



Regularization for a fractional sideways heat equation



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ARTICLE INFO

Article history:

Received 28 April 2012

Received in revised form 29 March 2013

Keywords:

Fractional diffusion equation

Ill-posedness

Regularization

Optimal error estimate

ABSTRACT

We consider a sideways problem for a fractional heat equation which is highly ill-posed. This study gives a new dynamic method for choosing a regularization parameter. By using the spectral methods, some convergence rates on the temperature and heat flow are given. For illustration, several numerical examples are constructed to show the feasibility and efficiency of the proposed methods. Comparing with the traditional stationary methods for choosing regularization parameter, the proposal methods are more accurate and effective.

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1. Introduction

In recent years, fractional calculus and derivatives have encountered much success in many fields of science, for instance, mechanical engineering [1], viscoelasticity [2], control theory, Lévy motion [3], electron transportation [4], dissipation [5], heat conduction [6–8] and high-frequency financial data [9]. A number of experiments in early stages show that to describe practical phenomena, for example Brownian motion, the fractional derivative is more accurate than the traditional integer derivative. Fractional derivatives also have been found to be quite flexible in describing viscoelastic behaviors. Hence, in the last decades, the study of fractional diffusion equations has attracted intense attention.

If the initial concentration distribution and boundary conditions are given, a complete recovery of the unknown solution is attainable from solving a well-posed forward problem. However, in some practical problems, the boundary data can only be measured on a portion of the boundary or some points in the solution domain. This leads to an ill-posed problem of the fractional heat equation, which means the solution does not depend continuously on the given known conditions, see [10]. In this paper, we investigate a sideways problem for the fractional diffusion equation in a semi-unbounded domain, which is a serious ill-posed problem. This kind of ill-posed problem is important in many branches of engineering sciences [11]. This is usually referred to as the ill-posed backward determination problem, which is in nature “unstable” because the unknown solution and its derivatives have to be determined from indirect observable data which contain a measurement error. The major difficulty in establishing any numerical algorithm for approximating the solution is due to the severe ill-posedness of the problem.

Due to the difficulty of the fractional derivative and the ill-posedness, to best of the the authors' knowledge, the results on inverse problems for fractional diffusion equations are very few. The uniqueness of an inverse problem for a one-dimensional fractional diffusion equation was given by Cheng et al. [12]. Zheng and Wei [13] gave a regularization method for a Cauchy problem of the time fractional advection–dispersion equation in a space-bounded domain. Numerical results by using difference methods were given by Bondarenko [14] and Murio [15]. In this paper, we focus on the sideways problem for the fractional diffusion equation. This is a problem defined in a semi-unbounded domain, in which the Fourier transform is a powerful tool to analyze. Most of the current researches on the ill-posed problem are based on solving the

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corresponding direct problem iteratively. The iterations and the initial guess value are important and sensitive in these iterative computational methods. By using a dynamic spectral regularization method, in this paper, a one-stage direct computational method for this ill-posed problem is obtained.

In a theoretical aspect, order optimal error estimations of temperature and heat flow are proved in Sections 3 and 4. A logarithmic estimation on the inaccessible boundary is also obtained. In particular, by using a new dynamic method for choosing the regularization parameter, which we call the dynamic spectral regularization method, an optimal error estimation on temperature is obtained. Comparing with most of the static spectral regularization method, the regularization parameters are changing with respect to all points in space. The proposed dynamic method could give higher accuracy not only in theory, but also in numerical experiments. Numerical examples show the comparison between static and dynamic spectral regularization methods and support the theoretical results.

The paper is organized as follows. First we will give an analysis of the ill-posedness of the sideways problem of the fractional order diffusion equation in Section 2, then some regularization solutions are constructed and the error estimates are proved. In the last section, we give some numerical examples to show the validity of the regularization method.

2. Ill-posedness of the sideways problem

In several engineering contexts, it is sometimes necessary to estimate the surface temperature or heat flux in a body from a measured temperature history at a fixed location inside a body. This is the so-called inverse heat conduction problem (abbr. IHCP). For the numerical methods, we can refer to [16–18] and the references therein.

Let us consider the following sideways problem for the fractional heat equation:

$$\begin{aligned} \frac{\partial^\beta u}{\partial t^\beta} - u_{xx} &= 0, \quad x > 0, \quad t > 0, \\ u(x, 0) &= 0, \quad x > 0, \\ u(1, t) &= g(t), \quad t > 0, \\ u(x, t)|_{x \rightarrow \infty} &\text{bounded}, \end{aligned} \quad (2.1)$$

where the time fractional derivative $\frac{\partial^\beta u}{\partial t^\beta}$ is the Caputo fractional derivative of order β ($0 < \beta \leq 1$) defined in [19]

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t-s)^\beta}, \quad 0 < \beta < 1, \quad (2.2)$$

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial u(x, t)}{\partial t}, \quad \beta = 1. \quad (2.3)$$

We need to seek the solutions $u(x, \cdot) \in L^2(\mathbb{R})$ (or $u_x(x, \cdot) \in L^2(\mathbb{R})$) from the given data $g(t) \in L^2(\mathbb{R})$. Physically, $g(\cdot)$ can only be measured, there should be measurement errors, and we would actually have some data functions $g_\delta(\cdot) \in L^2(\mathbb{R})$, for which

$$\|g_\delta - g\| \leq \delta, \quad (2.4)$$

where the constant $\delta > 0$ represents a bound on the measurement error, $\|\cdot\|$ denotes the L^2 -norm, and there exists a constant $E > 0$, such that the following a-priori bound exists for the problem

$$\|u(0, \cdot)\| \leq E. \quad (2.5)$$

Throughout this paper, we extend all the functions to the whole line $-\infty < t < \infty$ by setting the functions to be zero for $t < 0$ if necessary. Let

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt \quad (2.6)$$

be the Fourier transform of the function $f(t) \in L^2(\mathbb{R})$. The solution of (2.1) can be formulated in the frequency domain:

$$\hat{u}(x, \xi) = e^{(1-x)(i\xi)^{\frac{\beta}{2}}} \hat{g}(\xi), \quad (2.7)$$

where

$$\eta := (i\xi)^{\frac{\beta}{2}} = |\xi|^{\frac{\beta}{2}} \left(\cos\left(\frac{\beta\pi}{4}\right) + i \operatorname{sign}(\xi) \sin\left(\frac{\beta\pi}{4}\right) \right). \quad (2.8)$$

Denote the real part a and imaginary part b of η as

$$a := \Re(\eta); \quad b := \Im(\eta). \quad (2.9)$$

We notice that for a fixed $0 < x \leq 1$, $e^{(1-x)a}$ is unbounded as $|\xi| \rightarrow \infty$. Hence we call $e^{(1-x)(i\xi)^{\frac{\beta}{2}}}$ the amplified factor of the given data. Here we want to seek a solution $u(x, \cdot) \in L^2(\mathbb{R})$, this implies that the exact data $\hat{g}(\xi)$ must decay rapidly as $|\xi| \rightarrow \infty$. But in practice, we can only get the noisy data $\hat{g}_\delta(\cdot) \in L^2(\mathbb{R})$. Thus for the noisy data $\hat{g}_\delta(\cdot)$, we cannot obtain a meaningful solution. It shows that the ill-posedness of problem (2.1) is caused by the high frequency of \hat{g} . A natural idea is to eliminate all high frequencies from the solution. This idea was first proposed in [20] where a classical sideways heat equation is solved by the cut-off method. Following the idea of the cut-off method, we can get the regularization solution in the frequency domain

$$\hat{u}_{\xi_{\max}}(x, \xi) = e^{(1-x)(i\xi)^{\frac{\beta}{2}}} \hat{g}(\xi) \chi_{\xi_{\max}}(\xi), \quad (2.10)$$

where $\chi_{\xi_{\max}}(\xi)$ is a characteristic function satisfying

$$\chi_{\xi_{\max}}(\xi) = \begin{cases} 1, & |\xi| \leq \xi_{\max}, \\ 0, & |\xi| > \xi_{\max}, \end{cases}$$

where the ξ_{\max} is a regularization parameter. However, in this study, we will not consider this method for the reconstruction of temperature. Instead, we consider that the ξ_{\max} depends on the reconstruction location x , i.e., the regularization parameter ξ_{\max} is dynamic. We will deal with the dynamic regularization method in the next section.

3. Temperature: dynamic spectral regularization and error estimate

By previous analysis, the cause of the ill-posedness of problem (2.1) lies in the amplified factor of data. To stabilize the problem, a natural idea is to construct a function that approaches the amplified factor. The constructed function must satisfy the following properties:

- (I) The modulus of constructed function should be less than or equal to that of the amplified factor.
- (II) We need to preserve the information of low-frequency components and eliminate the information of high-frequency components as well.

By introducing a parameter which serves as the regularization parameter, we can construct a series of methods:

Method 1:

$$\hat{u}_\alpha(x, \xi) = \begin{cases} e^{(1-x)(a+bi)} \hat{g}_\delta(\xi), & e^{-a(1-x)} \geq \sqrt{\alpha}, \\ \frac{e^{-2a(1-x)}}{\alpha} e^{(1-x)(a+bi)} \hat{g}_\delta(\xi), & e^{-a(1-x)} < \sqrt{\alpha}. \end{cases} \quad (3.1)$$

Method 2:

$$\hat{v}_\alpha(x, \xi) = \begin{cases} e^{(1-x)(a+bi)} \hat{g}_\delta(\xi), & e^{-a(1-x)} \geq \sqrt{\alpha}, \\ \frac{e^{-a(1-x)}}{\sqrt{\alpha}} e^{(1-x)(a+bi)} \hat{g}_\delta(\xi), & e^{-a(1-x)} < \sqrt{\alpha}. \end{cases} \quad (3.2)$$

Method 3:

$$\hat{w}_\alpha(x, \xi) = \begin{cases} e^{(1-x)(a+bi)} \hat{g}_\delta(\xi), & e^{-a(1-x)} \geq \sqrt{\alpha}, \\ \frac{e^{-\frac{a}{2}(1-x)}}{\alpha^{\frac{1}{4}}} e^{(1-x)(a+bi)} \hat{g}_\delta(\xi), & e^{-a(1-x)} < \sqrt{\alpha}. \end{cases} \quad (3.3)$$

Generally,

$$\hat{u}_\alpha(x, \xi) = \begin{cases} e^{(1-x)(a+bi)} \hat{g}_\delta(\xi), & e^{-a(1-x)} \geq \sqrt{\alpha}, \\ \frac{e^{-\nu a(1-x)}}{\sqrt{\alpha}^\nu} e^{(1-x)(a+bi)} \hat{g}_\delta(\xi), & e^{-a(1-x)} < \sqrt{\alpha}, \end{cases} \quad (3.4)$$

where $\nu > 0$ is a real number.

Obviously, if $\alpha \rightarrow 0$ as $\delta \rightarrow 0$, the regularization solutions approach the exact solution. In general, the regularization parameter α does not depend on the location x . Here we will cancel this restriction and instead select α dynamically. We only deal with Methods 1 and 2 because subsequently we find that Method 2 is an “optimal” regularization method. Here the “optimal” means that the error estimate on Method 2 is “optimal” in the sense of the worst-case error. Please refer to Remark 3.8.

Lemma 3.1. For $0 < x < 1$, suppose that $u_\alpha(x, t)$ is the solution whose Fourier transform is given by Method 1 with exact data $g(t)$, $u_\alpha^\delta(x, t)$ is the solution whose Fourier transform is given by Method 1 with noisy data $g_\delta(t)$, (2.4) holds, we have

$$\|u_\alpha^\delta(x, \cdot) - u_\alpha(x, \cdot)\| \leq \alpha^{-\frac{1}{2}} \delta. \quad (3.5)$$

Proof. By Parseval's equality and (3.1), then

$$\begin{aligned} \|u_\alpha(x, \cdot) - u_\alpha^\delta(x, \cdot)\| &= \|\hat{u}_\alpha(x, \cdot) - \hat{u}_\alpha^\delta(x, \cdot)\| \\ &= \left\| \min \left\{ 1, \frac{e^{-2a(1-x)}}{\alpha} \right\} e^{(1-x)(a+bi)} (\hat{g} - \hat{g}_\delta) \right\| \leq \alpha^{-\frac{1}{2}} \delta. \quad \square \end{aligned}$$

Lemma 3.2. For $0 < x < 1$, suppose that $u(x, t)$ is the exact solution with exact data $g(t)$, $u_\alpha(x, t)$ is the solution whose Fourier transform is given by Method 1 with exact data $g(t)$, (2.5) holds, we have

$$\|u_\alpha(x, \cdot) - u(x, \cdot)\| \leq \frac{2(1-x)}{2-x} \left(\frac{x}{2-x} \right)^{\frac{x}{2(1-x)}} \alpha^{\frac{x}{2(1-x)}} E. \quad (3.6)$$

Proof. By Parseval's equality and using $\hat{g}(\xi) = e^{-(a+bi)} \hat{u}(0, \xi)$, then

$$\begin{aligned} \|u_\alpha(x, \cdot) - u(x, \cdot)\| &= \|\hat{u}_\alpha(x, \cdot) - \hat{u}(x, \cdot)\| \\ &= \left\| \min \left\{ 1, \frac{e^{-2a(1-x)}}{\alpha} \right\} e^{(1-x)(a+bi)} \hat{g} - e^{(1-x)(a+bi)} \hat{g} \right\| \\ &= \left\| \left(1 - \min \left\{ 1, \frac{e^{-2a(1-x)}}{\alpha} \right\} \right) e^{(1-x)(a+bi)} \hat{g} \right\| \\ &= \left\| \left(1 - \min \left\{ 1, \frac{e^{-2a(1-x)}}{\alpha} \right\} \right) e^{(1-x)(a+bi)} e^{-(a+bi)} \hat{u}(0, \cdot) \right\| \\ &\leq \sup_{e^{-2a(1-x)} \leq \alpha} \left(1 - \frac{e^{-2a(1-x)}}{\alpha} \right) e^{-ax} \|\hat{u}(0, \cdot)\|. \end{aligned}$$

Now we seek the maximum of $F(a) := (1 - \frac{e^{-2a(1-x)}}{\alpha}) e^{-ax}$ under the restriction $e^{-2a(1-x)} \leq \alpha$. It is easy to find the zero point a^* of $F'(a) = 0$, i.e., a^* satisfies

$$\frac{x\alpha}{2-x} = e^{-2a^*(1-x)}, \quad (3.7)$$

and a^* maximizes the function $F(a)$, hence $F(a) \leq F(a^*)$. Finally,

$$\|u_\alpha(x, \cdot) - u(x, \cdot)\| \leq F(a^*) E = \left(1 - \frac{x}{2-x} \right) \left(\frac{x}{2-x} \alpha \right)^{\frac{x}{2(1-x)}} E. \quad \square$$

Theorem 3.3. For $0 < x < 1$, suppose that $u(x, t)$ is the exact solution with exact data $g(t)$, $u_\alpha^\delta(x, t)$ is the solution whose Fourier transform is given by Method 1 with noisy data $g_\delta(t)$, (2.4) and (2.5) hold. If α is selected dynamically

$$\alpha(x) = 2^{2x-2} \left(\frac{x}{2-x} \right)^{x-2} \left(\frac{\delta}{E} \right)^{2(1-x)}, \quad (3.8)$$

then we have

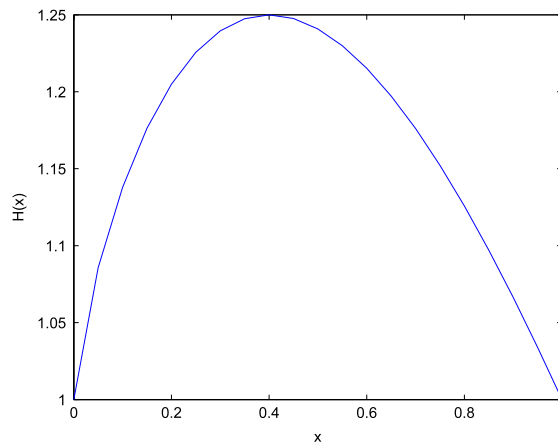
$$\|u_\alpha^\delta(x, \cdot) - u(x, \cdot)\| \leq \frac{2^{1-x}}{2-x} \left(\frac{x}{2-x} \right)^{-\frac{x}{2}} \delta^x E^{1-x}. \quad (3.9)$$

Proof. According to Lemmas 3.1 and 3.2, we have

$$\|u_\alpha^\delta(x, \cdot) - u(x, \cdot)\| \leq \alpha^{-\frac{1}{2}} \delta + \frac{2(1-x)}{2-x} \left(\frac{x}{2-x} \right)^{\frac{x}{2(1-x)}} \alpha^{\frac{x}{2(1-x)}} E.$$

Minimize the right-hand side of the above inequality with respect to α , we can get (3.8). Thus, (3.9) holds. \square

Remark 3.4. The coefficient function in (3.9) $H(x) := \frac{2^{1-x}}{2-x} \left(\frac{x}{2-x} \right)^{-\frac{x}{2}}$ is plotted as Fig. 1.

Fig. 1. $H(x)$.

Analogously, for Method 2, we find that it is just the optimal filtering method in [21].

Lemma 3.5. For $0 < x < 1$, suppose that $v_\alpha(x, t)$ is the solution whose Fourier transform is given by Method 2 with exact data $g(t)$, $v_\alpha^\delta(x, t)$ is the solution whose Fourier transform is given by Method 2 with noisy data $g_\delta(t)$, (2.4) holds, we have

$$\|v_\alpha^\delta(x, \cdot) - v_\alpha(x, \cdot)\| \leq \alpha^{-\frac{1}{2}} \delta. \quad (3.10)$$

Proof. By Parseval's equality and (3.2), then

$$\begin{aligned} \|v_\alpha(x, \cdot) - v_\alpha^\delta(x, \cdot)\| &= \|\hat{v}_\alpha(x, \cdot) - \hat{v}_\alpha^\delta(x, \cdot)\| \\ &= \left\| \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\alpha}} \right\} e^{(1-x)(a+bi)} (\hat{g} - \hat{g}_\delta) \right\| \leq \alpha^{-\frac{1}{2}} \delta. \quad \square \end{aligned}$$

Lemma 3.6. For $0 < x < 1$, suppose that $u(x, t)$ is the exact solution with exact data $g(t)$, $v_\alpha(x, t)$ is the solution whose Fourier transform is given by Method 2 with exact data $g(t)$, (2.5) holds, we have

$$\|v_\alpha(x, \cdot) - u(x, \cdot)\| \leq (1-x)x^{\frac{x}{1-x}} \alpha^{\frac{x}{2(1-x)}} E. \quad (3.11)$$

Proof. By Parseval's equality and using $\hat{g}(\xi) = e^{-(a+bi)} \hat{u}(0, \xi)$, then

$$\begin{aligned} \|v_\alpha(x, \cdot) - u(x, \cdot)\| &= \|\hat{v}_\alpha(x, \cdot) - \hat{u}(x, \cdot)\| \\ &= \left\| \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\alpha}} \right\} e^{(1-x)(a+bi)} \hat{g} - e^{(1-x)(a+bi)} \hat{g} \right\| \\ &= \left\| \left(1 - \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\alpha}} \right\} \right) e^{(1-x)(a+bi)} \hat{g} \right\| \\ &= \left\| \left(1 - \min \left\{ 1, \frac{e^{-a(1-x)}}{\sqrt{\alpha}} \right\} \right) e^{(1-x)(a+bi)} e^{-(a+bi)} \hat{u}(0, \cdot) \right\| \\ &\leq \sup_{e^{-2a(1-x)} \leq \alpha} \left(1 - \frac{e^{-a(1-x)}}{\sqrt{\alpha}} \right) e^{-ax} \|\hat{u}(0, \cdot)\|. \end{aligned}$$

Now we seek the maximum of $F(a) := (1 - \frac{e^{-a(1-x)}}{\sqrt{\alpha}}) e^{-ax}$ under the restriction $e^{-2a(1-x)} \leq \alpha$. It is easy to find the zero point a^* of $F'(a) = 0$, i.e., a^* satisfies

$$x\sqrt{\alpha} = e^{-a^*(1-x)}, \quad (3.12)$$

and a^* maximize the function $F(a)$, hence $F(a) \leq F(a^*)$. Finally,

$$\|v_\alpha(x, \cdot) - u(x, \cdot)\| \leq F(a^*)E = (1-x)(x\sqrt{\alpha})^{\frac{x}{1-x}} E. \quad \square$$

Theorem 3.7. For $0 < x < 1$, suppose that $u(x, t)$ is the exact solution with exact data $g(t)$, $v_\alpha^\delta(x, t)$ is the solution whose Fourier transform is given by Method 2 with noisy data $g_\delta(t)$, (2.4) and (2.5) hold. If α is selected dynamically

$$\alpha(x) = x^{-2} \left(\frac{\delta}{E} \right)^{2(1-x)}, \quad (3.13)$$

then we have

$$\|v_\alpha^\delta(x, \cdot) - u(x, \cdot)\| \leq \delta^x E^{1-x}. \quad (3.14)$$

Proof. According to Lemmas 3.5 and 3.6, we have

$$\|v_\alpha^\delta(x, \cdot) - u(x, \cdot)\| \leq \alpha^{-\frac{1}{2}} \delta + (1-x)x^{\frac{x}{1-x}} \alpha^{\frac{x}{2(1-x)}} E.$$

Minimize the right-hand side of the above inequality with respect to α , we can get (3.13). Thus, (3.14) holds. \square

Remark 3.8. Following the same idea from [22], we have the following conclusion: the “optimal” error bound of the problem for determining the temperature is given by $\delta^x E^{1-x}$. That is to say that the best possible error estimate for a regularization R_α^δ has the form of $\|R_\alpha^\delta u - u\| \leq \delta^x E^{1-x}$.

According the theory of regularization, any regularization method R_α^δ for the problem with noisy data is called

- (i) optimal on the a-priori condition (2.5) if the error estimate $\|R_\alpha^\delta u - u\| \leq \delta^x E^{1-x}$ holds;
- (ii) order optimal on the a-priori condition (2.5) if the error estimate $\|R_\alpha^\delta u - u\| \leq c \delta^x E^{1-x}$ holds with $c \geq 1$.

Therefore we know that the error estimate (3.14) is optimal, i.e., Method 2 is an “optimal” regularization method.

We find that the error estimate (3.14) is not valid for the location at $x = 0$. This is common in ill-posed problems. If a stronger a-priori condition is added

$$\|u(0, \cdot)\|_p \leq M, \quad p > 0, \quad (3.15)$$

where $\|\cdot\|_p$ denotes the norm of Sobolev space $H^p(\mathbb{R})$, the problem on the convergence at $x = 0$ can be resolved. For clarity, we rewrite the solution at $x = 0$ for Method 2:

$$\hat{v}_\alpha(0, \xi) = \begin{cases} e^{(a+bi)\hat{g}_\delta(\xi)}, & e^{-a} \geq \sqrt{\alpha}, \\ \frac{e^{-a}}{\sqrt{\alpha}} e^{a+bi\hat{g}_\delta(\xi)}, & e^{-a} < \sqrt{\alpha}. \end{cases} \quad (3.16)$$

Lemma 3.9. For $x = 0$, suppose that $v_\alpha(0, t)$ is the solution whose Fourier transform is given by (3.16) with exact data $g(t)$, $v_\alpha^\delta(0, t)$ is the solution whose Fourier transform is given by (3.16) with noisy data $g_\delta(t)$, (2.4) holds, we have

$$\|v_\alpha^\delta(0, \cdot) - v_\alpha(0, \cdot)\| \leq \alpha^{-\frac{1}{2}} \delta. \quad (3.17)$$

Proof. By Parseval's equality and (3.16), then

$$\begin{aligned} \|v_\alpha(0, \cdot) - v_\alpha^\delta(0, \cdot)\| &= \|\hat{v}_\alpha(0, \cdot) - \hat{v}_\alpha^\delta(0, \cdot)\| \\ &= \left\| \min \left\{ 1, \frac{e^{-a}}{\sqrt{\alpha}} \right\} e^{a+bi} (\hat{g} - \hat{g}_\delta) \right\| \leq \alpha^{-\frac{1}{2}} \delta. \quad \square \end{aligned}$$

Lemma 3.10. For $x = 0$, suppose that $u(0, t)$ is the exact solution with exact data $g(t)$, $v_\alpha(0, t)$ is the solution whose Fourier transform is given by (3.16) with exact data $g(t)$, (3.15) holds, we have

$$\|v_\alpha(0, \cdot) - u(0, \cdot)\| \leq (1 - C_*) \left[\ln \frac{\cos\left(\frac{\beta\pi}{4}\right)}{\sqrt{\alpha} C_*} \right]^{-\frac{2p}{\beta}} M, \quad (3.18)$$

where $C_* < 1$ is a constant.

Proof. By Parseval's equality and using $\hat{g}(\xi) = e^{-(a+bi)} \hat{u}(0, \xi)$, then

$$\begin{aligned} \|v_\alpha(0, \cdot) - u(0, \cdot)\| &= \|\hat{v}_\alpha(0, \cdot) - \hat{u}(0, \cdot)\| \\ &= \left\| \min \left\{ 1, \frac{e^{-a}}{\sqrt{\alpha}} \right\} e^{a+bi} \hat{g} - e^{a+bi} \hat{g} \right\| \\ &= \left\| \left(1 - \min \left\{ 1, \frac{e^{-a}}{\sqrt{\alpha}} \right\} \right) e^{a+bi} \hat{g} \right\| \end{aligned}$$

$$\begin{aligned}
&= \left\| \left(1 - \min \left\{ 1, \frac{e^{-a}}{\sqrt{\alpha}} \right\} \right) (1 + \xi^2)^{-p/2} (1 + \xi^2)^{p/2} \hat{u}(0, \cdot) \right\| \\
&\leq \sup_{e^{-a} \leq \sqrt{\alpha}} \left(1 - \frac{e^{-a}}{\sqrt{\alpha}} \right) (1 + \xi^2)^{-p/2} \|\hat{u}(0, \cdot)\|_p \\
&\leq \sup_{e^{-a} \leq \sqrt{\alpha}} \left(1 - \frac{e^{-a}}{\sqrt{\alpha}} \right) |\xi|^{-p} \|\hat{u}(0, \cdot)\|_p.
\end{aligned}$$

Noting that $a = |\xi|^{\frac{\beta}{2}} \cos\left(\frac{\beta\pi}{4}\right)$, now we seek the maximum of $F(\xi) := \frac{\sqrt{\alpha} - e^{-|\xi|^{\frac{\beta}{2}} \cos(\frac{\beta\pi}{4})}}{\sqrt{\alpha} |\xi|^p}$ under the restriction $e^{-|\xi|^{\frac{\beta}{2}} \cos(\frac{\beta\pi}{4})} \leq \sqrt{\alpha}$. It is easy to find the zero point ξ_* of $F'(\xi) = 0$, i.e., ξ_* satisfies

$$\frac{\sqrt{\alpha} p |\xi_*|^{p-1}}{p |\xi_*|^{p-1} + \beta \frac{1}{2\sqrt{\alpha}} |\xi_*|^{\beta/2+p-1} \cos(\pi\beta/4)} = e^{-|\xi_*|^{\frac{\beta}{2}} \cos(\frac{\beta\pi}{4})}. \quad (3.19)$$

Denote $C_* = \frac{p |\xi_*|^{p-1}}{p |\xi_*|^{p-1} + \beta \frac{1}{2\sqrt{\alpha}} |\xi_*|^{\beta/2+p-1} \cos(\pi\beta/4)}$, obviously, $C_* < 1$, and ξ_* maximizes the function $F(\xi)$, hence $F(\xi) \leq F(\xi_*)$. Finally,

$$\|v_\alpha(0, \cdot) - u(0, \cdot)\| \leq F(\xi_*) E = (1 - C_*) \left[\ln \frac{\cos(\frac{\beta\pi}{4})}{\sqrt{\alpha} C_*} \right]^{-\frac{2p}{\beta}} M. \quad \square$$

Theorem 3.11. For $x = 0$, suppose that $u(x, t)$ is the exact solution with exact data $g(t)$, $v_\alpha^\delta(x, t)$ is the solution whose Fourier transform is given by Method 2 with noisy data $g_\delta(t)$, (2.4) and (2.5) hold. If α is selected

$$\alpha = \left(\frac{\delta}{M} \right)^2, \quad (3.20)$$

then we have for $\delta \rightarrow 0$

$$\|v_\alpha^\delta(0, \cdot) - u(0, \cdot)\| \leq O \left(\left[\ln \left(\frac{M}{\delta} \right) \right]^{-\frac{2p}{\beta}} \right). \quad (3.21)$$

Proof. According to Lemmas 3.9 and 3.10, we have

$$\|u_\alpha^\delta(0, \cdot) - u(0, \cdot)\| \leq \alpha^{-\frac{1}{2}} \delta + (1 - C_*) \left[\ln \frac{\cos(\frac{\beta\pi}{4})}{\sqrt{\alpha} C_*} \right]^{-\frac{2p}{\beta}} M.$$

According to the selection of α , (3.21) holds. \square

4. Heat flux: cut-off regularization and error estimate

In some engineering contexts, sometimes it is necessary to estimate the heat flux at the inaccessible surface. It is a more difficult problem than the problem of estimating surface temperature. The cut-off method in the frequency domain is the simplest method for ill-posed problems and it is a rather effective and convenient method for solving ill-posed problems defined on a strip domain [20].

According to (2.7), the heat flux solution of (2.1) can be formulated in the frequency domain:

$$\hat{u}_x(x, \xi) = -(i\xi)^{\frac{\beta}{2}} e^{(1-x)(i\xi)^{\frac{\beta}{2}}} \hat{g}(\xi). \quad (4.1)$$

We can construct regularization methods for problem (2.1) with noisy data:

$$\hat{v}_x^{\xi_{\max}, \delta}(x, \xi) = -(a + bi) e^{(1-x)(a+bi)} \hat{g}_\delta(\xi) \chi_{\max}. \quad (4.2)$$

Assume that there exists a-priori condition

$$\|u(0, \cdot)\|_p \leq E, \quad p \geq 0, \quad (4.3)$$

Lemma 4.1. For $0 < x < 1$, suppose that $v_x^{\xi_{\max}}(x, t)$ is the solution by cut-off method with exact data $g(t)$, $v_x^{\xi_{\max}, \delta}(x, t)$ is the solution by cut-off method with noisy data $g_\delta(t)$, (2.4) holds, we have

$$\|v_x^{\xi_{\max}, \delta}(x, \cdot) - v_x^{\xi_{\max}}(x, \cdot)\| \leq \sup_{|\xi| \leq \xi_{\max}} (|\xi|^{\beta/2} e^{(1-x)|\xi|^{\beta/2}}) \delta. \quad (4.4)$$

Proof. By Parseval's equality and (4.2), then

$$\begin{aligned} \|v_x^{\xi_{\max}, \delta}(x, \cdot) - v_x^{\xi_{\max}}(x, \cdot)\| &= \|\hat{v}_x^{\xi_{\max}, \delta}(x, \cdot) - \hat{v}_x^{\xi_{\max}}(x, \cdot)\| \\ &= \left(\int_{|\xi| \leq \xi_{\max}} |(a+bi)e^{(1-x)(a+bi)}|^2 |g - g_\delta|^2 d\xi \right)^{1/2} \\ &\leq \sup_{|\xi| \leq \xi_{\max}} \left(|\xi|^{\beta/2} e^{(1-x)|\xi|^{\beta/2}} \right)^{\frac{\beta}{2} \cos(\beta\pi/4)} \delta \leq \sup_{|\xi| \leq \xi_{\max}} (|\xi|^{\beta/2} e^{(1-x)|\xi|^{\beta/2}}) \delta. \quad \square \end{aligned}$$

Lemma 4.2. For $0 < x < 1$, suppose that $u_x(x, t)$ is the exact solution with exact data $g(t)$, $v_x^{\xi_{\max}}(x, t)$ is the solution by cut-off method with exact data $g(t)$, (4.3) holds, we have

$$\|v_x^{\xi_{\max}}(x, \cdot) - u_x(x, \cdot)\| \leq \sup_{|\xi| > \xi_{\max}} \left(|\xi|^{\frac{\beta}{2}-p} e^{-\frac{x}{\sqrt{2}}|\xi|^{\beta/2}} \right) E. \quad (4.5)$$

Proof. By Parseval's equality and using $\hat{g}(\xi) = e^{-(a+bi)} \hat{u}(0, \xi)$, then

$$\begin{aligned} \|v_x^{\xi_{\max}}(x, \cdot) - u_x(x, \cdot)\| &= \|\hat{v}_x^{\xi_{\max}}(x, \cdot) - \hat{u}_x(x, \cdot)\| \\ &= \left(\int_{|\xi| > \xi_{\max}} |(a+bi)e^{(1-x)(a+bi)} g(\xi)|^2 d\xi \right)^{1/2} \\ &= \left(\int_{|\xi| > \xi_{\max}} |(a+bi)e^{(1-x)(a+bi)} e^{-(a+bi)} f(\xi)|^2 d\xi \right)^{1/2} \\ &= \left(\int_{|\xi| > \xi_{\max}} |(a+bi)e^{-x(a+bi)} (1+\xi^2)^{-p/2} (1+\xi^2)^{p/2} f(\xi)|^2 d\xi \right)^{1/2} \\ &\leq \sup_{|\xi| > \xi_{\max}} \left(|\xi|^{\frac{\beta}{2}-p} e^{-\frac{x}{\sqrt{2}}|\xi|^{\beta/2}} \right) E. \end{aligned}$$

Now set

$$\begin{aligned} A(s) &= s^{\beta/2} e^{(1-x)s^{\beta/2}} \\ B(s) &= s^{\beta/2-p} e^{-\frac{x}{\sqrt{2}}s^{\beta/2}}. \end{aligned}$$

Now we divide into two cases for p .

Case I. $0 \leq p < \beta/2$, $0 < x < 1$.

By an elementary calculation,

$$s_0 = \left(\frac{\sqrt{2}(\beta - 2p)}{x\beta} \right)^{\frac{2}{\beta}}$$

is the unique zero point of the first order derivative $B'(s)$, $B(s)$ increases for $s < s_0$ and decreases for $s > s_0$. Therefore,

$$\sup_{s > \xi_{\max}} B(s) = \begin{cases} s_0^{\beta/2-p} e^{-\frac{x}{\sqrt{2}}s_0^{\beta/2}} \leq s_0^{\beta/2-p} e^{-\frac{x}{\sqrt{2}}\xi_{\max}^{\beta/2}}, & \text{for } \xi_{\max} < s_0, \\ \xi_{\max}^{\beta/2-p} e^{-\frac{x}{\sqrt{2}}\xi_{\max}^{\beta/2}}, & \text{for } \xi_{\max} \geq s_0, \end{cases} \quad (4.6)$$

i.e.,

$$\sup_{s > \xi_{\max}} B(s) \leq \max \left\{ \left(\frac{\sqrt{2}(\beta - 2p)}{x\beta} \right)^{\frac{1-2p}{\beta}}, \xi_{\max}^{\beta/2-p} e^{-\frac{x}{\sqrt{2}}\xi_{\max}^{\beta/2}} \right\}. \quad (4.7)$$

If we choose $\xi_{\max} = (\ln \frac{E}{\delta})^{\frac{2}{\beta}}$, then

$$\sup_{s > \xi_{\max}} B(s) \leq \max \left\{ \left(\frac{\sqrt{2}(\beta - 2p)}{x\beta} \right)^{\frac{1-2p}{\beta}}, \left(\ln \frac{E}{\delta} \right)^{1-\frac{2p}{\beta}} \right\} \left(\frac{E}{\delta} \right)^{-\frac{x}{\sqrt{2}}}. \quad (4.8)$$

Case II. $p \geq \beta/2$, $0 < x < 1$. If we choose $\xi_{\max} = (\ln \frac{E}{\delta})^{\frac{2}{\beta}}$, then

$$\sup_{s > \xi_{\max}} B(s) \leq \left(\ln \frac{E}{\delta} \right)^{1-\frac{2p}{\beta}} \left(\frac{E}{\delta} \right)^{-\frac{x}{\sqrt{2}}}. \quad (4.9)$$

Obviously,

$$\sup_{s < \xi_{\max}} A(s) \leq \xi_{\max}^{\beta/2} e^{(1-x)\xi_{\max}^{\beta/2}} = \left(\ln \frac{E}{\delta} \right)^{1-\frac{2p}{\beta}} \left(\frac{E}{\delta} \right)^{1-x}. \quad (4.10)$$

Hence, we have

$$\begin{aligned} \|v_x^{\xi_{\max}, \delta}(x, \cdot) - u_x(x, \cdot)\| &\leq \sup_{s < \xi_{\max}} A(s)\delta + \sup_{s > \xi_{\max}} B(s)E \\ &\leq O\left(\left(\ln \frac{E}{\delta}\right)^{1-\frac{2p}{\beta}} \delta^x E^{1-x} + \left(\ln \frac{E}{\delta}\right)^{1-\frac{2p}{\beta}} \delta^{\frac{x}{\sqrt{2}}} E^{1-\frac{x}{\sqrt{2}}}\right), \quad \text{for } \delta \rightarrow 0. \quad \square \end{aligned}$$

Now summarize what we have:

Theorem 4.3. For $0 < x < 1$, suppose that $u_x(x, t)$ is the exact solution with exact data $g(t)$, $v_x^{\xi_{\max}, \delta}(x, t)$ is the solution by cut-off method with noisy data $g_\delta(t)$, (2.4) and (4.3) hold. If ξ_{\max} is selected

$$\xi_{\max} = \left(\ln \frac{E}{\delta} \right)^{\frac{2}{\beta}}, \quad (4.11)$$

then we have for $\delta \rightarrow 0$

$$\|v_x^{\alpha, \delta}(x, \cdot) - u_x(x, \cdot)\| \leq O\left(\left(\ln \frac{E}{\delta}\right)^{1-\frac{2p}{\beta}} \delta^x E^{1-x} + \left(\ln \frac{E}{\delta}\right)^{1-\frac{2p}{\beta}} \delta^{\frac{x}{\sqrt{2}}} E^{1-\frac{x}{\sqrt{2}}}\right). \quad (4.12)$$

If a stronger a-priori condition is added

$$\|u(0, \cdot)\|_p \leq M, \quad p > \beta/2, \quad (4.13)$$

where $\|\cdot\|_p$ denotes the norm of Sobolev space $H^p(\mathbb{R})$, then we can get the convergence at $x = 0$. We rewrite the solution at $x = 0$:

$$\hat{v}_x^{\xi_{\max}, \delta}(0, \xi) = -(a + bi)e^{a+bi} \hat{g}_\delta(\xi) \chi_{\max}. \quad (4.14)$$

Similarly, we have:

Theorem 4.4. For $x = 0$, suppose that $u_x(0, t)$ is the exact solution with exact data $g(t)$, $v_x^{\xi_{\max}, \delta}(0, t)$ is the solution by cut-off method with noisy data $g_\delta(t)$, (2.4) and (4.13), If ξ_{\max} is selected

$$\xi_{\max} = \left(\ln \left(\frac{M}{\delta} \left(\ln \frac{M}{\delta} \right)^{-\frac{2p}{\beta}} \right) \right)^{\frac{2}{\beta}}, \quad (4.15)$$

then we have for $\delta \rightarrow 0$

$$\|v_x^{\alpha, \delta}(0, \cdot) - u_x(0, \cdot)\| \leq O\left(\left[\ln \left(\frac{M}{\delta} \right)\right]^{1-\frac{2p}{\beta}}\right). \quad (4.16)$$

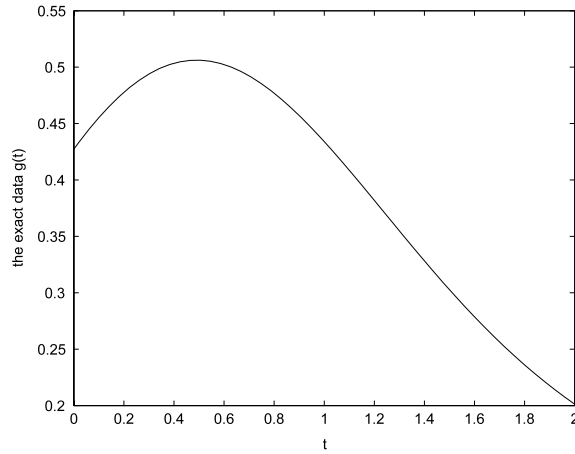


Fig. 2. Example 1: the exact input data $g(t)$.

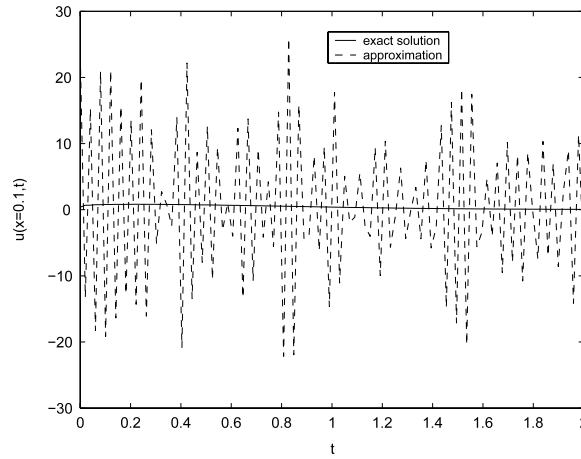


Fig. 3. The un-regularized solution and exact solution with $\delta = 3\%$.

5. Numerical examples

To demonstrate the effectiveness and stability of the proposed regularization methods, three examples with smooth, continuous and stepwise functions to be reconstructed, are presented in this section. To test the example, firstly we consider the following direct problem for some given data $f(t)$:

$$\begin{aligned} \frac{\partial^\beta u}{\partial t^\beta} - u_{xx} &= 0, \quad x > 0, \quad t > 0, \\ u(0, t) &= f(t), \quad t > 0, \\ u(x, 0) &= 0, \quad x > 0. \end{aligned} \quad (5.1)$$

This problem is a well-posed problem and its solution at $x = 1$ is given by

$$g(t) := u(1, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi t} e^{-(i\xi)^{\frac{\beta}{2}}} \hat{f}(\xi) d\xi. \quad (5.2)$$

In numerical implementation, we give the data $f(t)$ and sample at an equidistant grid, then carry out discrete Fourier transformation. We obtain the data $g(t)$ via inverse discrete Fourier transformation according to (5.2), then generate the noisy data g_δ :

$$g_\delta = g + g_{\max} * \delta \text{ rand}(\text{size}(g)), \quad (5.3)$$

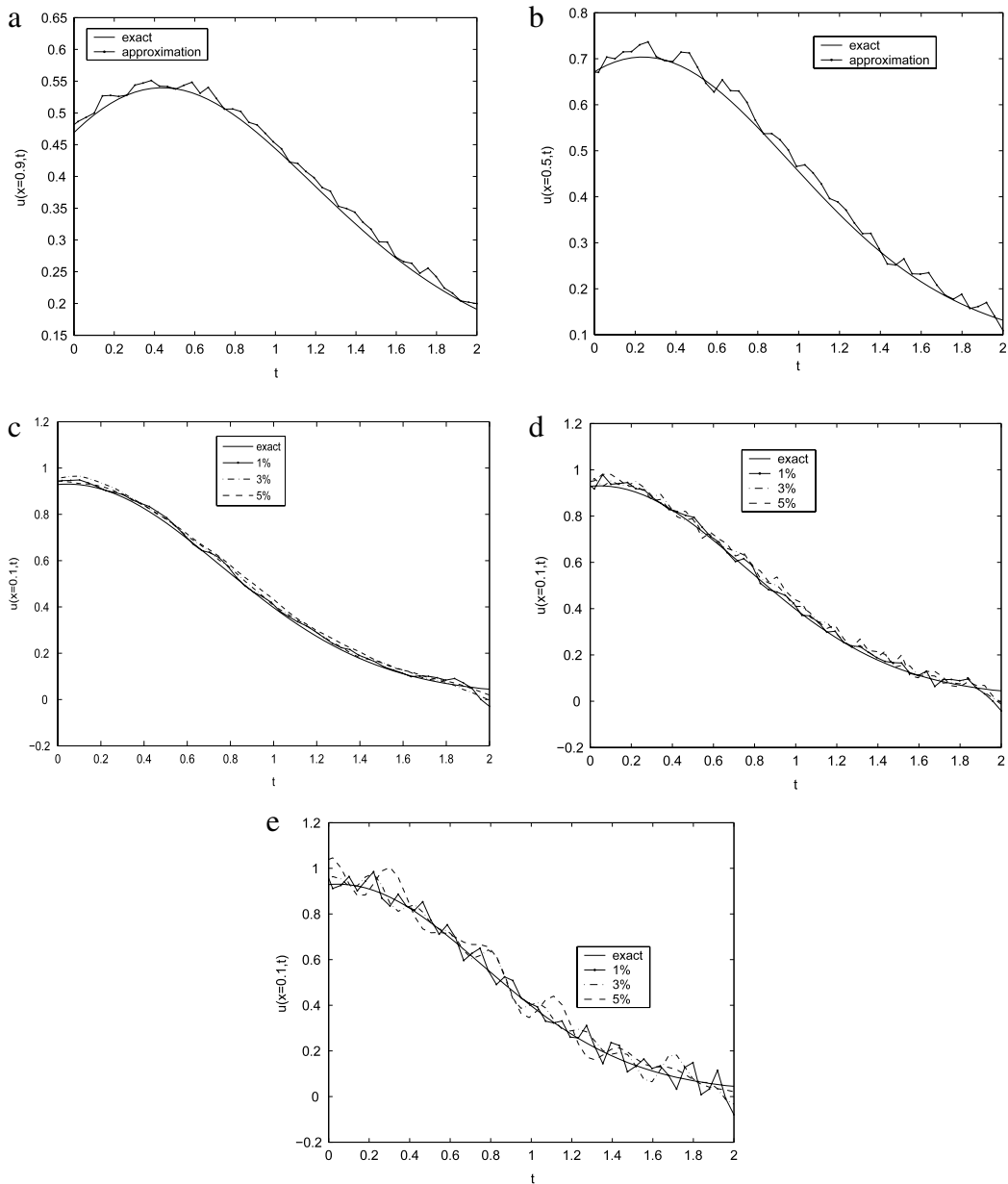


Fig. 4. (4a): reconstruction of temperature at $x = 0.9$; (4b): reconstruction of temperature at $x = 0.5$; (4c): reconstruction of temperature by Method 1 at $x = 0.1$; (4d): reconstruction of temperature by Method 2 at $x = 0.1$; (4e): reconstruction of temperature by cut-off method at $x = 0.1$.

δ indicates the error level of g , g_{\max} is the maximum one of sampled data g , RMS denotes the root mean square for a sampled function W which is defined by

$$\text{RMS}(W) = \sqrt{\frac{1}{s} \sum_{j=1}^s (W(t_j))^2}, \quad (5.4)$$

s is the total number of test points. Similarly, we can define the root mean square error (RMSE) for the computed data and exact data. The symbol $\text{rand}(\text{size}(\cdot))$ is a random number between $[0, 1]$.

Numerical implementation is completed by Matlab in IEEE double precision with unit round-off $1.1 \cdot 10^{-16}$. The regularized solutions were computed by the discrete Fast Fourier Transform (FFT) and inverse discrete Fast Fourier Transform technique according to the formulas in Sections 3 and 4. In the numerical experiment, we fix $\beta = 0.90$ except some specification, $s = 101$.

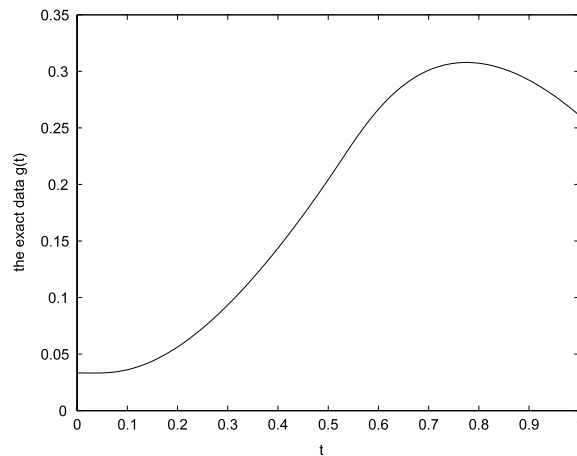


Fig. 5. Example 2: the input exact data $g(t)$.

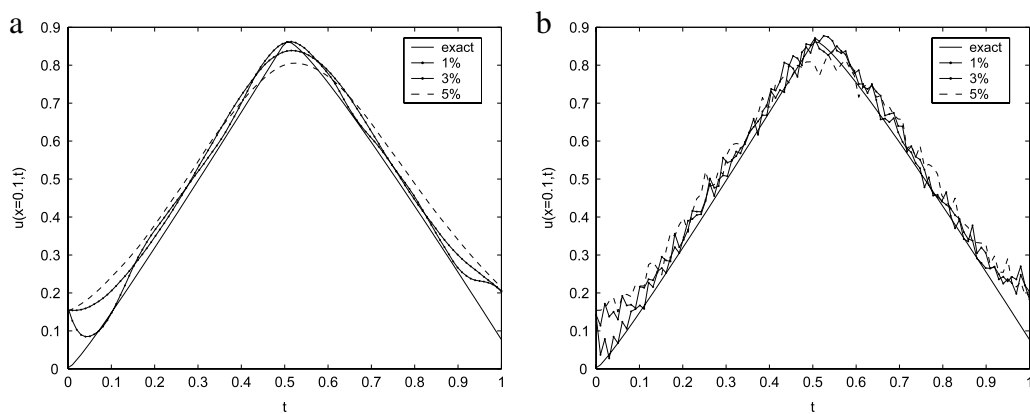


Fig. 6. Example 2. (6a): reconstruction by Method 1 at $x = 0.1$ with $\alpha = 0.005$; (6b): reconstruction by Method 2 at $x = 0.1$ with $\alpha = 0.006$.

Table 1
Error behavior for Method 1.

x	0.9	0.5	0.1
α	0.48	0.04	0.05
$\text{RMSE}(u - u_\alpha^\delta)$	0.01	0.02	0.03

5.1. Reconstruction of temperature at different locations

Example 1. Let $u(0, t) := f(t) = e^{-t^2}$.

Fig. 2 shows the input exact data $g(t)$.

Fig. 3 shows the result with a noise level $\delta = 3\%$ without using regularization methods. From Fig. 3, we can see that the computed solution is unavailable. Due to the ill-posedness of problem (2.1), the regularization method must be applied.

Fig. 4(a) and (b) show the results computed by Method 1 with $\delta = 3\%$ for $x = 0.9, 0.5$. Fig. 4(c) and (d) show the results computed by Methods 1 and 2 with different noisy levels for $x = 0.1$. Fig. 4(e) show the results computed by cut-off method with different noisy levels for $x = 0.1$.

From Fig. 4(c)–(e) we can see that Methods 1 and 2 are comparable, but both are better than the cut-off method.

For Method 1, the regularization parameter α and RMSE are listed in Table 1 ($\delta = 3\%$).

Example 2. Furthermore, to illustrate the effectiveness of our proposed methods, the following nonsmooth function is considered in this example:

$$f(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 0.5, \\ 2(1-t), & \text{if } 0.5 \leq t \leq 1. \end{cases}$$

Fig. 5 shows the input data $g(t)$.

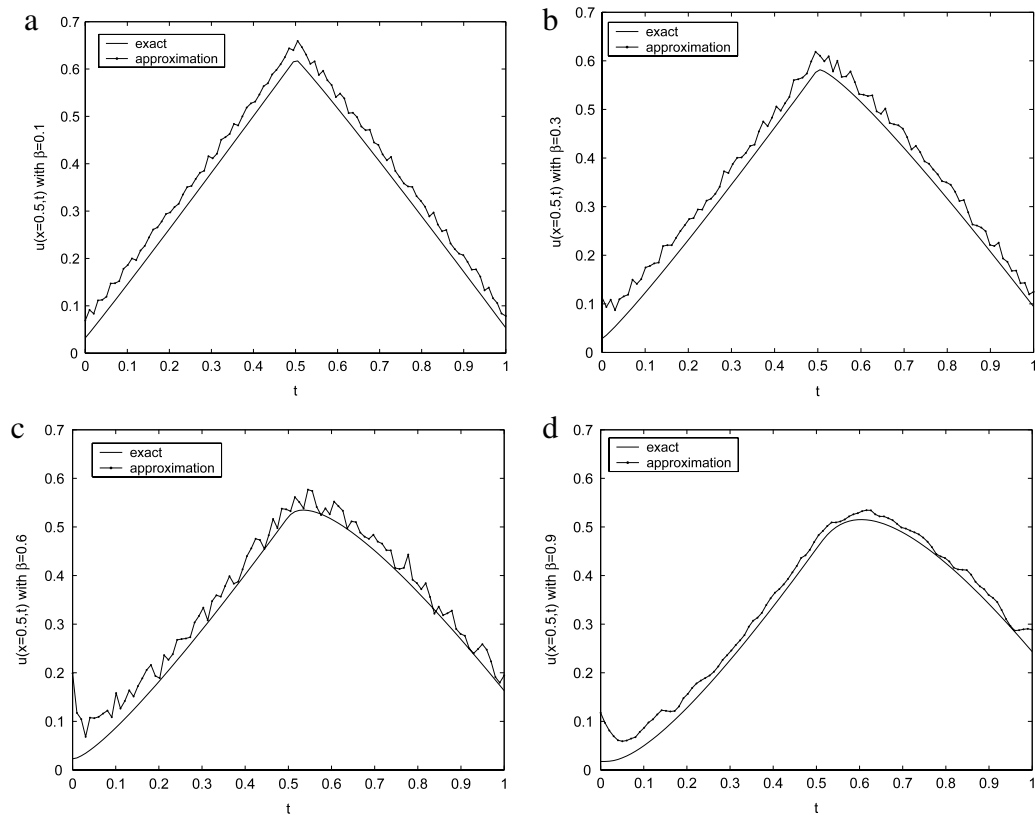


Fig. 7. Example 2. Reconstruction of temperature at $x = 0.5$ with $\alpha = 0.02$, $\delta = 3\%$. (7a): $\beta = 0.1$; (7b): $\beta = 0.3$; (7c): $\beta = 0.6$; (7d): $\beta = 0.9$.

Fig. 6 shows the results by Methods 1 and 2 with different noise levels at $x = 0.1$. It is easy to see that the proposed methods perform very well for this example.

Fig. 7 shows the results by Method 1 with different β values. From Fig. 7, we can see that the solutions continuously depend on the index of time fractional derivative.

Example 3. In this example, we consider the following direct problem

$$\begin{aligned} \frac{\partial^\beta u}{\partial t^\beta} - u_{xx} &= 0, \quad x > 0, \quad t > 0, \\ u(x, 0) &= 0, \quad t > 0, \\ u(0, t) &= H(t - 0.2) - H(t - 0.6), \quad t > 0, \end{aligned}$$

where $H(t)$ denotes the Heaviside function.

Fig. 8 shows the input exact data $g(t)$.

Fig. 9 shows the results by Methods 1 and 2 with different noise levels at $x = 0.5$.

Fig. 10 shows the results by Methods 1 and 2 with different noise levels at $x = 0$. Numerical results show that the proposed methods can also give reasonable results for this discontinuous example.

5.2. Reconstruction of Heat flux at $x = 0$

In this subsection, we will focus on the reconstruction of heat flux. Here we are interested in the results at $x = 0$. The above three examples are implemented via (4.14).

When $\beta = 0.9$, the following results are less encouraging except Example 1 (see Fig. 11).

However, when $\beta = 0.2$, we find the results are satisfactory for Examples 2 and 3 (see Fig. 12). This is because when β becomes smaller, the degree of ill-posedness of the problem becomes smaller.

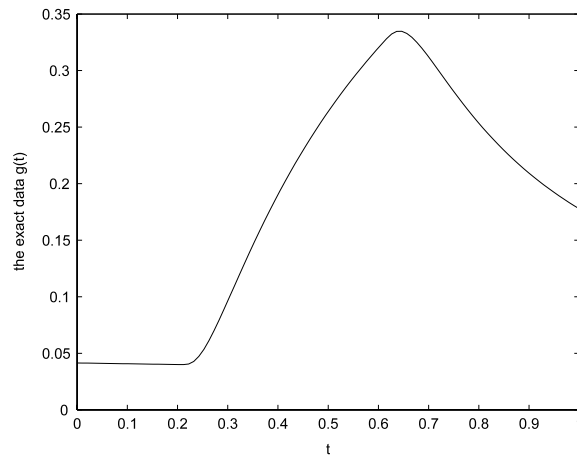


Fig. 8. Example 3: the exact input data $g(t)$.

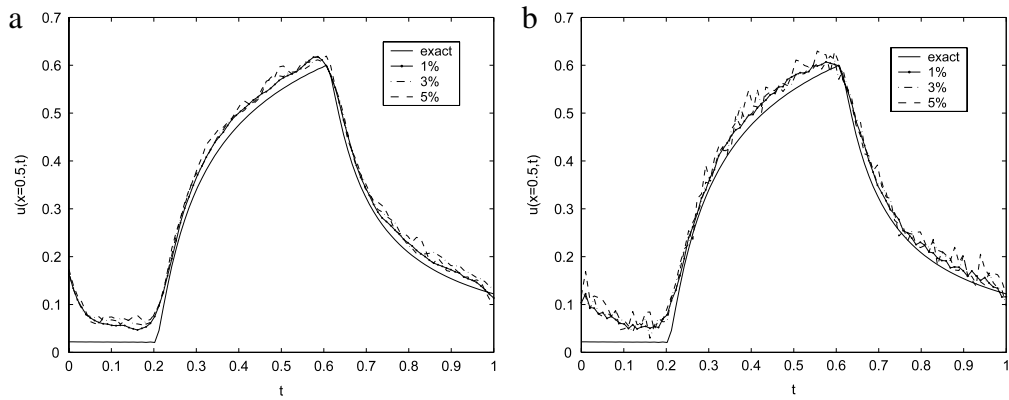


Fig. 9. Example 3. (9a): reconstruction by Method 1 at $x = 0.5$ with $\alpha = 0.02$; (9b): reconstruction by Method 2 at $x = 0.5$ with $\alpha = 0.06$.

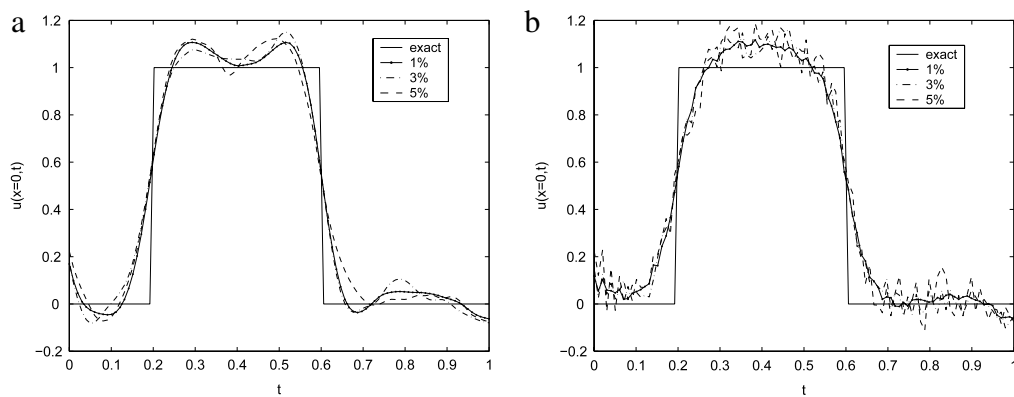


Fig. 10. Example 3. (10a): reconstruction by Method 1 at $x = 0$ with $\alpha = 0.001$; (10b): reconstruction by Method 2 at $x = 0$ with $\alpha = 0.006$.

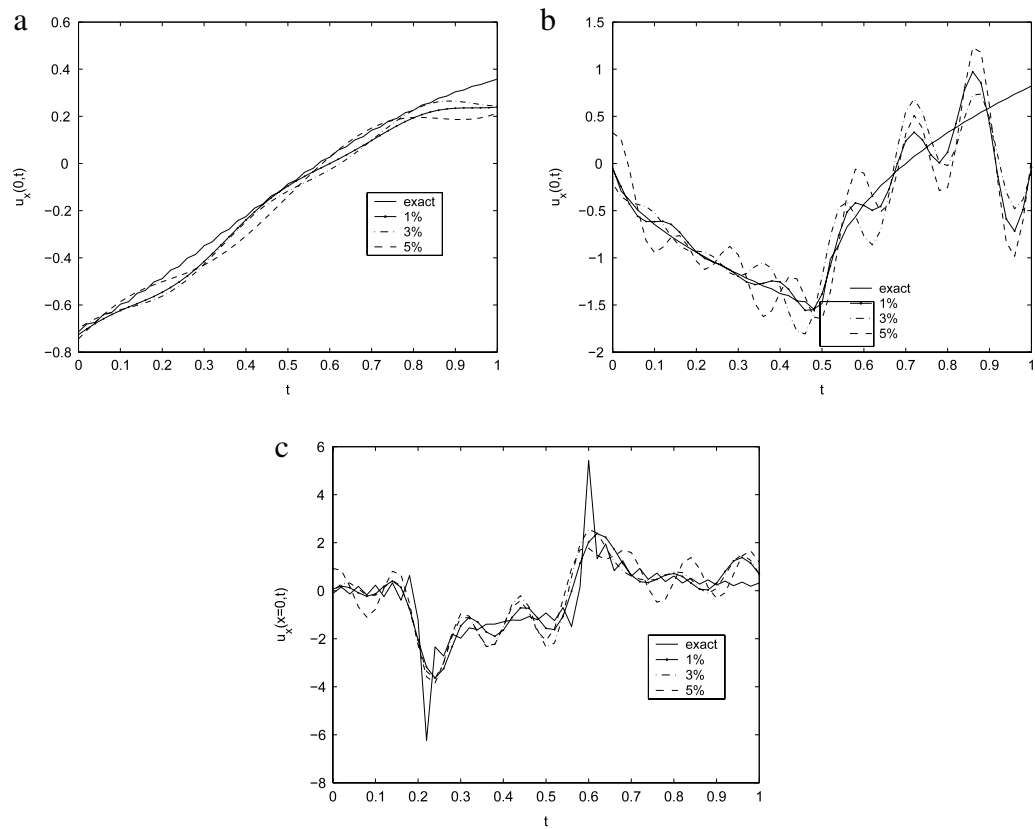


Fig. 11. Reconstruction of heat flux by cut-off method. (11a): Example 1 with $\xi_{\max} = 15$; (11b) Example 2 with $\xi_{\max} = 45$; (11c) Example 3 with $\xi_{\max} = 50$.

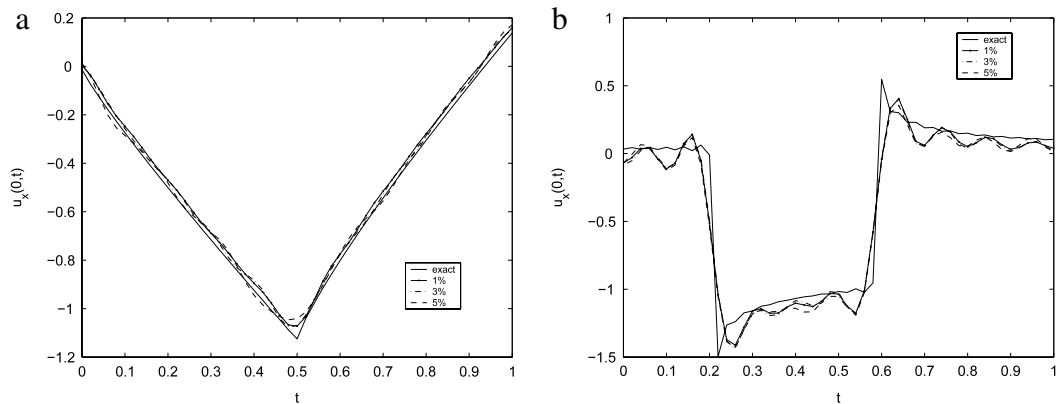


Fig. 12. Reconstruction of heat flux by cut-off method. (12a): Example 2 with $\xi_{\max} = 65$; (12b): Example 3 with $\xi_{\max} = 60$.

Acknowledgments

The authors would like to thank the reviewers for their comments and suggestions that greatly helped improve the readability of the paper. This work was partially supported by a grant from the National Natural Science Foundation of China (No. 11001223), the Research Fund for the Doctoral Program of Higher Education of China (No. 20106203120001), the Key (Keygrant) Project of Chinese Ministry of Education (No. 212179), and the Doctoral Foundation of Northwest Normal University, China (No. 5002-577).

References

- [1] W. Chen, L.J. Ye, H.G. Sun, Fractional diffusion equations by the Kansa method, *Computers and Mathematics with Applications* 59 (2010) 1614–1620.
- [2] Z. Yu, J. Lin, Numerical research on the coherent structure in the viscoelastic second-order mixing layers, *Applied Mathematics and Mechanics* 19 (1998) 671–677.
- [3] N. Laskin, I. Lambadaris, F.C. Harmantzis, M. Devetsikiotis, Fractional Levy motion and its application to network traffic modeling, *Computer Networks* 40 (3) (2002) 363–375.
- [4] H. Scher, E.W. Montroll, Anomalous transit-time dispersion in amorphous, *Physical Review B* 12 (6) (1975) 2455–2477.
- [5] T.L. Szabo, J. Wu, A model for longitudinal and shear wave propagation in viscoelastic media, *Journal of the Acoustical Society of America* 107 (5) (2000) 2437–2446.
- [6] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, P. Paradisi, Discrete random walk models for space–time fractional diffusion, *Chemical Physics* 284 (2002) 521–541.
- [7] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Physics Reports* 339 (2000) 1–77.
- [8] I.M. Sokolov, J. Klafter, A. Blumen, Fractional kinetics, *Physics Today* 55 (2002) 48–54.
- [9] R.V. Mendes, A fractional calculus interpretation of the fractional volatility model, *Nonlinear Dynamics* 55 (2009) 395–399.
- [10] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover Publications, New York, 1953.
- [11] Y.C. Hon, M. Li, A computational method for inverse free boundary determination problem, *International Journal for Numerical Methods in Engineering* 73 (2008) 1291–1309.
- [12] J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, Uniqueness in an inverse problem for one-dimensional fractional diffusion equation, *Inverse Problems* 16 (2009) 115002 (16pp).
- [13] G.H. Zheng, T. Wei, Spectral regularization method for a Cauchy problem of the time fractional advection–dispersion equation, *Journal of Computational and Applied Mathematics* 233 (2010) 2631–2640.
- [14] A.N. Bondarenko, D.S. Ivaschenko, Numerical methods for solving inverse problems for time fractional diffusion equation with variable coefficient, *Journal of Inverse and Ill-Posed Problems* 17 (5) (2009) 419–440.
- [15] D.A. Murio, Stable numerical solution of a fractional-diffusion inverse heat conduction problem, *Computers & Mathematics with Applications* 53 (2007) 1492–1501.
- [16] Y.C. Hon, T. Wei, A fundamental solution method for inverse heat conduction problem, *Engineering Analysis with Boundary Elements* 28 (5) (2004) 489–495.
- [17] Y.C. Hon, T. Wei, The method of fundamental solutions for solving multidimensional inverse heat conduction problems, *Computer Modeling in Engineering & Sciences* 7 (2) (2005) 119–132.
- [18] F. Berntsson, A spectral method for solving the sideways heat equation, *Inverse Problems* 15 (1999) 891–906.
- [19] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [20] L. Eldén, F. Berntsson, T. Regińska, Wavelet and Fourier methods for solving the sideways heat equation, *SIAM Journal on Scientific Computing* 21 (6) (2000) 2187–2205.
- [21] T.I. Seidman, L. Elden, An 'optimal filtering' method for the sideways heat equation, *Inverse Problems* 6 (1990) 681–696.
- [22] U. Tautenhahn, Optimality for ill-posed problems under general source conditions, *Numerical Functional Analysis and Optimization* 19 (1998) 377–398.