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A New Family of Marshall-Olkin Extended Distributions

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Abstract

The purpose of this paper is to introduce, discuss and analyze a new family of Marshall-Olkin extended distributions. This new family generalizes the following Marshall-Olkin extended distributions: (i) exponential, (ii) Rayleigh, (iii) linear failure rate, and (iv) Weibull. Some statistical and reliability properties of the new family are discussed. The maximum likelihood estimates of its unknown parameters are obtained. The obtained results are validated using a real data set and it is shown that the new family provide a better fit than some other known distributions.

Key words: Marshall-Olkin, Weibull distribution, Linear failure rate distribution, Rayleigh distribution, Exponential distribution, Maximum likelihood method.

1 Introduction

Many of researchers are interested in search that introduce new families of distributions or generalized some of the presented distributions which can be used to describe the lifetimes of some devices or to describe sets of real data. Exponential, Rayleigh, Weibull and linear failure rate are some of the important distributions widely used in reliability theory and survival analysis. However, these distributions have a limited range of applicability and cannot represent all situations found in applications. For example, although the exponential distribution is often described as flexible, its hazard function is constant. The limitations of standard distributions often arouse the interest of researchers in finding new distributions by extending existing ones. The procedure of expanding a family of distributions for added flexibility or constructing covariates models is a well-known technique in the literature. For instance, the family of Weibull distributions contains exponential distribution and is constructed by taking powers of exponentially distributed random variables.

Marshall and Olkin [1] introduced a new method of adding a parameter into a family of distributions. The resulting distribution, known as Marshall-Olkin (M-O) extended distribution, includes the baseline distribution as a special case and gives more flexibility to model various types of data. According to them if $\bar{F}(x)$ denote the survival or reliability function of a continuous random variable X , then the timely honored device of adding a new parameter results in another survival function (SF) $\bar{G}(x)$ defined by

$$\bar{G}(x; \alpha) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha} \bar{F}(x)}, \quad -\infty \leq x \leq \infty, \quad \alpha \geq 0, \quad \bar{\alpha} = 1 - \alpha. \quad (1)$$

The M-O extended distributions offer a wide range of behavior than the basic distributions from which they are derived. The property that the extended form of distributions can have an interesting hazard function depending on the value of the added parameter and therefore can be used to model real situations in a better manner than the basic distribution (cf. [1] - [13]). Let $h(x)$ and $r(x)$ denotes the hazard rate functions of the transformed distribution and the original distribution, respectively. Marshall and Olkin [1] have called the additional shape parameter "tilt parameter", since the hazard rate of the new family is shifted below ($\alpha > 1$) or above ($0 < \alpha \leq 1$) the hazard rate of the underlying distribution, that is, for all $x \geq 0$, $h(x) \leq r(x)$ when $\alpha > 1$, and $h(x) \geq r(x)$ when $0 < \alpha \leq 1$.

Marshall and Olkin [1] introduced a two-parameter extension of the exponential [M-OEE (α, σ)] distribution with SF

$$\bar{G}(x; \alpha, \sigma) = \frac{\alpha}{e^{\sigma x} - \bar{\alpha}}, \quad x > 0, \quad \alpha, \sigma \geq 0;$$

three-parameter extension of the Weibull distribution [M-OEW (α, β, γ)] with SF

$$\bar{G}(x; \alpha, \beta, \gamma) = \frac{\alpha e^{-\beta x^\gamma}}{1 - \bar{\alpha} e^{-\beta x^\gamma}}, \quad x > 0, \quad \alpha, \beta, \gamma > 0,$$

and two parameter extension of the Rayleigh distribution [M-OER (α, σ, β)] with SF

$$\bar{G}(x; \alpha, \beta) = \frac{\alpha e^{-\beta x^2}}{1 - \bar{\alpha} e^{-\beta x^2}}, \quad x > 0, \quad \alpha, \sigma, \beta > 0.$$

In addition, based on M-O method, Ghitany and Kotz [2] introduced and studied extension of the linear failure rate distribution [M-OELFR (α, σ, β)] with SF

$$\bar{G}(x; \alpha, \sigma, \beta) = \frac{\alpha e^{-\sigma x - \beta x^2}}{1 - \bar{\alpha} e^{-\sigma x - \beta x^2}}, \quad x > 0, \quad \alpha, \sigma, \beta > 0.$$

All the M-O distributions mentioned above are special cases of the new family that is introduced and studied in Section 2 below. In Section 3, some statistical and reliability properties of the new family are discussed. In Section 4, the method of maximum likelihood estimation is used to estimate the unknown parameters. In addition, simulation is utilized to calculate the unknown shape parameter and to study its properties. Section 5 give some applications to explain how a real data set can be modeled by the new family. Finally, in Section 6, some conclusions and remarks of the current and future research are presented.

2 Density and hazard rate of the new family

Let X follows the modified Weibull (MW) distribution with SF

$$\bar{F}(x; \sigma, \beta, \gamma) = e^{-\sigma x - \beta x^\gamma}, \quad x > 0, \quad (2)$$

where $\gamma > 0$, $\sigma, \beta \geq 0$ such that $\sigma + \beta > 0$ (Sarhan and Zaindin [14]). Substituting (2) in (1) we get a new family distributions called M-O extended modified Weibull distributions and denoted as M-OEMW $(\alpha, \sigma, \beta, \gamma)$, with the following SF

$$\bar{G}(x; \alpha, \sigma, \beta, \gamma) = \frac{\alpha e^{-\sigma x - \beta x^\gamma}}{1 - \bar{\alpha} e^{-\sigma x - \beta x^\gamma}}, \quad x > 0, \quad \alpha, \gamma > 0, \quad \sigma, \beta \geq 0, \quad \sigma + \beta > 0.$$

The corresponding cumulative distribution function (CDF) and probability distribution function (PDF) are obtained, respectively as follows:

$$G(x; \alpha, \sigma, \beta, \gamma) = \frac{1 - e^{-\sigma x - \beta x^\gamma}}{1 - \bar{\alpha} e^{-\sigma x - \beta x^\gamma}},$$

and

$$g(x; \alpha, \sigma, \beta, \gamma) = \frac{\alpha(\sigma + \beta\gamma x^{\gamma-1})e^{-\sigma x - \beta x^\gamma}}{(1 - \bar{\alpha} e^{-\sigma x - \beta x^\gamma})^2},$$

where $x > 0$, $\alpha, \gamma > 0$, $\sigma, \beta \geq 0$, $\sigma + \beta > 0$.

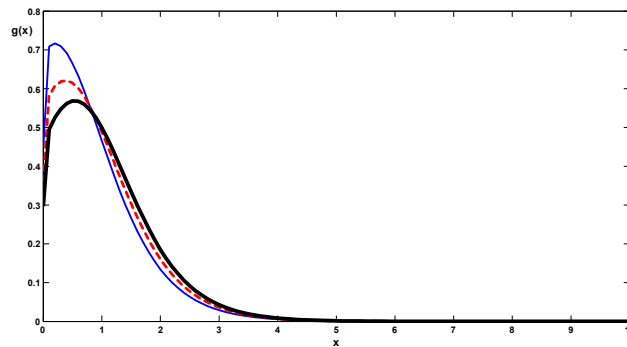


Figure 1. Plot of $g(x)$ for $\alpha = 2, 2.5, 3$ (plain, dashed, bold), $\sigma = 0.9$, $\beta = 0.6$, $\gamma = 1.1$

The next result provide the behavior of the PDF of the M-OEMW distribution and can be verified using elementary calculus.

Proposition 2.1.

Let $X \sim \text{M-OEMW}(\alpha, \sigma, \beta, \gamma)$, then X has

- (i) Increasing PDF provided $\bar{\alpha} < e^{\sigma x + \beta x^\gamma}$,
- (ii) Decreasing PDF provided $\bar{\alpha} > e^{\sigma x + \beta x^\gamma}$.

The following example shows that the new family generalizes the M-O extended distributions of exponential, Weibull, Rayleigh and linear failure rate.

Example 2.1.

Let $X \sim \text{M-OEMW}(\alpha, \sigma, \beta, \gamma)$, then

- (i) If $\beta = 0$, then $X \sim \text{M-OEE}(\alpha, \sigma)$;
- (ii) If $\sigma = 0$, then $X \sim \text{M-OEW}(\alpha, \beta, \gamma)$;
- (iii) If $\sigma = 0$ and $\gamma = 2$, then $X \sim \text{M-OER}(\alpha, \sigma, \beta)$;
- (iv) If $\gamma = 2$, then $X \sim \text{M-OELFR}(\alpha, \sigma, \beta)$.

The hazard rate (HR) and reversed hazard rate (RHR) functions of a lifetime random variable X with M-OEMW $(\alpha, \sigma, \beta, \gamma)$ distribution are given respectively, by

$$h(x; \alpha, \sigma, \beta, \gamma) = \frac{\sigma + \beta\gamma x^{\gamma-1}}{1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma}}, \quad x > 0,$$

and

$$r(x; \alpha, \sigma, \beta, \gamma) = \frac{\alpha(\sigma + \beta\gamma x^{\gamma-1})e^{-\sigma x - \beta x^\gamma}}{(1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma})(1 - e^{-\sigma x - \beta x^\gamma})}, \quad x > 0.$$

The following plots presents the possible shapes of the HR and the RHR of the M-OEMW distribution.

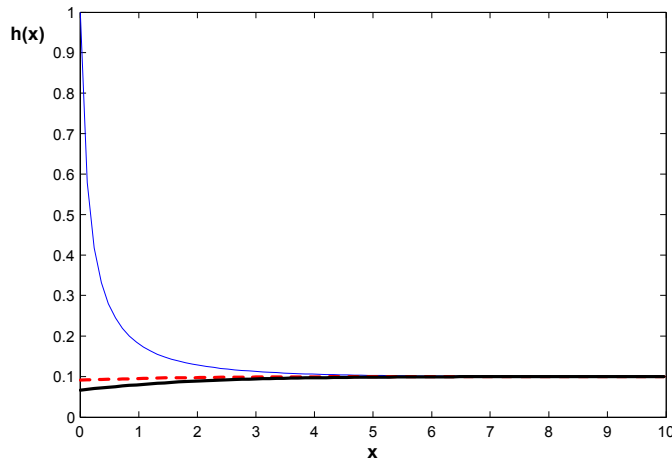


Figure 2. Plot of HR for $\alpha = 0.1, 1.1, 1.5$ (plain, dashed, bold), $\sigma = 0.2, \beta = 0.5, \gamma = 1$

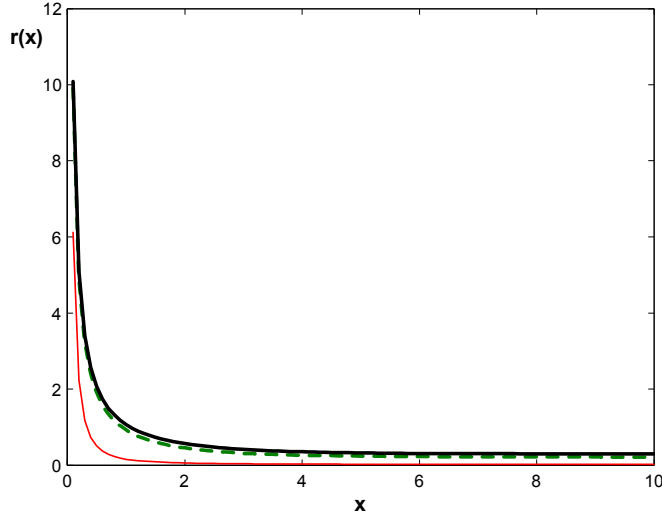


Figure 3. Plot of RHR for $\alpha = .1, 1.1, 1.5$ (plain, dashed, bold), $\sigma = 0.2, \beta = 0.5, \gamma = 1$

Remark 2.1.

It is observed that if $X \sim \text{M-OEMW}(\alpha, \sigma, \beta, \gamma)$, then X has:

- (i) Increasing HR provided $1 \leq \gamma < 3, \alpha > 1$, or $\gamma \geq 3$.
- (ii) Decreasing HR provided $\gamma < 1, \alpha < 1.3$, or $\gamma = 1, \alpha < 1$.

3 Reliability and statistical properties

In the literature, one can find many characterization of various lifetime distributions. These characterizations are meaningful because they throw a new light on the understanding of the intrinsic meaning of the reliability properties that are involved (cf. [15] - [18]). In this section, we study some statistical and reliability properties of this new family.

3.1 Mean residual life

The expected additional lifetime given that a component has survived until time t , called mean residual life (MRL). The MRL function is very important in reliability and survival analysis because it describes the aging process. It is well-known that the MRL uniquely determine the distribution function, i.e. it has all the information about the model. If the random variable X represents the life of a component, then the MRL is given by

$$\mu_T(t) = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(t)(x) dx, t \geq 0.$$

Although the shape of the HR function plays an important role, the MRL function is found to be more relevant than the HR function because the former summarizes the entire residual life

function, whereas the latter considers only the risk of instantaneous failure at some time t . The MRL function of a lifetime random variable X with M-OEMW $(\alpha, \sigma, \beta, \gamma)$ is given by

$$\mu_T(t) = \frac{1 - \bar{\alpha}e^{-\sigma t - \beta t^\gamma}}{\alpha e^{-\sigma t - \beta t^\gamma}} \int_t^\infty \frac{\alpha e^{-\sigma x - \beta x^\gamma}}{1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma}} dx.$$

Table 1 displays the MRL function at point $t = 5$ for M-OEMW $(\sigma = 0.2, \beta = .5, \gamma = 1)$ and different choices of parameter α . It is noted that the MRL is generally increasing for increasing values of α .

Table 1. Mean residual life of M-OEMW

α	σ	β	γ	MRL at $t = 5$
0.3	0.2	0.5	1	1.41337
0.7	0.2	0.5	1	1.42208
1.5	0.2	0.5	1	1.43930
2.0	0.2	0.5	1	1.44993

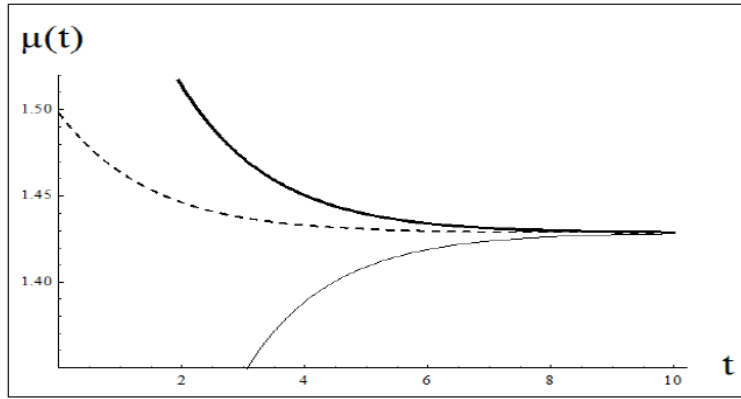


Figure 4. Plot of MRL for $\alpha = 0.1, 1.1, 1.5$ (plain, dashed, bold), $\sigma = 0.2, \beta = 0.5, \gamma = 1$

From the above figure, it observed that the MRL have different shape based on the values of α .

3.2 Mean inactivity time

The mean inactivity time (MIT) function is a well-known reliability measure which has applications in many disciplines such as reliability theory, survival analysis and actuarial studies. Let X be a lifetime random variable with distribution function F . The MIT function of X is defined by:

$$m(t) = \frac{1}{F(t)} \int_0^t F(x) dx, \quad t > 0.$$

The MIT is important characteristic in many applications to describe the time, which had elapsed since the failure. The MIT function of a lifetime random variable X with M-OEMW $(\alpha, \sigma, \beta, \gamma)$ distribution is:

$$m(t) = \frac{1 - \bar{\alpha}e^{-\sigma t - \beta t^\gamma}}{1 - e^{-\sigma t - \beta t^\gamma}} \int_0^t \frac{1 - e^{-\sigma x - \beta x^\gamma}}{1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma}} dx$$

Table 2 displays the MIT at the point $t = 5$ for M-OEMW ($\sigma = 0.2$, $\beta = .5$, $\gamma = 1$) and different choices of parameter α . It is noted that the MIT is generally increasing for decreasing values of α .

Table 2. Mean inactivity time of M-OEMW

α	σ	β	γ	MIT at $t=5$
2.0	0.2	0.5	1	3.29792
1.5	0.2	0.5	1	3.48189
0.7	0.2	0.5	1	3.92515
0.3	0.2	0.5	1	4.31590

Due to economic consequences and safety issues, it is necessary for the industry to perform systematic studies using reliability concepts. There exist plenty of scenarios where a statistical comparison of reliability measures is required in both reliability engineering and biomedical fields. Recently, Kayid and Izadkhah [16] introduced a new reliability function called strong mean inactivity time (SMIT) function defined as

$$M_X(t) = \frac{\int_0^t 2x F(x) dx}{F(t)}, \quad t > 0.$$

The SMIT function of a lifetime random variable X with M-OEMW ($\alpha, \sigma, \beta, \gamma$) is

$$M_X(t) = \frac{2(1 - \bar{\alpha}e^{-\sigma t - \beta t^\gamma})}{1 - e^{-\sigma t - \beta t^\gamma}} \int_0^t \frac{x - xe^{-\sigma x - \beta x^\gamma}}{1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma}} dx.$$

Table 3 displays the SMIT function at $t = 5$ for M-OEMW at $\sigma = 0.2$, $\beta = 0.5$, $\gamma = 1$ and different choices of parameter $\alpha > 1$. It is noted that the strong mean inactivity time is generally decreasing for increasing values of α .

Table 3. Strong mean inactivity time of M-OEMW

α	σ	β	γ	SMIT at $t=5$
1.1	0.2	0.5	1	21.954
2.0	0.2	0.5	1	20.5876
2.5	0.2	0.5	1	19.9985

3.3 Mean and variance

Let $X \sim \text{M-OEMW}(\alpha, \sigma, \beta, \gamma)$, then the mean and the variance of X are given, respectively by

$$E(X) = \int_0^\infty \frac{x\alpha(\sigma + \beta\gamma x^{\gamma-1})e^{-\sigma x - \beta x^\gamma}}{(1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma})^2} dx$$

and

$$\begin{aligned} Var(X) = & \int_0^\infty \frac{x^2\alpha(\sigma + \beta\gamma x^{\gamma-1})e^{-\sigma x - \beta x^\gamma}}{(1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma})^2} dx \\ & - \left[\int_0^\infty \frac{x\alpha(\sigma + \beta\gamma x^{\gamma-1})e^{-\sigma x - \beta x^\gamma}}{(1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma})^2} dx \right]^2 \end{aligned}$$

In general, the last integrals cannot be given explicitly in terms of $\alpha, \sigma, \beta, \gamma$. The variance of M-OEMW is shown graphically in the following figure for $\sigma = 0.2, \beta = 0.5, \gamma = 1$:

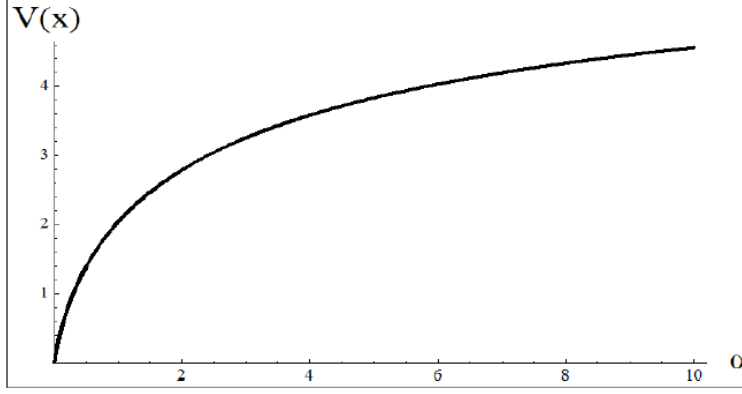


Figure 5. Plot of the variance of M-OEMW for different value of α and $\sigma = 0.2, \beta = 0.5, \gamma = 1$

From the above plot, it is noted that the variance increasing for increasing values of α .

3.4 Median

Let $X \sim \text{M-OEMW}(\alpha, \sigma, \beta, \gamma)$. The median of this distribution is the value d which satisfies

$$\int_0^d \frac{\alpha(\sigma + \beta\gamma x^{\gamma-1})e^{-\sigma x - \beta x^\gamma}}{(1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma})^2} dx = 0.5$$

Table 4 displays the median for M-OEMW at $\sigma = 0.2, \beta = 0.5, \gamma = 1$ and different choices of parameter α .

Table 4. Median of M-OEMW

α	σ	β	γ	Median
0.3	0.2	0.5	1	0.374806
0.7	0.2	0.5	1	0.75804
1.5	0.2	0.5	1	1.30899
2.0	0.2	0.5	1	1.56945

It is noted that the median is generally increasing for increasing the values of α .

3.5 Renyi entropy

Entropy has been used in various situations in science and engineering. The entropy of a random variable X with density function $g(x)$ is a measure of variation of the uncertainty. The Renyi entropy of order δ is defined by

$$H_\delta = \frac{1}{1-\delta} \log \left(\int_{-\infty}^{\infty} g(x)^\delta dx \right), \quad \delta \geq 0, \delta \neq 1.$$

Let $X \sim \text{M-OEMW}(\alpha, \sigma, \beta, \gamma)$, the corresponding Renyi entropy is obtained as

$$H_\delta = \frac{1}{1-\delta} \log \left(\int_0^\infty \left[\frac{\alpha(\sigma + \beta\gamma x^{\gamma-1})e^{-\sigma x - \beta x^\gamma}}{(1 - \bar{\alpha}e^{-\sigma x - \beta x^\gamma})^2} \right]^\delta dx \right), \quad \delta \geq 0, \delta \neq 1.$$

Table 5 displays the Renyi entropy for M-OEMW at $\delta = 3$, $\sigma = 0.2$, $\beta = 0.5$, $\gamma = 1$ and different choices of parameter $\alpha > 1$. It is noted that the Renyi entropy is generally increasing for increasing values of α .

Table 5. Renyi entropy of M-OEMW

α	σ	β	γ	Renyi entropy
1.1	0.2	0.5	1	0.97642
1.5	0.2	0.5	1	1.18997
2.0	0.2	0.5	1	1.36413
2.5	0.2	0.5	1	1.48199
3.0	0.2	0.5	1	1.56686

3.6 Quantile function

The quantile function is obtained by solving:

$$G[\varphi(u)] = u,$$

where $G(\cdot)$ is the CDF of M-OEMW ($\alpha, \sigma, \beta, \gamma$). In this case, the quantile function is the solution of the nonlinear equation

$$\beta\varphi(u)^\gamma + \sigma\varphi(u) + \log\left[\frac{1-u}{1-\alpha u}\right] = 0$$

3.7 Stochastic orderings

Stochastic orders have been used during the last forty years, at an accelerated rate, in many diverse areas of probability and statistics. Such areas include reliability theory, queuing theory, survival analysis, biology, economics, insurance and actuarial science (cf. Shaked and Shanthikumar [19]). Let X and Y be two random variables having distribution functions F and G , respectively, and denote by $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ their respective survival functions, with corresponding probability densities f, g . The random variable X is said to be smaller than Y in the:

(i) stochastic order (denoted as $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x ;

(ii) likelihood ratio order (denoted as $X \leq_{lr} Y$) if

$$f(x)/g(x) \text{ is decreasing in } x \geq 0;$$

(iii) hazard rate order (denoted as $X \leq_{hr} Y$) if

$$\bar{F}(x)/\bar{G}(x) \text{ is decreasing in } x \geq 0;$$

(iv) reversed hazard rate order (denoted as $X \leq_{rhr} Y$) if

$$F(x)/G(x) \text{ is decreasing in } x \geq 0;$$

The four stochastic orders defined above are related to each other, as the following implications (Shaked and Shanthikumar [19]):

$$X \leq_{rhr} Y \Leftarrow X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y. \quad (3)$$

The next theorem shows that the M-OEMW distributions are ordered with respect to the strongest likelihood ratio ordering when appropriate assumptions are satisfied.

Theorem 2.3.

Let $X \sim \text{M-OEMW}(\alpha_1, \sigma, \beta, \gamma)$ and let $Y \sim \text{M-OEMW}(\alpha_2, \sigma, \beta, \gamma)$. If $\alpha_1 < \alpha_2$, then

$$X \leq_{lr} Y \quad (X \leq_{hr} Y, X \leq_{rhr} Y, X \leq_{st} Y).$$

Proof.

First note that

$$\frac{f(x)}{g(x)} = \frac{\alpha_1}{\alpha_2} \left[\frac{1 - \bar{\alpha}_2 e^{-\sigma x - \beta x^\gamma}}{1 - \bar{\alpha}_1 e^{-\sigma x - \beta x^\gamma}} \right]^2.$$

Since, $\alpha_1 < \alpha_2$,

$$\begin{aligned} \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] &= 2 \frac{\alpha_1}{\alpha_2} (\alpha_1 - \alpha_2) \frac{(\sigma + \beta \gamma x^{\gamma-1}) e^{-\sigma x - \beta x^\gamma} (1 - \bar{\alpha}_2 e^{-\sigma x - \beta x^\gamma})}{(1 - \bar{\alpha}_1 e^{-\sigma x - \beta x^\gamma})^3} \\ &< 0. \end{aligned}$$

Hence, $f(x)/g(x)$ is decreasing in x . That is $X \leq_{lr} Y$. The remaining statements follows from the implications (3).

4 Maximum likelihood estimators

Let X_1, \dots, X_n be a random sample from M-OEMW $(\alpha, \sigma, \beta, \gamma)$ distribution, the likelihood function is given by:

$$L(X_1, \dots, X_n | \alpha, \sigma, \beta, \gamma) = \frac{\alpha^n \left[\prod_{i=1}^n (\sigma + \beta \gamma x_i^{\gamma-1}) \right] \exp \left(-\sigma \sum_{i=1}^n x_i - \beta \sum_{i=1}^n x_i^\gamma \right)}{\prod_{i=1}^n (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2}$$

The logarithm of the likelihood function is then

$$\begin{aligned} \ell(X_1, \dots, X_n | \alpha, \sigma, \beta, \gamma) &= n \ln \alpha - \sigma \sum_{i=1}^n x_i - \beta \sum_{i=1}^n x_i^\gamma + \sum_{i=1}^n \ln(\sigma + \beta \gamma x_i^{\gamma-1}) \\ &\quad - 2 \sum_{i=1}^n \ln(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}) \end{aligned}$$

The maximum likelihood estimators (MLEs) of α, σ, β and γ can be obtained by solving the following nonlinear equations:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{e^{-\sigma x_i - \beta x_i^\gamma}}{1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}} = 0 \quad (4)$$

$$\frac{\partial \ell}{\partial \sigma} = -\sum_{i=1}^n x_i + \sum_{i=1}^n \frac{1}{\sigma + \beta \gamma x_i^{\gamma-1}} - 2 \sum_{i=1}^n \frac{\bar{\alpha} x_i e^{-\sigma x_i - \beta x_i^\gamma}}{1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}} = 0 \quad (5)$$

$$\frac{\partial \ell}{\partial \beta} = -\sum_{i=1}^n x_i^\gamma + \sum_{i=1}^n \frac{\gamma x_i^{\gamma-1}}{\sigma + \beta \gamma x_i^{\gamma-1}} - 2 \sum_{i=1}^n \frac{\bar{\alpha} x_i^\gamma e^{-\sigma x_i - \beta x_i^\gamma}}{1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}} = 0 \quad (6)$$

$$\begin{aligned} \frac{\partial \ell}{\partial \gamma} &= -\beta \sum_{i=1}^n x_i^\gamma \ln x_i + \sum_{i=1}^n \frac{\beta \gamma x_i^{\gamma-1} \ln x_i + \beta x_i^{\gamma-1}}{\sigma + \beta \gamma x_i^{\gamma-1}} \\ &\quad - 2 \sum_{i=1}^n \frac{\beta \bar{\alpha} x_i^\gamma \ln x_i e^{-\sigma x_i - \beta x_i^\gamma}}{1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}} = 0 \end{aligned} \quad (7)$$

There is no explicit solution for Eqs. (4) - (7), so they need to be solved numerically. For a given known scale parameter ($\sigma = 1.5$) and known shape parameter ($\gamma = 3$), 1000 different samples are simulated from M-OEMW with different sizes and different values of the scale parameter α . We studied the behavior of the MLEs from unknown scale parameter α and shape parameter β . The values of α are taken as 0.5, 1.5 and 3, while the value of β is 2. Tables 6 and 7 below represent MLEs of parameter α and β , respectively.

Table 6. MLE of the parameter α

α	n	Estimate	Bias	MSE
0.5	30	0.553379	0.0533785	0.0400819
	50	0.531094	0.0310937	0.0225253
	70	0.520772	0.0207705	0.0152253
	150	0.512701	0.0127011	0.0057691
1.5	30	1.103151	-0.3685012	0.7163073
	50	1.160290	-0.3397093	0.5184824
	70	1.287741	-0.2122591	0.3447991
	150	1.513332	0.0133275	0.0596226
3.0	30	2.446983	-0.5530214	0.0158158
	50	2.857041	-0.1429615	0.0121988
	70	3.159194	0.1591872	0.0088691
	150	3.095032	0.0950291	0.0059484

Table 7. MLE of the parameter β

β	n	Estimate	Bias	MSE
$\alpha = 0.5$	30	1.216251	-0.783755	1.64312
	50	2.449001	0.449004	1.55223
	70	2.301832	0.301826	0.84236
	150	2.123334	0.123332	0.32487
$\alpha = 1.5$	30	1.139052	-0.860949	1.57492
	50	1.772512	-0.227505	0.69889
	70	1.897883	-0.102124	0.42643
	150	2.052381	0.052378	0.12237
$\alpha = 3.0$	30	1.685843	-0.314156	0.83074
	50	2.857044	-0.142962	0.43348
	70	2.097091	0.097090	0.20481
	150	2.041753	0.041746	0.08719

From Tables 6 and 7 it is observed that the mean square error and the absolute value of bias for the estimates of α and β are decreasing when the sample size n is increasing.

The second derivatives of the log likelihood function of M-OEMW with respect to α , σ , β and γ are given by

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2} + 2 \sum_{i=1}^n \frac{e^{-2\sigma x_i - 2\beta x_i^\gamma}}{(1 - \alpha e^{-\sigma x_i - \beta x_i^\gamma})^2}.$$

$$\frac{\partial^2 \ell}{\partial \sigma^2} = - \sum_{i=1}^n \frac{1}{(\sigma + \beta \gamma x_i^{\gamma-1})^2}$$

$$+ 2 \sum_{i=1}^n \frac{\bar{\alpha} x_i^2 e^{-2\sigma x_i - 2\beta x_i^\gamma} + \bar{\alpha} x_i^2 e^{-\sigma x_i - \beta x_i^\gamma} (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2}$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = - \sum_{i=1}^n \frac{\gamma^2 x_i^{2(\gamma-1)}}{(\sigma + \beta \gamma x_i^{\gamma-1})^2}$$

$$+ 2 \sum_{i=1}^n \left\{ \frac{\bar{\alpha} x_i^{2\gamma} e^{-\sigma x_i - \beta x_i^\gamma} (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2} + \frac{\bar{\alpha}^2 x_i^{2\gamma} e^{-2\sigma x_i - 2\beta x_i^\gamma}}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2} \right\}.$$

$$\frac{\partial^2 \ell}{\partial \gamma^2} = -\beta \sum_{i=1}^n x_i^\gamma (\ln x_i)^2$$

$$+ \sum_{i=1}^n \frac{1}{(\sigma + \beta \gamma x_i^{\gamma-1})^2} \left\{ (\sigma + \beta \gamma x_i^{\gamma-1}) (\beta \gamma x_i^{\gamma-1} (\ln x_i)^2 + 2\beta x_i^{\gamma-1} \ln x_i) - (\beta \gamma x_i^{\gamma-1} \ln x_i + \beta x_i^{\gamma-1})^2 \right\}$$

$$- 2 \sum_{i=1}^n \frac{1}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2} \left\{ (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}) \times \right.$$

$$\left. (\bar{\alpha} \beta x_i^\gamma (\ln x_i)^2 e^{-\sigma x_i - \beta x_i^\gamma} - \bar{\alpha} (\beta x_i^\gamma \ln x_i)^2) e^{-\sigma x_i - \beta x_i^\gamma} - (\bar{\alpha} \beta x_i^\gamma e^{-\sigma x_i - \beta x_i^\gamma} \ln x_i)^2 \right\}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \sigma} = 2 \sum_{i=1}^n \frac{(x_i e^{-\sigma x_i - \beta x_i^\gamma}) (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}) + \bar{\alpha} x_i e^{-2\sigma x_i - 2\beta x_i^\gamma}}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2}.$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = 2 \sum_{i=1}^n \frac{(x_i^\gamma e^{-\sigma x_i - \beta x_i^\gamma}) (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}) + \bar{\alpha} x_i^\gamma e^{-2\sigma x_i - 2\beta x_i^\gamma}}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2}.$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \gamma} = 2 \sum_{i=1}^n \frac{(x_i^\gamma e^{-\sigma x_i - \beta x_i^\gamma} \beta \ln x_i) (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}) + \bar{\alpha} x_i^\gamma e^{-2\sigma x_i - 2\beta x_i^\gamma} \beta \ln x_i}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2}.$$

$$\frac{\partial^2 \ell}{\partial \sigma \partial \beta} = - \sum_{i=1}^n \frac{\gamma x_i^{\gamma-1}}{(\sigma + \beta \gamma x_i^{\gamma-1})^2} + 2 \sum_{i=1}^n \frac{\bar{\alpha}^2 x_i^{\gamma+1} e^{-2\sigma x_i - 2\beta x_i^\gamma} - (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}) \bar{\alpha} x_i^\gamma e^{-\sigma x_i - \beta x_i^\gamma}}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2}.$$

$$\frac{\partial^2 \ell}{\partial \sigma \partial \gamma} = - \sum_{i=1}^n \frac{\beta \gamma x_i^{\gamma-1} \ln x_i + \beta x_i^{\gamma-1}}{(\sigma + \beta \gamma x_i^{\gamma-1})^2} + 2 \sum_{i=1}^n \frac{\beta \bar{\alpha}^2 x_i^{\gamma+1} e^{-2\sigma x_i - 2\beta x_i^\gamma} \ln x_i + (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}) \bar{\alpha} x_i^{\gamma+1} \beta e^{-\sigma x_i - \beta x_i^\gamma} \ln x_i}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2}.$$

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \beta \partial \gamma} = & - \sum_{i=1}^n x_i^\gamma \ln x_i \\ & + \sum_{i=1}^n \frac{(\sigma + \beta \gamma x_i^{\gamma-1}) (\gamma x_i^{\gamma-1} \ln x_i + x_i^{\gamma-1}) - (\beta \gamma x_i^{\gamma-1} \ln x_i + \beta x_i^{\gamma-1}) (\gamma x_i^{\gamma-1})}{(\sigma + \beta \gamma x_i^{\gamma-1})^2} \\ & - 2 \sum_{i=1}^n \frac{1}{(1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma})^2} \times \left\{ (1 - \bar{\alpha} e^{-\sigma x_i - \beta x_i^\gamma}) \times \right. \\ & \left. (\bar{\alpha} x_i^\gamma e^{-\sigma x_i - \beta x_i^\gamma} \ln x_i - \bar{\alpha} \beta x_i^{2\gamma} e^{-\sigma x_i - \beta x_i^\gamma} \ln x_i) - (\bar{\alpha}^2 \beta x_i^{2\gamma} e^{-2\sigma x_i - 2\beta x_i^\gamma} \ln x_i) \right\}. \end{aligned}$$

If we denote the MLE of $\theta = (\alpha, \sigma, \beta, \gamma)$ by $\hat{\theta} = (\hat{\alpha}, \hat{\sigma}, \hat{\beta}, \hat{\gamma})$, then the observed information matrix is given by

$$I(\theta) = \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \sigma} & -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ell}{\partial \alpha \partial \gamma} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \sigma} & -\frac{\partial^2 \ell}{\partial \sigma^2} & -\frac{\partial^2 \ell}{\partial \sigma \partial \beta} & -\frac{\partial^2 \ell}{\partial \sigma \partial \gamma} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ell}{\partial \sigma \partial \beta} & -\frac{\partial^2 \ell}{\partial \beta^2} & -\frac{\partial^2 \ell}{\partial \beta \partial \gamma} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \gamma} & -\frac{\partial^2 \ell}{\partial \sigma \partial \gamma} & -\frac{\partial^2 \ell}{\partial \beta \partial \gamma} & -\frac{\partial^2 \ell}{\partial \gamma^2} \end{bmatrix},$$

and hence the variance covariance matrix would be $I^{-1}(\theta)$. The approximate $(1 - \delta)100\%$ confidence intervals (CIs) for the parameters α , σ , β and γ are $\hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{V(\hat{\alpha})}$, $\hat{\sigma} \pm Z_{\frac{\delta}{2}} \sqrt{V(\hat{\sigma})}$, $\hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{V(\hat{\beta})}$ and $\hat{\gamma} \pm Z_{\frac{\delta}{2}} \sqrt{V(\hat{\gamma})}$ respectively, where $V(\hat{\alpha})$, $V(\hat{\sigma})$, $V(\hat{\beta})$ and $V(\hat{\gamma})$ are the variances of $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\beta}$ and $\hat{\gamma}$, which are given by the diagonal elements of $I^{-1}(\theta)$, and $Z_{\frac{\delta}{2}}$ is the upper $(\delta/2)$ percentile of standard normal distribution.

5 Fitting reliability data

Here we provide a data set to show how the new model works in practice. The following data set represents the lifetimes of 50 devices (see Aarset [20]).

Data Set																
0.1	7	1	1	1	1	1	2	3	6	67	67	79	83	84	85	86
0.2	11	12	18	18	18	18	18	21	32	67	72	82	84	85	85	86
36	40	45	46	47	50	55	60	63	63	67	75	82	84	85	85	

Some properties of the data set were computed in Table 8.

Table 8. Some properties of data set

$E(X)$	$Var(X)$	Kurtosis	Skewness
45.686	1078.15	-1.61441	-0.136442

From the above table it is clear that the distribution of this data set is negatively skewed. The parameter of the sample is estimated numerically. We use Eqs. (4) - (7) to obtain MLEs estimate and the results are given in Table 9.

Table 9. MLE for data set

Parameter	MLE
α	2.660390
σ	0.002509
β	0.050309
γ	0.881773

The observed information of the data and the variance covariance matrices are respectively

$$I_0(\hat{\alpha}, \hat{\sigma}, \hat{\beta}, \hat{\gamma}) = \begin{bmatrix} 2.23018 & -784.217 & -528.088 & -86.8585 \\ -784.217 & 82925.7 & 19147. & 10170.4 \\ -528.088 & 19147.1 & 39158.2 & 10460.5 \\ -86.8585 & 10170.4 & 10460.5 & 2387.12 \end{bmatrix}$$

and

$$I_0^{-1}(\hat{\alpha}, \hat{\sigma}, \hat{\beta}, \hat{\gamma}) = \begin{bmatrix} 0.461930 & -0.0004975 & -0.0082997 & 0.055300 \\ -0.0004975 & 7.8722 \times 10^{-6} & -0.00001897 & 0.000032 \\ -0.0082997 & -0.000018971 & 0.00010565 & -0.000684 \\ 0.05530 & 0.000031495 & -0.0006842 & 0.005295 \end{bmatrix}$$

The 99% CIs for the parameters α , σ , β and γ are [0.909598, 4.41119], [0.00, 0.00973706], [0.023831, 0.0767884] and [0.694319, 1.06923], respectively.

To test how well the M-OEMW distribution fits these data, the hypotheses is $H_0 : F = F_{M-OEMW}$ versus $H_1 : F \neq F_{M-OEMW}$. We use the Kolmogorov-Smirnov ($K-S$) distances between the empirical distribution function and the fitted distribution function to determine the appropriateness of the model. The $K-S$ value and corresponding p -value are respectively 0.169299 and 0.230. The small $K-S$ distance, and the large p -value for the test indicate that these data fit the M-OEMW quite well. In addition, we use likelihood ratio test (LRT) to determine the appropriateness of the model. The hypotheses are as follows:

$$H_0 : \alpha = 1 \text{ (MW)} \quad \text{versus} \quad H_1 : \alpha \neq 1 \text{ (M-OEMW)}$$

Log-likelihood value, likelihood ratio statistic (Λ) and corresponding p value are respectively -238.595, 15.3029 and 0.0000915. We observed that the calculated LRT statistic is greater than the critical point for this test, which is 6.635, also the p -value is very small. According to the LRT, we conclude that this data fits the M-OEMW distribution much better than the MW distribution.

On the other hand, the linear failure rate distribution (LFRD) is a special case of MWD. We want to test if these data fit the M-OEMW or the LFRD, using the likelihood ratio test (LRT). The hypotheses are as follows:

$$H_0 : \alpha = 1, \gamma = 2 \text{ (LFRD)} \quad \text{versus} \quad H_1 : \alpha \neq 1, \gamma \neq 2 \text{ (M-OEMW)}$$

Log-likelihood value, likelihood ratio statistic (Λ) and corresponding p value are respectively -238.595, 15271.353 and 7.374×10^{-3317} . We observed that the calculated LRT statistic is greater than the critical point for this test, which is 9.210. Also, from the p -value it is clear that we reject the null hypotheses. Sarhan and Zaindin [14] have fitted MWD to this data and they concluded that this data fits the MWD much better than the LFRD by using LRT and obtained p -value as 1.047×10^{-4} . Hence if we compare between two tests and from corresponding p -value, we conclude that this data fits the M-OEMW distribution much better than the MW distribution.

6 Conclusion

In this paper we achieved two goals. The first one is to introduce and study a new family of M-O extended distributions and study its statistical and reliability properties. The second goal is to estimate the unknown parameters of the new family and provide some applications in the context of statistics and reliability. The obtained results are validated using a real data set and it is shown that the new family provide a better fit than some other known distributions. Further properties and applications of the new family can be considered in the future of this research. In particular the following topics are interesting, and still remain as open problems:

- (i) Discuss the Bayesian analysis of the new family.
- (ii) Introduce and study a new class of weighted M-O bivariate modified Weibull distribution.

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1. Figure 1. Plot of $g(x)$ for $\alpha=2, 2.5, 3$ (plain, dashed, bold), $\sigma=0.9$, $\beta=0.6$, $\gamma=1.1$
2. Figure 2. Plot of HR for $\alpha=0.1, 1.1, 1.5$ (plain, dashed, bold), $\sigma=0.2$, $\beta=0.5$, $\gamma=1$
3. Figure 3. Plot of RHR for $\alpha=.1, 1.1, 1.5$ (plain, dashed, bold), $\sigma=0.2$, $\beta=0.5$, $\gamma=1$
4. Figure 4. Plot of MRL for $\alpha=0.1, 1.1, 1.5$ (plain, dashed, bold), $\sigma=0.2$, $\beta=0.5$, $\gamma=1$
5. Figure 5. Plot of the variance of M-OEMW for different value of α and $\sigma=0.2$,
 $\beta=0.5$, $\gamma=1$