrho_N : M \to \mathcal{V}_N$ such that, for each $\underline{w} \in M$, $\varrho_N(\underline{w}) \to \underline{w}$ for $N \to \infty$,

$$\lim_{N\to\infty} j_N\left(\varrho_N(\underline{w})\right) = j(\underline{w}),$$

and $\varrho_N(\underline{w}) \in \mathcal{K}_N$ for all $N \geq N_0(w)$ for some $N_0(\underline{w}) > 0$.

Classical *h*-FEM convergence for the variational problem under study is already treated in [18], where Newton–Cotes formulas in numerical quadrature are used instead of Gauss–Lobatto quadrature. Inspecting the proof of [18, Theorem 4.1] shows that the norm convergence for a fixed quadrature order hinges on the positiveness of the quadrature weights, what is satisfied for all quadrature orders with Gauss–Lobatto quadrature. Therefore in the following we can focus to the case where $h_{N,Q}$ is fixed for all $Q \in \mathcal{T}_N$ and $\min_{Q\in\mathcal{T}_N} p_{N,Q} \to \infty$.

To verify H1 it is enough to show that for any $\lambda \in C^0(\Gamma)$ with $\lambda|\Gamma_c \geq 0$,

$$\int_{\Gamma_c} (\underline{v}_n - d)\lambda\, ds \leq 0, \tag{8}$$

and to show that for any $\mu \in C^0(\Gamma)$ with $|\mu| \leq 1$ on Γ_c there holds

$$\int_{\Gamma_C} g\, \underline{v}_t\, \mu\, ds \leq \liminf_{N\to\infty} j_N(\underline{v}_N), \tag{9}$$

since by duality with respect to (L^1, L^∞) and density

$$j(\underline{v}) = \sup\left\{\int_{\Gamma_c} g\, \underline{v}_t \mu\, ds : \mu \in C^0(\Gamma), |\mu| \leq 1\right\}.$$

Moreover, since the mesh $\mathcal{T}_N$ is independent of N, we can simply consider the above integrals on any fixed edge $E \in \mathcal{E}_{c,N}$. Thus fix $\lambda, \mu \in C^0[\overline{E}]$ with $\lambda \geq 0$, $|\mu| \leq 1$ and also $q := q_{N,E}$. Similarly as [11] approximate these functions by a combination of Bernstein polynomials B_q with the local mapping $F_E : [-1, 1] \to \overline{E}$ to define $\lambda_q := B_q\lambda \circ F_E$, $\mu_q := B_q\mu \circ F_E$ via

$$\lambda_q(t) = (B_q\lambda \circ F_E)(t) := \sum_{k=0}^{q} \binom{q}{k} \left(\frac{1+t}{2}\right)^k \left(\frac{1-t}{2}\right)^{q-k} (\lambda \circ F_E)\left(\frac{2k}{q} - 1\right).$$

Since the Bernstein operators are monotone, $\lambda_q \geq 0$ and $|\mu_q| \leq 1$. By [28, Chapter 1, Theorem 2.3],

$$\lim_{q\to\infty} \|\lambda_q - \lambda\|_{L^\infty(E)} = \lim_{q\to\infty} \|\mu_q - \mu\|_{L^\infty(E)} = 0. \tag{10}$$

For the obstacle function $d \in H^{1/2+\varepsilon}(\Gamma_c)$ we use the interpolate $d_N := i_{E,q}\, d$ as approximation. By Theorem 2.1(i) with $r = 0, s = \frac{1}{2} + \varepsilon$

$$\lim_{N\to\infty} \|d_N - d\|_{L^2(E)} = 0. \tag{11}$$

Since the embedding $H^{1/2}(\Gamma) \hookrightarrow L^1(E)$ is weakly continuous, $\underline{v}_N \rightharpoonup \underline{v}$ in $L^1(E)$ and $\|\underline{v}_N\|_{(L^1(E))^2}$ is bounded. Therefore from

$$\left|\int_E \left[(\underline{v}_{N,n} - d_N)\lambda_{q_{N,E}-1} - (\underline{v}_n - d)\lambda\right] dt\right| \leq \|\underline{v}_{N,n} - d_N\|_{L^1(E)}\, \|\lambda_{q_{N,E}-1} - \lambda\|_{L^\infty(E)} + \left|\int_E \left[(\underline{v}_{N,n} - d_N) - (\underline{v}_n - d)\right] \lambda\, dt\right|;$$

$$\left| \int_E [\underline{v}_{N,t}\, \mu_{q_{N,E}-1} - \underline{v}_t \mu]\, dt \right| \le \|\underline{v}_{N,t}\|_{L^1(E)}\, \|\mu_{q_{N,E}-1} - \mu\|_{L^\infty(E)} + \left| \int_E [\underline{v}_{N,t} - \underline{v}_t]\mu\, dt \right|,$$

(10), (11) and using $\lambda, \mu \in L^\infty(e) = (L_1(e))^*$, we conclude

$$\lim_{N\to\infty} \int_E (\underline{v}_{N,n} - d_N)\, \lambda_{q_{N,E}-1}\, dt = \int_E (\underline{v}_n - d)\, \lambda\, dt, \tag{12}$$

$$\lim_{N\to\infty} \int_E \underline{v}_{N,t}\, \mu_{q_{N,E}-1} dt = \int_E \underline{v}_t\, \mu\, dt. \tag{13}$$

On the other hand, $(\underline{v}_{N,n} - d_N)|E\, \lambda_{q_{N,E}-1}$ and $\underline{v}_{N,t}|E\, \mu_{q_{N,E}-1}$ are polynomials of degree $2q-1$. Hence the above integrals can be evaluated exactly by the Gauss–Lobatto quadrature formula to obtain

$$\int_E (\underline{v}_{N,n} - d_N)\, \lambda_{q_{N,E}-1}\, dt = \sum_{j=0}^{q} \omega_j^{q+1} \left[(\underline{v}_{N,n} - d_N)\, \lambda_{q-1} \right] \circ F_E(\xi_j^{q+1}),$$

$$\int_E \underline{v}_{N,t}\, \mu_{q_{N,E}-1}\, dt = \sum_{j=0}^{q} \omega_j^{q+1} (\underline{v}_{N,t}\, \mu_{q-1}) \circ F_E(\xi_j^{q+1}).$$

Since the weights $\omega_j^{q+1} > 0$, $\lambda_{q-1} \ge 0$, $(\underline{v}_{N,n} - d_N) \circ F_E(\xi_j^{q+1}) \le 0$ by $\underline{v}_N \in \mathcal{K}_N$, respectively $|\mu_{q-1}| \le 1$, $g_e \ge 0$ we arrive at

$$\int_E (\underline{v}_{N,n} - d_N)\lambda_{q_{N,E}-1}\, dt \le 0;$$

$$g_E \int_E \underline{v}_{N,t}\, \mu_{q_{N,E}-1}\, dt \le g_E \sum_{j=0}^{q} \omega_j^{q+1} |\underline{v}_{N,t} \circ F_E(\xi_j^{q+1})| =: j_{E,N}(\underline{v}_N),$$

$$\sum_{E\in\mathcal{E}_{c,N}} j_{E,N}(\underline{v}_N) = j_N(\underline{v}_N).$$

In view of (12) and (13) this proves our claim (8), (9).

In the last step let us prove H2.

By the finiteness assumption we have due to [8] the density relation

$$\overline{\mathcal{K} \cap [C^\infty(\Omega)]^2} = \mathcal{K}.$$

Therefore we can take $M = \mathcal{K} \cap [C^\infty(\Omega)]^2$ and define $\varrho_N : M \to \mathcal{V}_N$ by $\varrho_N := i_N$. Moreover, since $\underline{w} \in M$ satisfies the constraints in $\mathcal{K}$ pointwise, $\varrho_N \underline{w} \in \mathcal{K}_N$ for all $\underline{w} \in M$. By Theorem 2.1(ii), $\varrho_N \underline{w} \to \underline{w}$ in $\underline{H}^1(\Omega)$. Finally by $j_N(\underline{w}) = j(\varrho_N \underline{w})$, we conclude for $N \to \infty$,

$$\begin{aligned} |j(\underline{w}) - j_N(\varrho_N \underline{w})| &\le |j(\underline{w}) - j(\varrho_N \underline{w})| + |j_N(\underline{w}) - j_N(\varrho_N \underline{w})| \\ &\le \|g\|_{L^\infty(\Gamma_c)} \left[\|\underline{w}_t - (\varrho_N \underline{w})_t\|_{L^1(\Gamma_c)} + \|\underline{w}_t - (\varrho_N \underline{w})_t\|_{L^\infty(\Gamma_c)} \right] \to 0. \quad \square \end{aligned}$$

4. An a priori *hp*-error estimate

In this section we provide an a priori error estimate for the *hp*-approximate $\underline{u}_N$ of the variational problem (π) under the regularity assumptions of [12], which in particular via the trace theorem amounts to $H^2(\Omega)$ regularity of the solution $\underline{u}$.

To this end, we apply an abstract Céa–Falk lemma for variational inequalities of the second kind taken from [15] together with Falk's cutting technique [29], adapt some arguments of Maischak and Stephan [12], and use some results of operator interpolation theory [20,25].

Let us first recall the abstract Céa–Falk lemma from [15]. Consider real normed vector spaces $(E, \|.\|_E)$, $(G, \|.\|_G)$ and their duals E^*, G^* such that $E \subset G$ continuously. Let E_N be a subspace of E $(N \in \mathbb{N})$ with the embedding $\iota_N \in \mathcal{L}(E_N, E)$. Let $K \subseteq E$ and $K_N \subseteq E_N$ be convex sets with some $x_0 \in K \cap \bigcap_{N\in\mathbb{N}} K_N$. Let $f^* \in E^*$ and for simplicity, $f_N^* = \iota_n^* f^*$. Let $B \in \mathcal{L}(E, E^*)$ be positive definite with respect to $\|\cdot\|_E$; i.e., there exist some $\underline{c}_B, \overline{c}_B > 0$ such that

$$\underline{c}_B \|v\|_E^2 \le B(v)(v) \le \overline{c}_B \|v\|_E^2 \quad (\forall\, v \in E).$$

Lemma 4.1. *Let the preceding assumptions on E, G, K, K_N, f^*, B be satisfied; let $u \in K$ and $u_N \in K_N$ such that*

$$(Bu - f^*) \in G^*,$$
$$B(u)(v-u) + j(v) - j(u) \ge f^*(v-u) \quad (\forall\, v \in K),$$
$$B(u_N)(v_N - u_N) + j_N(v_N) - j_N(u_N) \ge f_N^*(v_N - u_N) \quad (\forall\, v_N \in K_N).$$

Then there exists a constant $c > 0$ which depends on $\underline{c}_B, \overline{c}_B, x_0, f^*, B$ but not on N such that

$$c\|u - u_N\|_E^2 \le \inf_{v\in K} \left\{\|Bu - f^*\|_{G^*} \|u_N - v\|_G + |j_N(u_N) - j(v)|\right\}$$
$$+ \inf_{v_N\in K_N} \left\{\|u - v_N\|_E^2 + |j(u) - j_N(v_N)| + \|Bu - f^*\|_{G^*}\|u - v_N\|_G\right\}.$$

We apply the lemma in the setting: $E \equiv \underline{H}^1(\Omega) := (H^1(\Omega))^2 \subset G \equiv \underline{H}^{1/2}(\Omega)$, $B\underline{u} - f^* \equiv a(\underline{u}, \cdot) - l(\cdot)$. Thus we obtain the main result of this section.

Theorem 4.2. *Let $\underline{u} \in \mathcal{K}$ be the solution of the coercive problem (π) with meas $(\Gamma_0) > 0$. Assume $\underline{u} \in \underline{H}^2(\Omega)$, $d \in H^{3/2}(\Gamma_c)$, $a(\underline{u}, \cdot) - L(\cdot) \in (\underline{H}^{1/2}(\Omega))'$. Then there exists $c = c(\underline{u}, d, \underline{F}, \underline{T}, g) > 0$, independent of N such that*

$$\|\underline{u} - \underline{u}_N\|_{H^1(\Omega)} \le c \max_{Q\in\mathcal{T}_N} h_{N,Q}^{1/4}\, p_{N,Q}^{-1/4}.$$

Proof. Let us write $h = \max_{Q\in\mathcal{T}_N} h_{N,Q}$, $p = \min_{Q\in\mathcal{T}_N} p_{N,Q}$ for short. Lemma 4.1 splits the error under study into two different error terms:

$$c\|\underline{u} - \underline{u}_N\|^2_{\underline{H}^1(\Omega)} \le \inf_{\underline{v}\in\mathcal{K}} \left\{\|B\underline{u} - f^*\|_{\underline{H}^{1/2}(\Omega)'}\, \|\underline{u}_N - \underline{v}\|_{\underline{H}^{1/2}(\Omega)} + |j_N(\underline{u}_N - j(\underline{v}))|\right\}$$
$$+ \inf_{\underline{v}_N\in\mathcal{K}_N} \left\{\|\underline{u} - \underline{v}_N\|^2_{\underline{H}^1(\Omega)} + |j(\underline{u}) - j_N(\underline{v}_N)| + \|B\underline{u} - f^*\|_{\underline{H}^{1/2}(\Omega)'}\, \|\underline{u} - \underline{v}_N\|_{\underline{H}^{1/2}(\Omega)}\right\}. \tag{14}$$

To bound first the approximation error $\inf\{\dots|\underline{v}_N \in \mathcal{K}_N\}$ take $\underline{v}_N = \underline{u}_N^* := i_N\underline{u} \in \mathcal{K}_N$, the interpolate of $\underline{u} \in \underline{H}^2(\Omega) \subset (C^0(\Omega))^2$. By Theorem 2.1(ii) and by affine equivalence of $Q \in \mathcal{T}_N$ to $\hat{Q}$ using [20, Theorem 4.4.20], there are constants $c_1, c_2 > 0$ independent of $\underline{u}$ and N such that

$$\|\underline{u} - \underline{u}_N^*\|_{\underline{H}^1(\Omega)} \le c_1 \frac{h}{p} \|\underline{u}\|_{\underline{H}^2(\Omega)},$$
$$\|\underline{u} - \underline{u}_N^*\|_{\underline{H}^{1/2}(\Omega)} \le c_2 \left(\frac{h}{p}\right)^{3/2} \|\underline{u}\|_{\underline{H}^2(\Omega)}.$$

Further, by construction,

$$j_N(\underline{u}_N^*) = \sum_{E\in\mathcal{E}_{c,N}} g_E \sum_{j=0}^{q_{N,E}} \omega_j^{q_{N,E}+1}\, |(\underline{u}_{N,t}^* \circ F_E)\,(\xi_j^{q_{N,E}+1})|$$
$$= \sum_{E\in\mathcal{E}_{c,N}} g_E \sum_{j=0}^{q_{N,E}} \omega_j^{q_{N,E}+1}\, |(\underline{u}_t \circ F_E)\,(\xi_j^{q_{N,E}+1})| = j_N(\underline{u}).$$

Hence using the interpolation operators $i_{E,N}$, $i_{c,N}$ and the exactness of Gauss–Lobatto quadrature,

$$|j(\underline{u}) - j_N(\underline{u}_N^*)| \le \sum_{E\in\mathcal{E}_{c,N}} g_e \left|\int_E |\underline{u}_t| ds - \int_E i_{E,N}\left(|\underline{u}_t|\right) ds\right|$$
$$\le \|g\|_{L^\infty(\Gamma_c)} \int_{\Gamma_c} \left|\, |\underline{u}_t| - i_{c,N}\left(|\underline{u}_t|\right)\right| ds$$
$$\le \text{meas}\,(\Gamma_c)^{1/2}\|g\|_{L^\infty(\Gamma_c)} \left\||\underline{u}_t| - i_{c,N}\left(|\underline{u}_t|\right)\right\|_{L^2(\Gamma_c)}. \tag{15}$$

Since $|\underline{u}_t|$ can only be guaranteed to lie in $H^1(\Gamma)$ (as the max of the two absolutely continuous functions $\underline{u}_t, -\underline{u}_t$, see also [30, Corollary A.6]) we can use the regularity assumption to derive only

$$\||\underline{u}_t|\|_{H^1(\Gamma)} \le \|\underline{u}_t\|_{H^1(\Gamma)}. \tag{16}$$

Therefore we can conclude by Theorem 2.1(ii) and by affine equivalence of $E \in \mathcal{E}_{c,N}$ to $\hat{E}$ using [20, Theorem 4.4.20],

$$|j(u) - j_p(u_p^*)| \le \tilde{c}\, \frac{h}{p}\, \||\underline{u}_t|\|_{H^1(\Gamma)} \le \tilde{c}\, \frac{h}{p}\, \|\underline{u}_t\|_{H^1(\Gamma)}.$$

Thus for the approximation error,

$$\inf\left\{\dots|\underline{v}_N \in \mathcal{K}_N\right\} \le c_I(\underline{u}, g)\, \frac{h}{p}. \tag{17}$$

To bound the consistency error $\inf\left\{\ldots \mid \underline{v} \in \mathcal{K}\right\}$ we use Falk's cutting technique [29] and similarly as [12] define

$$\underline{v}^*_\Gamma = \begin{cases} \underline{0} & \text{on } \Gamma_0 \\ \gamma_T\, \underline{u}_N & \text{on } \Gamma_T \\ (\gamma_c\, \underline{u}_N)_t + [d + \inf((\gamma_c\, \underline{u}_N)_n - d_N, 0)]\, \underline{n} & \text{on } \Gamma_c \end{cases}$$

where $d_N := i_{c,N}\, d$ interpolates the gap function d.

To show that $\underline{v}^*_\Gamma \in \underline{H}^1(\Gamma)$ we use the arguments of [12]: For any E, the polynomial $\underline{u}_{N,n} - d_N \mid E$, which is of degree $q_{N,E}$, has at most $q_{N,E}$ zeros on E. Hence, the level set $\{x \in \Gamma_c : (\underline{u}_{N,n} - d_N)(x) < 0\}$ is the finite union of open subintervals and $\delta_N := \inf(\underline{u}_{N,n} - d_N, 0)$ is continuous and piecewise a polynomial on Γ_c. Therefore $\delta_N \in H^1(\Gamma_c)$ and $\underline{v}^*_\Gamma \in \underline{H}^1(\Gamma)$ as claimed.

Now we use that the trace map γ possesses a right inverse χ and that $\chi : L^2(\Gamma) \to H^{1/2}(\Omega)$ is bounded, see [31,32]. We let $\underline{v}^* = \chi\, \underline{v}^*_\Gamma$. Then $\underline{v}^* \in \mathcal{K}$ and using the estimates (20)–(24) in [12, proof of Theorem 2] we obtain

$$\begin{aligned} \|\underline{v}^* - \underline{u}_N\|_{\underline{H}^{1/2}(\Omega)} &\le c_\chi \|\underline{v}^*_\Gamma - \gamma\, \underline{u}_N\|_{(L^2(\Gamma_C))^2} \\ &\le c\, h^{1/2} p^{-1/2} \left(\|d\|_{H^{1/2}(\Gamma_C)} + \|u\|_{\underline{H}^1(\Omega)} \right). \end{aligned} \tag{18}$$

Finally we prove that $|j_N(\underline{u}_N) - j(\underline{v}^*)|$ has the same error order of $h^{1/2} p^{-1/2}$. To this end we use real interpolation between $H^1(\Gamma_c)$ and $L^2(\Gamma_c)$, see [20,25].

First note that by construction the tangential components of $\underline{v}^*_\Gamma$ and $\gamma_c\, \underline{u}_N$ coincide, hence we have $j_N(\underline{v}^*) = j_N(\underline{u}_N)$.

On the other hand, by Theorem 2.1(i),

$$\begin{aligned} \|\underline{v}^*_\Gamma\|_{\underline{H}^1(\Gamma_C)} &\le \|(\gamma_c\, \underline{u}_N)_n - d_N\|_{H^1(\Gamma_C)} + \|d\|_{H^1(\Gamma_C)} + \|(\gamma_c\, \underline{u}_N)_t\|_{H^1(\Gamma_C)} \\ &\le \|\gamma\, \underline{u}_N\|_{\underline{H}^1(\Gamma)} + 2\|d\|_{H^1(\Gamma_C)} + c h^{1/2} p^{-1/2} \|d\|_{H^{3/2}(\Gamma_C)}, \\ \|\underline{v}^*_\Gamma\|_{(L^2(\Gamma_C))^2} &\le \|(\gamma_c\, \underline{u}_N)_n - d_N\|_{L^2(\Gamma_C)} + \|d\|_{L^2(\Gamma_C)} + \|(\gamma_c\, \underline{u}_N)_t\|_{L^2(\Gamma_C)} \\ &\le \|\gamma\, \underline{u}_N\|_{(L^2(\Gamma))^2} + 2\|d\|_{L^2(\Gamma_C)} + c' h^{3/2} p^{-3/2} \|d\|_{H^{3/2}(\Gamma_C)}. \end{aligned}$$

Hence by real interpolation of the nonlocal H^t-norm and by the boundedness of $\|u_N\|$ in virtue of the a priori bound (5) or in virtue of Theorem 3.1, $\underline{v}^*_\Gamma$ is bounded, too, in $\underline{H}^{1/2}(\Gamma_C)$.

Then with similar arguments as above, see (15) and (16), we can conclude for $\underline{v}^*_t := \underline{v}^*_\Gamma \cdot \underline{t}$, $|\underline{v}^*_t| \in H^1(\Gamma_C)$,

$$\begin{aligned} |j(\underline{v}^*) - j_N(\underline{u}_N)| &\le \sum_{E \in \mathcal{E}_{c,N}} g_E \left| \int_E |\underline{v}^*_t| ds - \int_E i_{E,q_{N,E}} \left(|\underline{v}^*_t| \right) ds \right| \\ &\le \text{meas}\, (\Gamma_c)^{1/2} \|g\|_{L^\infty(\Gamma_c)} \left\| |\underline{v}^*_t| - i_{E,q_{N,E}} \left(|\underline{v}^*_t| \right) \right\|_{L^2(\Gamma_c)} \\ &\le c\, h\, p^{-1} \|\underline{v}^*_t\|_{H^1(\Gamma_c)}. \end{aligned}$$

To show the analogue estimate with respect to the L^2 norm we estimate separately:

$$j(\underline{v}^*_t) = \sum_{E \in \mathcal{E}_{c,N}} g_E \int_E |\underline{v}^*_t| ds \le \text{meas}\, (\Gamma_c)^{1/2} \|g\|_{L^\infty(\Gamma_c)} \|\underline{v}^*_t\|_{L^2(\Gamma_c)}$$

and with Cauchy–Schwarz inequality

$$\begin{aligned} j_N(\underline{v}^*_t) = j_N(\underline{u}_{N,t}) &= \sum_{E \in \mathcal{E}_{c,N}} g_E \sum_{j=0}^{q_{N,E}} \omega_j^{q_{N,E}+1} \left| \underline{u}_{N,t} \circ F_E(\xi_j^{q_{N,E}+1}) \right| \\ &\le \|g\|_{L^\infty(\Gamma_C)} \left[\sum_E \sum_{j=0}^{q_{N,E}} \omega_j^{q_{N,E}+1} \right]^{1/2} \left[\sum_E \sum_{j=0}^{q_{N,E}} \omega_j^{q_{N,E}+1} \{\underline{u}_{N,t} \circ F_E(\xi_j^{q_{N,E}+1})\}^2 \right]^{1/2} \\ &\le 3\|g\|_{L^\infty(\Gamma_c)}\, \text{meas}\, (\Gamma_c)^{1/2}\, \|\underline{u}_N\|_{(L^2(\Gamma_c))^2}, \end{aligned}$$

where we use the L^2 stability of Gauss–Lobatto quadrature, see [33, Lemma 2.2]. Therefore by real interpolation we arrive at

$$|j(\underline{v}^*) - j_N(\underline{u}_N)| \le c\, h^{1/2} p^{-1/2} \|\underline{v}^*_t\|_{H^{1/2}(\Gamma_c)},$$

hence for the consistency error,

$$\inf\left\{\ldots \mid \underline{v} \in \mathcal{K}\right\} \le c_{II}(\underline{u}, f^*, d, g)\, h^{1/2} p^{-1/2}. \tag{19}$$

Altogether, (14), (17) and (19) yield the claimed error estimate. □

Thus for our more general problem (π) we arrive at the same convergence order of $h^{1/4}p^{-1/4}$ as Maischak and Stephan in [12]. Under the weaker regularity assumption $\underline{u} \in H^{3/2}(\Omega)$, Dörsek and Melenk proved for the pure frictional problem a weaker convergence order with an additional log term. Both error estimates are suboptimal because of the consideration of the consistency error in the nonconforming approximation scheme and because of the regularity threshold in unilateral problems [30].

5. Some concluding remarks

It is noteworthy that although the present paper is only concerned with the *primal hp*-finite element method, *duality methods* come into play in the proof of Theorem 3.1. — Also the proof shows the usefulness of *Bernstein operators* — a well-known topic in classic approximation theory — in the numerical analysis of unilateral nonsmooth problems. In this connection let us refer to the recent paper of Ainsworth, Andriamaro, and Davydov [34] which employs *Bernstein–Bézier polynomials* to derive optimal assembly procedures for the finite element method. — Finally let us point out the decisive rôle of the *high integration order* and of the *positivity of the quadrature weights* in the Gauss–Lobatto quadrature. This extends to higher dimensional rectangles via tensor products in a straightforward way. Therefore our results pertain to quadrilateral meshes. It may be worthwhile to invent such an effective quadrature rule on a triangle. However recently, Helenbrook [35] has shown that an integration rule similar to Gauss–Lobatto quadrature does not exist on triangles and that a possible remedy is the use of a nonnodal basis. Moreover, Yuan Xu [36] has strengthened these negative results by giving a lower bound for the number of nodes of such quadrature rules and advocates a new method of constructing Lobatto-type cubature rules on triangles. Thus the efficient treatment of unilateral nonsmooth problems by the *hp*-FEM on *triangular meshes* seems to be wide open.

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