Journal of Computational and Applied Mathematics 261 (2014) 378–386

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Convergence of the two-step combined method and uniqueness of the solution of nonlinear operator equations

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ARTICLE INFO

Article history:
Received 20 April 2013
Received in revised form 7 November 2013

MSC:
65H10
65J15
47H17

Keywords:
Nonlinear equation
Iterative process
The convergence order
Radius of convergence
Nondifferentiable operator
Generalized Lipschitz condition

ABSTRACT

Local convergence of the two-step differential-difference method for solving nonlinear operator equations for generalized Lipschitz conditions for Fréchet derivatives of the first and second order and divided differences of the first order has been proven. There have been found estimations of the convergence ball's radii of this method and the uniqueness ball of solution of nonlinear equations. There has been established the superquadratical order of the convergence of the two-step combined method and a comparison of the results with the known ones has been made.

Contents

1. Introduction

Consider the nonlinear equation

$$F(x) = 0, \tag{1}$$

where F is the continuous operator defined on an open convex subset D of a Banach space X with values in a Banach space Y. The best known method for solving Eq. (1) is the classical Newton's method [1,2], which has a quadratic order of convergence in the case of the Lipschitz continuity of the first order derivative of the operator F. In [3] M. Bartish first proposed the method which is a two-step modification of Newton's method, conducted the research of its semilocal convergence (under

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http://dx.doi.org/10.1016/j.cam.2013.11.018

Kantorovich type conditions) and established the order of convergence $1+\sqrt{2}$ of the method. This iterative process for solving (1) has the form [4–8]

$$
\begin{aligned}
x_{k+1} &= x_k - \left[F'\left(\frac{x_k+y_k}{2}\right)\right]^{-1} F\left(x_k\right), \\
y_{k+1} &= x_{k+1} - \left[F'\left(\frac{x_k+y_k}{2}\right)\right]^{-1} F\left(x_{k+1}\right), \quad k=0,1,2,\ldots.
\end{aligned}
\tag{2}
$$

Here x_0, y_0 are initial values.

Difference analogous of the method (2)

$$
\begin{aligned}
x_{k+1} &= x_k - \left[\delta F\left(x_k, y_k\right)\right]^{-1} F\left(x_k\right), \\
y_{k+1} &= x_{k+1} - \left[\delta F\left(x_k, y_k\right)\right]^{-1} F\left(x_{k+1}\right), \quad k=0,1,2,\ldots,
\end{aligned}
\tag{3}
$$

where x_0, y_0 are given, has been studied in the papers [9,10], where there has been established the order of convergence (3), which also equals $1+\sqrt{2}$.

In work [11] during the investigation of Newton's method there have been proposed rather weak Lipschitz conditions for the derivative operator, in which instead of constant L, there has been used some positive integrable function. The author in his work [12] proposed the following Lipschitz conditions for the first order divided difference operator, and under these conditions investigated the convergence of the Secant method [12] and the two-step difference method (3) for operator equations [13], and in [14]—the two-step Newton type method for generalized Lipschitz conditions for the derivatives of the first and second orders of the nonlinear operator.

In works [15–21] there have been studied the combined iterative processes that just as the difference methods [16,17] can be applied for solving nonlinear equations with nondifferentiable operators, namely equations

$$
H(x) \equiv F(x) + G(x) = 0, \tag{4}
$$

where F and G are defined on an open convex subset D of the Banach space X with values in a Banach space Y; F is the Frechet differentiable operator, and G is the continuous operator, differentiability of which, generally speaking, is not required. In the first instance these are the methods that are a combination of the Newton and Secant method. They have the order of convergence not higher than $\left(1+\sqrt{5}\right)/2$. In particular, for the one-step method

$$
x_{k+1} = x_k - A_k^{-1}(F(x_k) + G(x_k)), \quad k=0,1,2,\ldots \tag{5}
$$

while $A_k = F'(x_k)$ let us conduct the analysis of the convergence of [20–22]; in works [16,17,19] there have been studied the one-step modifications with $A_k = F'(x_k) + \delta G(x_k, x_{k-1})$ and other selections of A_k. In [19] we explored the semilocal convergence of the one-step method for

$$
A_k = F'(x_k) + \delta G(2x_k - x_{k-1}, x_{k-1}).
$$

Difference methods for solving equations with nondifferentiable operators were studied in [23,24].

In work [18] we first proposed a method that is built on the basis of methods (2) and (3) [3,6,9,10]. Its iterative formula is:

$$
\begin{aligned}
x_{k+1} &= x_k - \left[F'\left(\frac{x_k+y_k}{2}\right) + \delta G(x_k, y_k)\right]^{-1} \left(F\left(x_k\right) + G(x_k)\right), \\
y_{k+1} &= x_{k+1} - \left[F'\left(\frac{x_k+y_k}{2}\right) + \delta G(x_k, y_k)\right]^{-1} \left(F\left(x_{k+1}\right) + G(x_{k+1})\right), \quad k=0,1,2,\ldots,
\end{aligned}
\tag{6}
$$

x_0, y_0 are given.

Just like Newton's–Secant method which was studied by many authors, the new method requires at each iteration the computation of one operator A_k and its inverse. The operator A_k consists of a combination of a Frechet derivative of one part of the operator (the differentiable part) and the divided difference of the second part (generally speaking, nondifferentiable). The number of computations of the function value increases by one at each step. Therefore, the number of computations at one iteration is almost identical in both methods. However, the convergence order of the new method is higher ($1+\sqrt{2} \approx 2.41$ vs. $\frac{1+\sqrt{5}}{2} \approx 1.62$).

The investigations of method (6) in [18] have been conducted under Hölder's conditions for the second order derivatives from F and usual Lipschitz conditions for the first order divided differences of G of Eq. (4).

In this paper, we study the method (6) under relatively weak, generalized Lipschitz conditions for the second order derivatives and divided differences of the first order.

2. Definitions and auxiliary lemma

Definition 1. Let F be a nonlinear operator defined on a subset D of a Banach space X with values in a Banach space Y and x, y be two points of D. A linear operator from X into Y, denoted by $\delta G(x, y)$, which satisfies the condition

$$\delta G(x,y)(x-y) = G(x) - G(y) \tag{7}$$

is called a divided difference of G at the points x and y.

Let us denote as $B(x_0, r) = \{x : \|x - x_0\| < r\}$ an open, and as $\overline{B(x_0, r)} = \{x : \|x - x_0\| \le r\}$ a closed ball of radius r with center at the point x_0.

In the study of iterative methods traditional are the Lipschitz conditions with constant L. However L in Lipschitz conditions does not need to be a constant, and can be a positive integrable function. In this case, we consider the conditions

$$\|F(x) - F(y)\| \le \int_0^{\|x-y\|} L(u)\, du \quad \forall\, x, y \in B(x_0, r), \tag{8}$$

and

$$\|\delta G(x,y) - \delta G(u,v)\| \le \int_0^{\|x-u\|+\|y-v\|} M(z)\, dz \quad \forall\, x, y, u, v \in B(x_0, r), \tag{9}$$

where L and M are positive integrable functions. Lipschitz conditions (8) and (9) we will call generalized Lipschitz conditions or Lipschitz conditions with the L (or M) average. Note that in the case of constant L, M we obtain from (8) and (9) the classical Lipschitz conditions.

Lemma 1 ([11]). *Let $h(t) = \frac{1}{t}\int_0^t L(u)\, du,\ 0 \le t \le r$, where $L(u)$ is a positive integrable function that is nondecreasing monotonically in $[0, r]$. Then $h(t)$ is nondecreasing monotonically with respect to t.*

Lemma 2. *Let $g(t) = \frac{1}{t^3}\int_0^t N(u)(t-u)^2\, du,\ 0 \le t \le r$, where $N(u)$ is a positive integrable function that is nondecreasing monotonically in $[0, r]$. Then $g(t)$ is a nondecreasing monotonically with respect to t.*

Proof. Indeed, in the case of monotonousness of N for $0 < t_1 < t_2$ we have

$$\begin{aligned}
g(t_2) - g(t_1) &= \frac{1}{t_2^3}\int_0^{t_2} N(u)\ (t_2-u)^2\, du - \frac{1}{t_1^3}\int_0^{t_1} N(u)\ (t_1-u)^2\, du \\
&= \frac{1}{t_2^3}\int_{t_1}^{t_2} N(u)\ (t_2-u)^2\, du + \frac{1}{t_2^3}\int_0^{t_1} N(u)\ (t_1-u)^2\, du - \frac{1}{t_1^3}\int_0^{t_1} N(u)\ (t_1-u)^2\, du \\
&= \frac{1}{t_2^3}\int_{t_1}^{t_2} N(u)\ (t_2-u)^2\, du + \left(\frac{1}{t_2^3} - \frac{1}{t_1^3}\right)\int_0^{t_1} N(u)\ (t_1-u)^2\, du \\
&\ge N(t_1)\left[\frac{1}{t_2^3}\int_{t_1}^{t_2} (t_2-u)^2\, du + \left(\frac{1}{t_2^3} - \frac{1}{t_1^3}\right)\int_0^{t_1} (t_1-u)^2\, du\right] \\
&= N(t_1)\left[\frac{1}{t_2^3}\int_0^{t_2} (t_2-u)^2\, du - \frac{1}{t_1^3}\int_0^{t_1} (t_1-u)^2\, du\right] = 0.
\end{aligned}$$

Therefore, $g(t) = \frac{1}{t^3}\int_0^t N(u)(t-u)^2\, du,\ 0 \le t \le r$ is nondecreasing monotonically with respect to t.

Lemma 3 ([11]). *Let $p(t) = \frac{1}{t}\int_0^t M(u)\,(t-u)\, du,\ 0 \le t \le r$, where M is a positive integrable function that is nondecreasing monotonically in $[0, r]$. Then $p(t)$ is increasing monotonically with respect to t.*

3. Local convergence of the iterative process (6)

Let us study the convergence of method (6). Let $D \subset X$ be an open convex subset, $B(x^*, r) \subseteq D$, r_0, r_1 are the solutions of the systems of equations

$$\begin{aligned}
&J\tau_0^2 + C\tau_1 = 1, \\
&\left[J\tau_0^2 + E(2\tau_0 + \tau_1)\right]\tau_0 = \tau_1
\end{aligned} \tag{10}$$

on the interval $(0, r)$, constants r, J, C, E are defined below.

The radius of the convergence ball and the order of convergence of the method (6) are defined by the following theorem.

Theorem 1. *Let $H(x) \equiv F(x) + G(x)$ be a nonlinear operator, defined in an open convex subset D of space X with values in the space Y. Suppose that*

(i) *$H(x) = 0$ has a solution $x^* \in D$, in which there exists the Frechet derivative $H'(x^*)$ and it is invertible;*
(ii) *there exist Frechet derivatives F' and F'' in $B(x^*, r) \subset D$, satisfying Lipschitz conditions with L and N average*

$$\left\| H'\left(x^*\right)^{-1}\left(F'(x) - F'\left(x^*\right)\right)\right\| \le \int_0^{\rho(x)} L(u)\,du,$$
$$\left\| H'\left(x^*\right)^{-1}\left(F''(x) - F''(y)\right)\right\| \le \int_0^{\|x-y\|} N(u)\,du, \tag{11}$$

and divided differences satisfy the Lipschitz condition with M average

$$\left\| H'\left(x^*\right)^{-1}\left(\delta G(x,y) - \delta G(u,v)\right)\right\| \le \int_0^{\|x-u\|+\|y-v\|} M(z)\,dz \tag{12}$$

where $x, y, u, v \in B(x^*, r)$, $\rho(x) = \|x - x^*\|$ and L, M and N are positive integrable functions and are nondecreasing monotonically;
(iii) *let $r > 0$ satisfy the equality*

$$\frac{\frac{1}{8}\int_0^r N(u)(r-u)^2\,du + r\int_0^{3r/2} L(u)\,du + r\int_0^{3r} M(u)\,du}{r\left(1 - \int_0^r L(u)\,du - \int_0^{2r} M(u)\,du\right)} = 1. \tag{13}$$

Then for all $x_0 \in B(x^*, r_0)$ and $y_0 \in B(x^*, r_1)$ sequences $\{x_k\}_{k\in N}$ and $\{y_k\}_{k\in N}$, are defined according to formulas (6), converge to the solution x^*, $x_k \in B(x^*, r_0)$ and $y_k \in B(x^*, r_1)$ for $k = 0, 1, 2, \ldots$ and the estimations are being fulfilled

$$\rho(x_{k+1}) = \left\|x_{k+1} - x^*\right\| \le J\rho(x_k)^3 + C\rho(x_k)\rho(y_k), \quad k = 0, 1, 2, \ldots,$$
$$\rho(y_{k+1}) = \left\|y_{k+1} - x^*\right\| \le J\rho(x_{k+1})^3 + E\left(\rho(x_k) + \rho(x_{k+1}) + \rho(y_k)\right)\rho(x_{k+1}), \quad k = 0, 1, 2, \ldots, \tag{14}$$

where $z_0 = \rho(x_0) + \rho(y_0) + \rho(x_1)$,

$$J = \frac{Q_0}{8\rho(x_0)^3}\int_0^{\rho(x_0)} N(u)\left(\rho(x_0) - u\right)^2 du; \qquad E = \frac{Q_0}{z_0}\left(\int_0^{z_0/2} L(u)du + \int_0^{z_0} M(u)du\right);$$
$$C = \frac{Q_0}{\rho(y_0)}\left(\int_0^{\rho(y_0)/2} L(u)du + \int_0^{\rho(y_0)} M(u)du\right);$$
$$Q_0 = \left(1 - \int_0^{\rho(x_0+y_0)/2} L(u)du - \int_0^{\rho(x_0)+\rho(y_0)} M(u)du\right)^{-1}. \tag{15}$$

The order of convergence of sequences $\{x_k\}_{k\in N}$ and $\{y_k\}_{k\in N}$ to the solution x^* equals $1 + \sqrt{2}$.

Proof. Let us chose arbitrarily $x_0 \in B(x^*, r_0)$ and $y_0 \in B(x^*, r_1)$, where r satisfies (13). Then in the case of monotonousness L, M and N according to Lemmas 1 and 2 we have, that $\frac{1}{t}\int_0^t L(u)\,du$, $\frac{1}{t}\int_0^t M(u)\,du$ and $\frac{1}{t^3}\int_0^t N(u)(t-u)^2\,du$ are nondecreasing with respect to t.

Let us denote as $A = A(x, y)$ the linear operator $A = F'\left(\frac{x+y}{2}\right) + \delta G(x, y)$ with $x, y \in B(x, r)$. Then, by (11)–(12), we get

$$\begin{aligned}\left\|I - H'(x^*)^{-1}A\right\| &= \left\|H'(x^*)^{-1}\left(H'(x^*) - A\right)\right\| \\ &= \left\|H'(x^*)^{-1}\left[F'(x^*) + \delta G(x^*, x^*) - F'\left(\frac{x+y}{2}\right) - \delta G(x, y)\right]\right\| \\ &\le \int_0^{\rho\left(\frac{x+y}{2}\right)} L(u)du + \int_0^{\rho(x)+\rho(y)} M(u)du.\end{aligned}$$

From the definition r it follows that

$$\int_0^r L(u)du + \int_0^{2r} M(u)du = 1 - \frac{1}{8r}\int_0^r N(u)(r-u)^2\,du - \int_0^{3r/2} L(u)\,du - \int_0^{3r} M(u)\,du < 1.$$

Then from the identity

$$\left\|A^{-1}H'(x^*)\right\| = \left\|\left(I - \left(I - H'(x^*)^{-1}A\right)\right)^{-1}\right\|$$

according to the Banach lemma on the invertible operator [1,21] it follows that A is invertible and

$$\left\|A^{-1}H'(x^*)\right\| \le \left(1 - \int_0^{\rho\left(\frac{x+y}{2}\right)} L(u)du - \int_0^{\rho(x)+\rho(y)} M(u)du\right)^{-1}.$$

Let us now suppose that $x_k \in B\left(x^*, r_0\right), y_k \in B\left(x^*, r_1\right)$. Set $A_k = A(x_k, y_k)$. Then A_k is invertible and, according to (6), we can write

$$\begin{aligned}
x_{k+1} - x^* &= x_k - x^* - A_k^{-1}\left(F\left(x_k\right) + G(x_k) - F(x^*) - G(x^*)\right) \\
&= -A_k^{-1}\left[F\left(x_k\right) + G(x_k) - F(x^*) - G(x^*) - \left(F'\left(\frac{x_k+y_k}{2}\right) + \delta G(x_k, y_k)\right)\left(x_k - x^*\right)\right] \\
&= A_k^{-1}H'(x^*)\left\{H'(x^*)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right)\left(x_k - x^*\right) - F\left(x_k\right) + F(x^*)\right]\right. \\
&\quad \left. + H'\left(x^*\right)^{-1}\left[\delta G\left(x_k, x^*\right) - \delta G\left(x_k, y_k\right)\right]\left(x_k - x^*\right)\right\} \\
&= A_k^{-1}H'(x^*)\left\{H'(x^*)^{-1}\left[F'\left(\frac{x_k+x^*}{2}\right)\left(x_k - x^*\right) - F\left(x_k\right) + F(x^*)\right]\right. \\
&\quad + H'\left(x^*\right)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right) - F'\left(\frac{x_k+x^*}{2}\right)\right]\left(x_k - x^*\right) \\
&\quad \left. + H'\left(x^*\right)^{-1}\left[\delta G\left(x_k, x^*\right) - \delta G\left(x_k, y_k\right)\right]\left(x_k - x^*\right)\right\}.
\end{aligned} \tag{16}$$

Let us write down the identity [6, Lemma 1, p. 336] for values $\omega = 1/2$

$$\begin{aligned}
F(x) - F(y) - F'\left(\frac{x+y}{2}\right)(x-y) &= \frac{1}{4}\int_0^1 (1-t)\left[F''\left(\frac{x+y}{2} + \frac{t}{2}(x-y)\right)\right. \\
&\quad \left. - F''\left(\frac{x+y}{2} + \frac{t}{2}(y-x)\right)\right](x-y)(x-y)dt.
\end{aligned}$$

Putting in this equality $x = x^*$, $y = x$, we will obtain the estimation

$$\begin{aligned}
&\left\|H'\left(x^*\right)^{-1}\left[F\left(x^*\right) - F\left(x_k\right) - F'\left(\frac{x_k+x^*}{2}\right)\left(x^* - x_k\right)\right]\right\| \\
&= \frac{1}{4}\left\|\int_0^1 (1-t)H'\left(x^*\right)^{-1}\left[F''\left(\frac{x_k+x^*}{2} + \frac{t}{2}\left(x^* - x_k\right)\right)\right.\right. \\
&\quad \left.\left. - F''\left(\frac{x_k+x^*}{2} + \frac{t}{2}\left(x_k - x^*\right)\right)\left(x^* - x_k\right)\ \left(x^* - x_k\right)\right]dt\right\| \\
&\le \frac{1}{4}\int_0^1 (1-t)\int_0^{t\left\|x_k-x^*\right\|} N(u)du\ \left\|x_k - x^*\right\|^2 dt \\
&= \frac{1}{8}\int_0^{\left\|x_k-x^*\right\|} N(u)\left(1 - \frac{u}{\|x_k - x^*\|}\right)^2 du\ \left\|x_k - x^*\right\|^2 \\
&= \frac{1}{8}\int_0^{\rho(x_k)} N(u)\left(\rho\left(x_k\right) - u\right)^2 du,
\end{aligned}$$

and also

$$\left\|H'\left(x^*\right)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right) - F'\left(\frac{x_k+x^*}{2}\right)\right]\right\| \le \int_0^{\rho(y_k)/2} L(u)du;$$

$$\left\|H'\left(x^*\right)^{-1}\left[\delta G\left(x_k, x^*\right) - \delta G\left(x_k, y_k\right)\right]\left(x_k - x^*\right)\right\| \le \int_0^{\rho(y_k)} M(u)du \left\|x_k - x^*\right\|.$$

Then according to Lemmas 1–3 and conditions (11)–(13) taking into account the latter inequalities we obtain

$$\begin{aligned}
\left\|x_{k+1}-x^*\right\| \le\ & \left\|A_k^{-1}H'(x^*)\right\| \left\{ \left\| H'(x^*)^{-1}\left[F'\left(\frac{x_k+x^*}{2}\right)\left(x_k-x^*\right)-F\left(x_k\right)+F(x^*)\right]\right\| \right.\\
& + \left\|H'\left(x^*\right)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right)-F'\left(\frac{x_k+x^*}{2}\right)\right]\left(x_k-x^*\right)\right\| \\
& \left. + \left\|H'\left(x^*\right)^{-1}\left[\delta G\left(x_k,x^*\right)-\delta G\left(x_k,y_k\right)\right]\left(x_k-x^*\right)\right\| \right\} \\
\le\ & \left\|A_k^{-1}H'(x^*)\right\| \left\{ \frac{1}{4}\int_0^1 (1-t)\left\|H'(x^*)^{-1}\left[F''\left(\frac{x_k+x^*}{2}+\frac{t}{2}\left(x_k-x^*\right)\right)\right.\right.\right.\\
& \left.\left. - F''\left(\frac{x_k+x^*}{2}+\frac{t}{2}\left(x^*-x_k\right)\right)\right]\left(x_k-x^*\right)\left(x_k-x^*\right)\right\| dt \\
& + \left\|H'(x^*)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right)-F'\left(\frac{x_k+x^*}{2}\right)\right]\left(x_k-x^*\right)\right\| \\
& \left. + \left\|H'\left(x^*\right)^{-1}\left[\delta G\left(x_k,x^*\right)-\delta G\left(x_k,y_k\right)\right]\left(x_k-x^*\right)\right\| \right\} \\
\le\ & \left\|A_k^{-1}H'(x^*)\right\| \left\{ \frac{1}{8}\int_0^{\rho(x_k)} N(u)\left(\rho\left(x_k\right)-u\right)^2 du + \int_0^{\rho(y_k)/2} L(u)du\,\rho\left(x_k\right) + \int_0^{\rho(y_k)} M(u)du\,\rho\left(x_k\right) \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left\|x_{k+1}-x^*\right\| \le\ & Q_k \left\{ \frac{1}{8\rho\left(x_k\right)^3}\int_0^{\rho(x_k)} N(u)\left(\rho\left(x_k\right)-u\right)^2 du\rho\left(x_k\right)^3 \right.\\
& \left. + \frac{2}{\rho\left(y_k\right)}\int_0^{\rho(y_k)/2} L(u)du\,\rho\left(x_k\right)\rho\left(y_k\right)/2 + \frac{1}{\rho\left(y_k\right)}\int_0^{\rho(y_k)} M(u)du\,\rho\left(x_k\right)\rho\left(y_k\right) \right\} \\
\le\ & Q_0 \left\{ \frac{1}{8\rho\left(x_0\right)^3}\int_0^{\rho(x_0)} N(u)\left(\rho\left(x_0\right)-u\right)^2 du\rho\left(x_k\right)^3 \right.\\
& \left. + \frac{2}{\rho\left(y_0\right)}\int_0^{\rho(y_0)/2} L(u)du\,\rho\left(x_k\right)\rho\left(y_k\right)/2 + \frac{1}{\rho\left(y_0\right)}\int_0^{\rho(y_0)} M(u)du\,\rho\left(x_k\right)\rho\left(y_k\right) \right\} \\
\le\ & J\rho\left(x_k\right)^3 + C\rho\left(x_k\right)\rho\left(y_k\right) < \left(Jr_0^2+Cr_1\right)\rho\left(x_k\right) = \rho\left(x_k\right) < r_0,
\end{aligned} \tag{17}$$

where $Q_k = \left(1-\int_0^{\rho\left(\frac{x_k+y_k}{2}\right)} L(u)du - \int_0^{\rho(x_k)+\rho(y_k)} M(u)du\right)^{-1}$.

Analogously

$$\begin{aligned}
y_{k+1}-x^* &= x_{k+1}-x^*-A_k^{-1}\left(F\left(x_{k+1}\right)+G(x_{k+1})-F(x^*)-G(x^*)\right) \\
&= A_k^{-1}\left[A_k(x_{k+1}-x^*)-F\left(x_{k+1}\right)+F(x^*)-G(x_{k+1})+G(x^*)\right] \\
&= A_k^{-1}H'(x^*)\left\{ H'(x^*)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right)\left(x_{k+1}-x^*\right)-F\left(x_{k+1}\right)+F(x^*)\right] \right.\\
&\quad \left. + H'\left(x^*\right)^{-1}\left[\delta G\left(x_k,y_k\right)-\delta G\left(x_{k+1},x^*\right)\right]\left(x_{k+1}-x^*\right) \right\} \\
&= A_k^{-1}H'(x^*)\left\{ H'(x^*)^{-1}\left[F'\left(\frac{x_{k+1}+x^*}{2}\right)\left(x_{k+1}-x^*\right)-F\left(x_{k+1}\right)+F(x^*)\right] \right.\\
&\quad + H'\left(x^*\right)^{-1}\left[F'\left(\frac{x_k+y_k}{2}\right)-F'\left(\frac{x_{k+1}+x^*}{2}\right)\right]\left(x_{k+1}-x^*\right) \\
&\quad \left. + H'\left(x^*\right)^{-1}\left[\delta G\left(x_k,y_k\right)-\delta G\left(x_{k+1},x^*\right)\right]\left(x_{k+1}-x^*\right) \right\}.
\end{aligned}$$

Out of that

$$\begin{aligned}\left\|y_{k+1}-x^*\right\| &\le Q_k\left\{\frac{1}{8}\int_0^{\rho(x_{k+1})}N(u)\left(\rho\left(x_{k+1}\right)-u\right)^2du+\int_0^{z_k/2}L(u)du\,\rho\left(x_{k+1}\right)+\int_0^{z_k}M(u)du\,\rho\left(x_{k+1}\right)\right\}\\
&= Q_k\left\{\frac{1}{8\rho\left(x_{k+1}\right)^3}\int_0^{\rho(x_{k+1})}N(u)\left(\rho\left(x_{k+1}\right)-u\right)^2du\rho\left(x_{k+1}\right)^3\right.\\
&\quad\left.+\frac{2}{z_k}\int_0^{z_k/2}L(u)du\,\rho\left(x_{k+1}\right)z_k/2+\frac{1}{z_k}\int_0^{z_k}M(u)du\,\rho\left(x_{k+1}\right)z_k\right\}\\
&\le Q_0\left\{\frac{1}{8\rho\left(x_0\right)^3}\int_0^{\rho(x_0)}N(u)\left(\rho\left(x_0\right)-u\right)^2du\rho\left(x_{k+1}\right)^3\right.\\
&\quad\left.+\frac{2}{z_0}\int_0^{z_0/2}L(u)du\,\rho\left(x_{k+1}\right)z_k/2+\frac{1}{z_0}\int_0^{z_0}M(u)du\,\rho\left(x_{k+1}\right)z_k\right\}\\
&\le J\rho\left(x_{k+1}\right)^3+E\left(\rho\left(x_k\right)+\rho\left(y_k\right)+\rho\left(x_{k+1}\right)\right)\rho\left(x_{k+1}\right)\\
&< \left[Jr_0^2+E\left(2r_0+r_1\right)\right]\rho\left(x_{k+1}\right)=\frac{r_1}{r_0}\rho\left(x_{k+1}\right)<r_1,\end{aligned}\tag{18}$$

where $z_k=\rho\left(x_k\right)+\rho\left(y_k\right)+\rho\left(x_{k+1}\right)$.

In particular, sequence $\{\|x_k-x^*\|\}$ according to (17) monotonically converges to a limit value a, $0\le a<r_0$. From inequality (17) comes $a\le Ja^3+Car_1$. For $a\ne 0$ we receive a contradiction

$$1\le Ja^2+Cr_1<Jr_0^2+Cr_1=1.$$

From this we receive: $\lim_{k\to\infty}x_k=x^*=\lim_{k\to\infty}y_k$.

We set $a_k=\rho\left(x_k\right),\ b_k=\rho\left(y_k\right),\ k=0,1,2,\ldots$. From inequalities (17), (18) we get

$$a_{k+1}\le Ja_k^3+Ca_kb_k,\quad k=0,1,2,\ldots,\tag{19}$$

$$\begin{aligned}b_{k+1} &\le a_{k+1}\min\left\{\frac{r_1}{r_0},\ Ja_{k+1}^2+E\left(a_k+a_{k+1}+b_k\right)\right\}\\
&\le a_{k+1}\min\left\{\frac{r_1}{r_0},\ (2E+Ja_k)a_k+Eb_k\right\}\\
&\le a_{k+1}\min\left\{\frac{r_1}{r_0},\ \left(2E+Jr_0+E\frac{r_1}{r_0}\right)a_k\right\}\le\left(Jr_0+E\left(2+\frac{r_1}{r_0}\right)\right)a_{k+1}a_k.\end{aligned}\tag{20}$$

From the estimations (19) and (20) for big enough k with a certain positive constant C_1 follows $a_{k+1}\le C_1a_k^2a_{k-1}$.

From the last inequality we obtain the equation for determining the order of convergence $\rho^2-2\rho-1=0$, positive root of which $\rho^*=1+\sqrt{2}$ and is the order of the convergence of (6).

4. The uniqueness ball for the solution of equations

The uniqueness ball for the solution is defined in the following theorem.

Theorem 2. *Let us assume that $H(x^*)\equiv F(x^*)+G(x^*)=0$, F has a continuous derivative in $B\left(x^*,r\right)$, $H'\left(x^*\right)^{-1}$ exists, and F' satisfies the Lipschitz condition with L average*

$$\left\|H'\left(x^*\right)^{-1}\left(F'(x)-F'(y)\right)\right\|\le\int_0^{\|x-y\|}L(u)du,\quad\forall x,y\in B\left(x^*,r\right),\tag{21}$$

the divided difference $\delta G(x,y)$ satisfies the Lipschitz condition with M average

$$\left\|H'\left(x^*\right)^{-1}\left(\delta G(x,y)-\delta G(u,v)\right)\right\|\le\int_0^{\|x-u\|+\|y-v\|}M(u)du,\quad\forall x,y,u,v\in B\left(x^*,r\right),\tag{22}$$

where $\rho(x)=\|x-x^*\|$ and L and M are the positive integrable functions. Let r satisfy

$$\frac{1}{r}\int_0^r(r-u)L(u)du+\int_0^rM(u)du\le 1.\tag{23}$$

Then the equation $H(x)=0$ has a unique solution x^* in $B\left(x^*,r\right)$.

Proof. Let us choose arbitrarily $x_0 \in B(x^*, r)$ and consider the iteration

$$x_{k+1} = x_k - H'(x^*)^{-1}(F(x_k) + G(x_k)), \quad k = 0, 1, 2, \ldots. \tag{24}$$

We obtain

$$\begin{aligned} x_1 - x^* &= x_0 - x^* - H'(x^*)^{-1}(F(x_0) + G(x_0)) \\ &= H'(x^*)^{-1}\left[H'(x^*)\left(x_0 - x^*\right) - F(x_0) + F(x^*) - G(x_0) + G(x^*)\right] \\ &= H'(x^*)^{-1}\left[F'(x^*)\left(x_0 - x^*\right) - F(x_0) + F(x^*) + \delta G(x^*, x^*)\left(x_0 - x^*\right) - G(x_0) + G(x^*)\right] \\ &= H'(x^*)^{-1}\left\{F'(x^*)\left(x_0 - x^*\right) - F(x_0) + F(x^*) + \left[\delta G(x^*, x^*) - \delta G(x_0, x^*)\right]\left(x_0 - x^*\right)\right\}. \end{aligned}$$

Let us estimate the rate of expression $H'(x^*)^{-1}\left\{F'(x^*)(x_0 - x^*) - F(x_0) + F(x^*)\right\}$.

$$\begin{aligned} \left\|H'(x^*)^{-1}\left\{F'(x^*)\left(x_0 - x^*\right) - F(x_0) + F(x^*)\right\}\right\| &\le \left\|\int_0^1 H'(x^*)^{-1}\left\{F'\left(x^* + t\left(x_0 - x^*\right)\right) - F'(x^*)\right\}\left(x_0 - x^*\right) dt\right\| \\ &\le \int_0^1 \int_0^{t\|x_0 - x^*\|} L(u)du \left\|x_0 - x^*\right\| dt \\ &\le \int_0^{\|x_0 - x^*\|} \left(1 - \frac{u}{\|x_0 - x^*\|}\right) L(u)du \left\|x_0 - x^*\right\| \\ &= \int_0^{\rho(x_0)} \left(\rho(x_0) - u\right) L(u)du. \end{aligned}$$

Then

$$\left\|x_1 - x^*\right\| \le \left(\frac{1}{\rho(x_0)} \int_0^{\rho(x_0)} \left(\rho(x_0) - u\right) L(u)du + \int_0^{\rho(x_0)} M(u)du\right) \left\|x_0 - x^*\right\| = q_0 \left\|x_0 - x^*\right\|, \tag{25}$$

where

$$q_0 = \frac{1}{\rho(x_0)} \int_0^{\rho(x_0)} \left(\rho(x_0) - u\right) L(u)du + \int_0^{\rho(x_0)} M(u)du < \frac{1}{r} \int_0^r (r - u) L(u)du + \int_0^r M(u)du \le 1.$$

According to (25)

$$\left\|x_1 - x^*\right\| \le q_0 \left\|x_0 - x^*\right\|.$$

Thus, the iteration (24) can be continued infinitely, and

$$\left\|x_k - x^*\right\| \le q_0^k \left\|x_0 - x^*\right\|, \quad k = 1, 2, \ldots.$$

Therefore, $\lim_{k\to\infty} x_k = x^*$. But if $H(x_0) = 0$, then from (24) $x_k = x_0$. So from this follows $x_0 = x^*$.

Let us denote that having set in (4) $G(x) \equiv 0$, from Theorem 1 we obtain Theorem 1 from [14], and from Theorem 2—Theorem 4.1 with [11] for Newton method, and having set in (4) $F(x) \equiv 0$, we obtain from Theorem 1 a corresponding Theorem 1 from [13] for method (3), but with more precise and obvious estimations. With Lipschitz constants we obtain the corresponding theorems from the works [6,10,18].

Remark. In our papers [18,19] we have described the results of numerical experiments for the methods of solving the nonlinear equation systems that contain differentiable and nondifferentiable parts. In particular, in [19] there have been presented results for the one-step methods, and in [18] there have been investigated a two-step method (6) and a comparison of it with the one-step methods (5). The obtained numerical results show that the combined methods (5) and (6) are more efficient than the difference methods. Methods that use only the value of the derivative of the differentiable part are not efficient, especially in the case of nonlinearity of the nondifferentiable operator part. The combined method (6) has a greater advantage over the combined one-step methods (5) in the case of finding a solution with high accuracy and for the worse initial approximation.

5. Conclusions

In this work it is investigated the local convergence of the combined method (6) for solving nonlinear operator equations under the generalized Lipschitz conditions for derivatives of the first and second order and divided differences of the first order, in which instead of Lipschitz constants some positive integrable functions are being used. The uniqueness ball of solution is established. In partial cases the received results of local convergence contain the results obtained in the works of other authors.

Acknowledgments

The author would like to thank the referees for their careful reading and valuable suggestions which have considerably improved the original manuscript of this paper.

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