



A note on compound renewal risk models with dependence



Hélène Cossette^a, Etienne Larrivée-Hardy^a, Etienne Marceau^a, Julien Trufin^{b,*}

^a École d'Actuariat, Université Laval, Québec, Canada

^b Department of Mathematics, Université Libre de Bruxelles (ULB), Bruxelles, Belgium

ARTICLE INFO

Article history:

Received 20 August 2014

Received in revised form 11 February 2015

Keywords:

Ruin measures

Bivariate distributions

Copulas

Light-tailed claim distributions

Change of measure techniques

Importance sampling

ABSTRACT

Over the last decade, there have been a significant amount of research works on compound renewal risk models with dependence. These risk models assume a dependence relation between interclaim times and claim amounts. In this paper, we pursue their investigation. We apply change of measure techniques within the compound renewal risk models with dependence to obtain exact expressions for the Gerber–Shiu discounted penalty function. We propose a more general approach than the usual one based on the random walk associated to the risk process as it is presented in the literature. More refined, our method keeps the embedded information in the sequence of claim amounts and interclaim times and enables us to derive an exact expression for the Gerber–Shiu discounted penalty function. Simulation is one of the advantages of change of measure techniques since we can find a new probability measure under which ruin occurs almost surely. In this paper, we investigate the importance sampling method based on change of measure techniques to compute several ruin measures. Numerical illustrations are carried out for specific bivariate distributions of the interclaim time and the claim amount to approximate interesting ruin measures.

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1. Introduction

Over the last decade, there have been a significant amount of research works on compound renewal risk models with dependence. These risk models assume a dependence relation between interclaim times and claim amounts. We begin here with a brief description of the model (see e.g. [1–4] for details). For an insurance portfolio, the surplus process is defined by $\underline{U} = \{U(t), t \geq 0\}$ where the surplus level at time t , $U(t)$ is given by

$$U(t) = u + ct - S(t),$$

where $U(0) = u$ is the initial surplus and c is the premium rate. The aggregate claim amount process, denoted by $\underline{S} = \{S(t), t \geq 0\}$ with $S(t) = \sum_{j=1}^{N(t)} X_j$ (\sum_a^b equals 0 if $b < a$), is a compound renewal process. The claim number process $\underline{N} = \{N(t), t \in \mathbb{R}^+\}$ is an ordinary renewal process where the interclaim times $\{W_j, j \in \mathbb{N}^+\}$ form a sequence of independent and strictly positive real-valued random variables (rvs). The time between the $(j-1)$ th and the j th claim ($j = 2, \dots$) is defined by the r.v. W_j with W_1 the time of the first claim. The rvs $\{W_j, j \in \mathbb{N}^+\}$, identically distributed as the canonical r.v. W , have a probability density function (pdf) f_W and a cumulative distribution function (cdf) F_W . The time of arrival of the j th claim is denoted $T_j = W_1 + \dots + W_j$. The claim amount rvs $\{X_j, j \in \mathbb{N}^+\}$, where X_j corresponds to the amount of the j th claim, are assumed to be a sequence of strictly positive, independent and identically distributed (i.i.d.) rvs with pdf f_X and cdf F_X .

* Corresponding author.

E-mail address: julien.trufin@ulb.ac.be (J. Trufin).

In compound renewal risk models with dependence, $\{(X_j, W_j), j \in \mathbb{N}^+\}$ form a sequence of i.i.d. random vectors distributed as the canonical random vector (X, W) , in which the components may be dependent. The joint pdf of (X, W) is denoted by $f_{X,W}$ and the joint cdf is denoted by $F_{X,W}$. The associated moment generating function (mgf), denoted by

$$M_{X,W}(r_1, r_2) = E[e^{r_1 X} e^{r_2 W}] = \int_0^\infty \int_0^\infty e^{r_1 x} e^{r_2 t} f_{X,W}(x, t) dx dt,$$

is assumed to exist throughout the paper.

The time of ruin is defined by the rv $\tau_u = \inf\{t \geq 0 : U(t) < 0\}$ with $\tau_u = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. The infinite-time ruin probability is $\psi(u) = \Pr(\tau_u < \infty | U(0) = u)$. The classical Gerber–Shiu discounted penalty function is defined by

$$m_\delta(u) = E[e^{-\delta \tau_u} w(U(\tau_u^-), |U(\tau_u)|) 1_{\{\tau_u < \infty\}} | U(0) = u], \quad (1)$$

where w is the so-called penalty function that depends on the surplus immediately prior to ruin $U(\tau_u^-)$ and the deficit at ruin $|U(\tau_u)|$. The scalar δ is the force of interest assumed to be nonnegative. The function $1_{\{A\}}$ is the usual indicator function where $1_{\{A\}} = 1$ if the event A occurs and 0 otherwise. Throughout the paper, we assume the positive security loading condition $E[cW - X] > 0$ which ensures that ruin will not occur almost surely.

If $\delta = 0$ and $w(x, y) = 1$ for all $x, y \in \mathbb{R}^+$, (1) corresponds to the infinite-time ruin probability $\psi(u)$. Also, for $\delta \geq 0$ and $w(x, y) = 1$ for all $x, y \in \mathbb{R}^+$, (1) becomes $\bar{G}_\delta(u) = E[e^{-\delta \tau_u} 1_{\{\tau_u < \infty\}} | U(0) = u]$, using the notation of Cheung et al. [4]. Note that $\bar{G}_\delta(u)$ can be interpreted either as the present value of 1 paid at ruin or the Laplace transform of the time of ruin. If the penalty is only function of the deficit at ruin (which corresponds to the overshoot of the random walk V over the surplus u), we write $w(x, y) = w_2(y)$ and hence (1) becomes $m_{\delta,2}(u) = E[e^{-\delta \tau_u} w_2(|U(\tau_u)|) 1_{\{\tau_u < \infty\}} | U(0) = u]$, under the notation of Cheung et al. [4].

As mentioned in e.g. [1], it is possible to identify the random walk embedded in the risk process. We consider the sequence of i.i.d. rvs $\underline{L} = \{L_j, j \in \mathbb{N}^+\}$, where $L_j = X_j - cW_j$ is the net loss at the j th claim, with $L_j \sim L$, for $j \in \mathbb{N}^+$. The premium rate c is fixed such that $E[L] = E[X - cW] < 0$ and we define $\eta = \frac{cE[W]}{E[X]} - 1 > 0$ as the relative security loading. Based on \underline{L} , we denote by $\underline{V} = \{V_j, k \in \mathbb{N}\}$ the random walk with negative drift, where $V_0 = 0$ and $V_j = \sum_{l=1}^j L_l$, $j \in \mathbb{N}^+$. The maximum net cumulative loss process associated to \underline{V} is defined by $\underline{Z} = \{Z_j, j \in \mathbb{N}\}$, where $Z_j = \max_{l=0,1,2,\dots,j} \{V_l\}$. We introduce the rv $Z_\infty = \max_{l \in \mathbb{N}^+} \{V_l\}$. An alternative definition for the infinite-time ruin probability is then provided by

$$\psi(u) = E[1_{\{\tau_u < \infty\}}] = \Pr(Z_\infty > u) = E[1_{\{Z_\infty > u\}}] = 1 - F_{Z_\infty}(u),$$

where F_{Z_∞} corresponds to the cumulative distribution function (cdf) of Z_∞ .

Let us now define the rv $\sigma_u = \inf_{j \in \mathbb{N}^+} \{j, Z_j > u\}$ with $\sigma_u = \infty$ if $Z_j \leq u$ for all $j \geq 1$ (i.e. when ruin does not occur). The rv σ_u corresponds to the claim number at which ruin occurs. If $\sigma_u < \infty$, we have $\tau_u = T_{\sigma_u}$. Note that the deficit at ruin is also given by $|U(\tau_u)| = V_{\sigma_u} - u$ and the surplus prior to ruin is given by $U(\tau_u^-) = X_{\sigma_u} - V_{\sigma_u} + u$. It implies that the Gerber–Shiu discounted penalty function defined in (1) can also be expressed as

$$m_\delta(u) = E[e^{-\delta \tau_u} w(X_{\sigma_u} - V_{\sigma_u} + u, V_{\sigma_u} - u) 1_{\{\tau_u < \infty\}} | U(0) = u].$$

In the following, we also use the general Lundberg equation which is fundamental in ruin theory. Within compound renewal risk models with dependence, its expression is given by

$$h_\delta(r) = E[e^{rL - \delta W}] = E[e^{r(X - cW) - \delta W}] = M_{X,W}(r, -cr - \delta) = 1. \quad (2)$$

We denote by ρ_δ the strictly positive solution to (2), if it exists, called the (Lundberg) adjustment coefficient. This coefficient is crucial in ruin theory (see e.g. [5–7]), and it can be seen as a measure of dangerousness of an insurance portfolio. Also, ρ_δ is useful to obtain exponential inequalities, exact expressions and asymptotic expressions for ruin measures.

As previously mentioned, one can find a vast literature in regard to compound renewal risk models with dependence. Albrecher and Teugels [1] consider an arbitrary dependence structure based on a copula for (X, Y) . Assuming the existence of ρ_0 , they examine notably the exponential behavior of $\psi(u)$ and obtain asymptotic expression for $\psi(u)$. Cheung et al. [4] examine the structure and some properties for $m_\delta(u)$ and propose a generalization of $m_\delta(u)$. Boudreault et al. [8] examine several properties of an extension of the classical compound Poisson risk model assuming a dependence structure where the distribution of X is defined in terms of W . Cossette et al. [2,3] investigate the family of risk models proposed by Albrecher and Teugels [1] with a dependence structure for (X, W) defined with a (generalized) Farlie Gumbel Morgenstern copula. Badescu et al. [9] consider a bivariate phase-type distribution for (X, W) and obtain explicit results for $\psi(u)$ and some special cases of $m_\delta(u)$. Ambagaspitiya [10] obtains, by means of Wiener Hopf factorization techniques, the expressions for $\psi(u)$ in the cases of two classes of bivariate distributions (X, W) . For Erlang(n) inter arrival times and a dependence structure based on the Farlie Gumbel Morgenstern copula, Chadjiconstantinidis and Vrontos [11] derive notably the Laplace transform of $m_\delta(u)$ and explicit expressions for the discounted joint and marginal distribution functions of $U(\tau_u^-)$ and $|U(\tau_u)|$.

In this paper, we pursue the investigation of compound renewal risk models with dependence. We aim to use the change of measure techniques to derive an exact expression for $m_\delta(u)$. Rolski et al. [6], Pham [12], Asmussen and Albrecher [7] and Schmidli [13] discuss several advantages in regard to these techniques. To do so, we propose a more general approach

than the usual approach (referred to as the “random walk approach”), which relies on \underline{L} . For an exposition of the random walk approach, see e.g. [6,7,12,14] to obtain exact expression of $\psi(u)$ within the classical compound Poisson risk model and the classical compound renewal risk model. See also [15] for the application of the random approach to derive the expressions for $\psi(u)$ and $m_{0,2}(u)$ in the context of the compound renewal risk model with dependence. Our approach, more refined, keeps the embedded information in $\{(X_j, W_j), j \in \mathbb{N}^+\}$ and enables us to derive the announced expression for $m_\delta(u)$. Moreover, we explain that our approach can be extended to derive the exact expressions for extensions of $m_\delta(u)$ as the ones proposed in e.g. [4]. What we propose differs from the so-called Markov additive approach which is applied by Asmussen and Albrecher [7] and Schmidli [13] to derive the exact expression of $m_\delta(u)$ within the more restrictive compound Poisson risk model and compound renewal risk model, respectively. We believe that our approach is simpler.

Simulation is one of the advantages of change of measure techniques since we can find a new probability measure under which ruin occurs almost surely. Glasserman [16], Pham [12], Sigman [14], and Asmussen and Albrecher [7] have explained how to apply the importance sampling method based on change of measure techniques to compute $\psi(u)$ in the context of the classical compound Poisson risk model. Asmussen and Albrecher [7] have also applied it to evaluate $\psi(u)$ in the context of the compound renewal risk model. In this paper, we investigate the importance sampling method based on change of measure techniques to compute several ruin measures. The performance of this method is compared on a theoretical basis and also through numerous examples. These examples are based on chosen bivariate distributions for (X, W) , which have to be redefined under the new probability measure.

Our paper is organized as follows. In Section 2, we use change of measure techniques to obtain exact expressions for $m_\delta(u)$. Afterwards, in Section 3, we investigate importance sampling method to compute ruin measures and study its quality. Finally, in Section 4, we examine specific bivariate distributions for (X, W) , and we derive their corresponding bivariate distributions resulting from the change of measure. Numerical illustrations are also provided in Section 4 to illustrate the importance sampling method.

2. Change of measure

2.1. Preliminaries

Rolski et al. [6], Pham [12], Asmussen and Albrecher [7] and Schmidli [13] discuss several advantages in regards to the use of change of measure techniques. In addition to getting exact expressions for $\psi(u)$ and $m_\delta(u)$, they provide a natural way to find Lundberg exponential bounds and permit the use of ordinary renewal theory to derive asymptotic expressions for $\psi(u)$ and $m_\delta(u)$. Moreover, the computation of these expressions can be relatively simple using simulation as discussed in Section 4. Indeed, it offers the possibility to compute different ruin measures for a variety of bivariate distributions of (X, W) which may be difficult, even impossible, to find otherwise. For a review on change of measure techniques and their application to simulation see e.g. [6,16,7,12,14]. See also [13] for an application of change of measure techniques to the investigation of the Gerber–Shiu function in the context of the classical compound renewal risk model. Links to large deviations results can be found in e.g. [12].

With a change of measure technique based on the random walk \underline{V} and its increments \underline{L} , Rolski et al. [6], Glasserman [16], Asmussen and Albrecher [7], Pham [12], and Sigman [14] derive the expression for the ruin probability $\psi(u)$ within the classical compound risk model. Asmussen and Albrecher [7] also use this technique within the compound renewal risk model. Cossette et al. [15] briefly recall this approach and show that it can be used within compound renewal risk models with dependence as well. They use the change of measure technique based on the sequence \underline{L} of net losses to obtain the exact expressions for $\psi(u)$ and $m_{0,2}(u)$. This approach is however not refined enough for the derivation of $m_\delta(u)$. For that reason, we propose a second approach that broadens the scope of application to $m_\delta(u)$ with no restriction on the choice of penalty function $w(x, y)$. Indeed, as mentioned in Example 4.6.3 of Glasserman [16], the specific form of $L_k = X_k - cW_k$ is dropped under the random walk approach. Their components and dependence structure are also ignored. Inspired from the multivariate setting exposed in [16, in 4.6.1], we rely here on the sequence $\{(X_k, W_k), k \in \mathbb{N}^+\}$ of claim amounts and interclaim times allowing to keep track of both the claim amounts and the interclaim times, and not only the increments of the random walk \underline{L} . This will make easier the derivation of the exact expression of $m_\delta(u)$ under the new probability measure $\mathbb{P}^{(\rho_\delta)}$.

Let us assume the rvs $X_1, W_1, \dots, X_n, W_n$ to be continuous and the premium rate c to be equal to 1. Obviously, the joint pdf of $(X_1, W_1, \dots, X_n, W_n)$ is given by

$$f_{X_1, W_1, \dots, X_n, W_n}(x_1, t_1, \dots, x_n, t_n) = \prod_{i=1}^n f_{X_i, W_i}(x_i, t_i) = \prod_{i=1}^n f_{X, W}(x_i, t_i).$$

Our objective is to evaluate $E[\phi(X_1, W_1, \dots, X_n, W_n)]$, given by

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \int_0^\infty \phi(x_1, t_1, \dots, x_n, t_n) f_{X_1, W_1, \dots, X_n, W_n}(x_1, t_1, \dots, x_n, t_n) dx_1 dt_1 \dots dx_n dt_n, \quad (3)$$

where ϕ is a function for which the expectation exists.

We define under a new probability measure $\mathbb{P}^{(r)}$ the joint pdf of $(X_1, W_1, \dots, X_n, W_n)$ by

$$f_{X_1, W_1, \dots, X_n, W_n}^{(r)}(x_1, t_1, \dots, x_n, t_n) = e^{r \sum_{i=1}^n x_i - r \sum_{i=1}^n t_i - \delta \sum_{i=1}^n t_i - n \Gamma(r)} f_{X_1, W_1, \dots, X_n, W_n}(x_1, t_1, \dots, x_n, t_n), \quad (4)$$

where $\Gamma(r) = \ln E[e^{r(X-W)-\delta W}]$ assuming $\Gamma(r) < \infty$ for values of $r \neq 0$. Using (4), the expression in (3) becomes

$$\begin{aligned} & \int_0^\infty \int_0^\infty \dots \int_0^\infty \int_0^\infty \phi(x_1, t_1, \dots, x_n, t_n) f_{X_1, W_1, \dots, X_n, W_n}^{(r)}(x_1, t_1, \dots, x_n, t_n) dx_1 dt_1 \dots dx_n dt_n \\ &= \int_0^\infty \int_0^\infty \dots \int_0^\infty \int_0^\infty \phi(x_1, t_1, \dots, x_n, t_n) \frac{f_{X_1, W_1, \dots, X_n, W_n}^{(r)}(x_1, t_1, \dots, x_n, t_n)}{e^{r \sum_{i=1}^n x_i - r \sum_{i=1}^n t_i - \delta \sum_{i=1}^n t_i - n \Gamma(r)}} dx_1 dt_1 \dots dx_n dt_n \\ &= E^{(r)} \left[\phi(X_1, W_1, \dots, X_n, W_n) e^{-r \sum_{i=1}^n X_i + r \sum_{i=1}^n W_i + \delta \sum_{i=1}^n W_i + n \Gamma(r)} \right] \\ &= E^{(r)} [\phi(X_1, W_1, \dots, X_n, W_n) e^{-rV_n + \delta T_n + n \Gamma(r)}]. \end{aligned} \quad (5)$$

With this new approach, the likelihood ratio corresponds to

$$\begin{aligned} R(X_1, W_1, \dots, X_n, W_n) &= \frac{f_{X_1, W_1, \dots, X_n, W_n}^{(r)}(X_1, W_1, \dots, X_n, W_n)}{f_{X_1, W_1, \dots, X_n, W_n}(X_1, W_1, \dots, X_n, W_n)} \\ &= e^{r \sum_{i=1}^n X_i - r \sum_{i=1}^n W_i - \delta \sum_{i=1}^n W_i - n \Gamma(r)}. \end{aligned}$$

Given that σ_u is a stopping time and (5), we have

$$E[\phi(X_1, W_1, \dots, X_{\sigma_u}, W_{\sigma_u}) 1_{\{\sigma_u < \infty\}}] = E^{(r)}[\phi(X_1, W_1, \dots, X_{\sigma_u}, W_{\sigma_u}) e^{-rV_{\sigma_u} + \delta T_{\sigma_u} + \sigma_u \Gamma(r)} 1_{\{\sigma_u < \infty\}}]. \quad (6)$$

2.2. Main result

The following proposition provides the expression for $m_\delta(u)$ under $\mathbb{P}^{(\rho_\delta)}$.

Proposition 1. Assume that ρ_δ exists for $\delta \geq 0$. Then, we have

$$\begin{aligned} m_\delta(u) &= E^{(\rho_\delta)}[w(X_{\sigma_u} - V_{\sigma_u} + u, V_{\sigma_u} - u) e^{-\rho_\delta V_{\sigma_u}}] \\ &= e^{-\rho_\delta u} E^{(\rho_\delta)}[w(X_{\sigma_u} - V_{\sigma_u} + u, V_{\sigma_u} - u) e^{-\rho_\delta (V_{\sigma_u} - u)}] \\ &= e^{-\rho_\delta u} E^{(\rho_\delta)}[w(U(\tau_u^-), |U(\tau_u)|) e^{\rho_\delta U(\tau_u)}]. \end{aligned} \quad (7)$$

Proof. In (6), we let

$$\phi(x_1, t_1, \dots, x_{\sigma_u}, t_{\sigma_u}) = e^{-\delta \sum_{j=1}^{\sigma_u} t_j} w\left(x_{\sigma_u} - \sum_{j=1}^{\sigma_u} (x_j - t_j) + u, \sum_{j=1}^{\sigma_u} (x_j - t_j) - u\right)$$

so that the Gerber–Shiu function becomes

$$\begin{aligned} m_\delta(u) &= E[e^{-\delta \tau_u} w(U(\tau_u^-), |U(\tau_u)|) 1_{\{\tau_u < \infty\}}] \\ &= E\left[e^{-\delta \sum_{j=1}^{\sigma_u} W_j} w\left(X_{\sigma_u} - \sum_{j=1}^{\sigma_u} (X_j - W_j) + u, \sum_{j=1}^{\sigma_u} (X_j - W_j) - u\right) 1_{\{\sigma_u < \infty\}}\right] \\ &= E[e^{-\delta T_{\sigma_u}} w(X_{\sigma_u} - V_{\sigma_u} + u, V_{\sigma_u} - u) 1_{\{\sigma_u < \infty\}}] \\ &= E^{(\rho_\delta)}[e^{-\delta T_{\sigma_u}} w(X_{\sigma_u} - V_{\sigma_u} + u, V_{\sigma_u} - u) e^{-\rho_\delta V_{\sigma_u} + \delta T_{\sigma_u} + \sigma_u \Gamma(\rho_\delta)}]. \end{aligned} \quad (8)$$

The random walk V has a positive drift under the new probability measure $\mathbb{P}^{(\rho_\delta)}$, and hence $\sigma_u < \infty$ or equivalently $1_{\{\sigma_u < \infty\}} = 1_{\{\tau_u < \infty\}} = 1$. Then, since $\Gamma(\rho_\delta) = 0$, (8) becomes

$$\begin{aligned} m_\delta(u) &= E[e^{-\delta \tau_u} w(U(\tau_u^-), |U(\tau_u)|) 1_{\{\tau_u < \infty\}}] \\ &= E^{(\rho_\delta)}[w(X_{\sigma_u} - V_{\sigma_u} + u, V_{\sigma_u} - u) e^{-\rho_\delta V_{\sigma_u}}] \end{aligned}$$

$$\begin{aligned}
 &= e^{-\rho_\delta u} E^{(\rho_\delta)} \left[w \left(X_{\sigma_u} - V_{\sigma_u} + u, V_{\sigma_u} - u \right) e^{-\rho_\delta (V_{\sigma_u} - u)} \right] \\
 &= e^{-\rho_\delta u} E^{(\rho_\delta)} \left[w \left(U \left(\tau_u^- \right), |U \left(\tau_u \right)| \right) e^{\rho_\delta U \left(\tau_u \right)} \right]. \quad \blacksquare
 \end{aligned}$$

2.3. Lundberg exponential bounds for $\psi(u)$, $\bar{G}_\delta(u)$, and $m_\delta(u)$

Proposition 1 provides a natural way to derive Lundberg exponential bounds for $\psi(u)$, $\bar{G}_\delta(u)$, and $m_\delta(u)$.

Corollary 2. In the context of **Proposition 1**, we have the following bounds for $\psi(u)$ and $\bar{G}_\delta(u)$:

$$\psi(u) \leq e^{-\rho_0 u} \quad (9)$$

and

$$\bar{G}_\delta(u) \leq e^{-\rho_\delta u}. \quad (10)$$

In addition, if $w(x, y)$ is bounded, we also get a bound for $m_\delta(u)$,

$$m_\delta(u) \leq \sup_{x, y \geq 0} \{w(x, y)\} e^{-\rho_\delta u}. \quad (11)$$

Proof. The inequalities in (9), (10), and (11) follow from (7) and $U(\tau_u) < 0$. \blacksquare

2.4. Asymptotic expression for $m_\delta(u)$

Proposition 3. Let (X, W) be a pair of continuous rvs. Assume that ρ_δ exists and that the penalty function w is continuous and bounded. Then, there is some constant $C_\delta > 0$ such that

$$\lim_{u \rightarrow \infty} m_\delta(u) = C_\delta e^{-\rho_\delta u}.$$

Proof. The proof is inspired from the one of Proposition XII.2.10 in [7], which is based on ordinary renewal theory. Let us define $m_\delta^*(u)$ by

$$m_\delta^*(u) = m_\delta(u) e^{\rho_\delta u} = E^{(\rho_\delta)} \left[w \left(U \left(\tau_u^- \right), |U \left(\tau_u \right)| \right) e^{\rho_\delta U \left(\tau_u \right)} \right].$$

In the following, we denote the joint pdf of $(U(\tau_0^-), |U(\tau_0)|)$ under $\mathbb{P}^{(\rho_\delta)}$ by $f_{U(\tau_0^-), |U(\tau_0)|}^{(\rho_\delta)}$. Then, we have

$$\begin{aligned}
 m_\delta^*(u) &= \int_0^u \int_0^\infty m_\delta^*(u-y) f_{U(\tau_0^-), |U(\tau_0)|}^{(\rho_\delta)}(x, y) dx dy \\
 &\quad + \int_u^\infty \int_0^\infty w(x+u, y-u) e^{-\rho_\delta(y-u)} f_{U(\tau_0^-), |U(\tau_0)|}^{(\rho_\delta)}(x, y) dx dy.
 \end{aligned} \quad (12)$$

Let us now define the proper pdf and cdf of $|U(\tau_0)|$ under $\mathbb{P}^{(\rho_\delta)}$ by

$$f_{|U(\tau_0)|}^{(\rho_\delta)}(x) = \int_0^\infty f_{U(\tau_0^-), |U(\tau_0)|}^{(\rho_\delta)}(x, y) dy \quad (13)$$

and $F_{|U(\tau_0)|}^{(\rho_\delta)}$. Combining (12) and (13) leads to

$$m_\delta^*(u) = \int_0^u m_\delta^*(u-y) f_{|U(\tau_0)|}^{(\rho_\delta)}(y) dy + \int_u^\infty \int_0^\infty w(x+u, y-u) e^{-\rho_\delta(y-u)} f_{U(\tau_0^-), |U(\tau_0)|}^{(\rho_\delta)}(x, y) dx dy, \quad (14)$$

which is a proper renewal equation of the form

$$m_\delta^*(u) = m_\delta^* * f_{|U(\tau_0)|}^{(\rho_\delta)}(u) + a^{(\rho_d)}(u),$$

with

$$a^{(\rho_d)}(u) = \int_u^\infty \int_0^\infty w(x+u, y-u) e^{-\rho_\delta(y-u)} f_{U(\tau_0^-), |U(\tau_0)|}^{(\rho_\delta)}(x, y) dx dy.$$

Then, we apply the key renewal theorem (see e.g. Theorem 6.1.10 of Rolski et al. [6]) for the solution to the proper renewal Eq. (14), which yields

$$\lim_{u \rightarrow \infty} m_{\delta}^*(u) = \frac{\int_0^{\infty} a^{(\rho_d)}(u) du}{\int_0^{\infty} \left(1 - F_{|U(\tau_0)|}^{(\rho_{\delta})}(y)\right) dy} = C_{\delta}. \quad (15)$$

For this result to hold, $a^{(\rho_d)}$ must be directly Riemann integrable. We recall that a function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is directly Riemann integrable if

$$\lim_{h \rightarrow 0} h \times \left(\sum_{k=1}^{\infty} \sup_{(k-1)h \leq t \leq kh} \varphi(t) \right) = \lim_{h \rightarrow 0} h \times \left(\sum_{k=1}^{\infty} \inf_{(k-1)h \leq t \leq kh} \varphi(t) \right),$$

i.e. if the limits of the upper and lower bounds exist and coincide. The integral is then equal to this limit (see e.g. Chapter 6 of Rolski et al. [6] for details). In order to show that $a^{(\rho_d)}$ is directly Riemann integrable, it suffices to show that there exists an upper bound to $a^{(\rho_d)}$ that is Riemann integrable. Since the penalty function w is continuous and since $0 \leq e^{-\rho_{\delta}(y-u)} \leq 1$, there exists a constant c such that $a^{(\rho_d)}(u) \leq c \times \left(1 - F_{U(\tau_0^-), |U(\tau_0)|}^{(\rho_{\delta})}(\infty, u)\right)$, for $u \geq 0$. Now, as the adjustment coefficient ρ_{δ} exists, the mgf of the claim amounts, as well as the mgf's of $U(\tau_0^-)$ and $|U(\tau_0)|$ exist. Consequently, all their moments also exist which enable us to conclude that $\left(1 - F_{U(\tau_0^-), |U(\tau_0)|}^{(\rho_{\delta})}(\infty, u)\right)$ is directly Riemann integrable and that $a^{(\rho_d)}$ is also directly Riemann integrable. Note that the denominator in (15) is finite since all moments of $|U(\tau_0)|$ exist. ■

Note that the derivation of analytical expressions for C_{δ} is rather difficult. In Section 3, an application of the above asymptotic expression is given.

2.5. Application of change of measure to extensions of $m_{\delta}(u)$

Cheung et al. [4] propose two additional ruin-related quantities, as extensions to $m_{\delta}(u)$, and study their properties. First, they introduce the minimal surplus prior to ruin which is defined by the rv $A_{1,\sigma_u} = \min(u, U(T_1), \dots, U(T_{\sigma_u-1}))$ if $\sigma_u > 1$ and $A_{1,\sigma_u} = u$, if $\sigma_u = 1$. The introduction of A_{1,σ_u} allows notably the analysis of the last ladder height with $A_{1,\sigma_u} + |U(\tau_u)|$. As a first extension of $m_{\delta}(u)$, Cheung et al. [4] define

$$\chi_{\delta,123}(u) = E \left[e^{-\delta \tau_u} w_{123}(U(\tau_u^-), |U(\tau_u)|, A_{1,\sigma_u}) 1_{\{\tau_u < \infty\}} | U(0) = u \right], \quad (16)$$

(see (4) in [4]).

Secondly, Cheung et al. [4] propose to study the rv $A_{2,\sigma_u} = U(T_{\sigma_u-1})$ if $\sigma_u > 1$ and $A_{2,\sigma_u} = u$, if $\sigma_u = 1$, which corresponds to the surplus immediately after the second last claim before ruin occurs (if ruin occurs at or after the second claim) or to u (if ruin occurs on the first claim). As a second extension of $m_{\delta}(u)$, Cheung et al. [4] define

$$\chi_{\delta,1234}(u) = E \left[e^{-\delta \tau_u} w_{1234}(U(\tau_u^-), |U(\tau_u)|, A_{1,\sigma_u}, A_{2,\sigma_u}) 1_{\{\tau_u < \infty\}} | U(0) = u \right], \quad (17)$$

(see (2) in [4]). The introduction of A_{2,σ_u} permits the analysis of the last interclaim time before ruin with $\frac{U(\tau_u^-) - A_{2,\sigma_u}}{c}$.

Inspired by the proof of Proposition 1, we apply (6) to derive the next corollary which allows us to derive the expressions of $\chi_{\delta,123}(u)$ and $\chi_{\delta,1234}(u)$ under $\mathbb{P}^{(\rho_{\delta})}$.

Corollary 4. Let us define the function

$$\chi_{\delta}(u) = E \left[\phi(X_1, W_1, \dots, X_{\sigma_u}, W_{\sigma_u}) 1_{\{\tau_u < \infty\}} | U(0) = u \right],$$

assuming that the expectation exists. Then, we have

$$\chi_{\delta}(u) = e^{-\rho_{\delta} u} E^{(\rho_{\delta})} \left[\phi(X_1, W_1, \dots, X_{\sigma_u}, W_{\sigma_u}) e^{-\rho_{\delta}(V_{\sigma_u} - u)} e^{\delta \tau_u} \right].$$

In Examples 5 and 6, we derive the expressions of $\chi_{\delta,123}(u)$ and $\chi_{\delta,1234}(u)$ under $\mathbb{P}^{(\rho_{\delta})}$.

Example 5. The rv A_{1,σ_u} can be rewritten as

$$A_{1,\sigma_u} = \begin{cases} u, & \sigma_u = 1 \\ \min \left(u, u - (X_1 - W_1), \dots, u - \sum_{j=1}^{\sigma_u-1} (X_j - W_j) \right), & \sigma_u > 1. \end{cases} \quad (18)$$

With [Corollary 4](#) and

$$\phi(X_1, W_1, \dots, X_{\sigma_u}, W_{\sigma_u}) = e^{-\delta \sum_{j=1}^{\sigma_u} W_j} w_{123} \left(X_{\sigma_u} - \sum_{j=1}^{\sigma_u} (X_j - W_j) + u, \sum_{j=1}^{\sigma_u} (X_j - W_j) - u, A_{1, \sigma_u} \right),$$

(16) becomes

$$\chi_{\delta, 123}(u) = e^{-\rho_\delta u} E^{(\rho_\delta)} \left[w_{123}(U(\tau_u^-), |U(\tau_u)|, A_{1, \sigma_u}) e^{-\rho_\delta (V_{\sigma_u} - u)} \right]. \quad \square$$

Example 6. The rv A_{2, σ_u} can be rewritten as

$$A_{1, \sigma_u} = \begin{cases} u, & \sigma_u = 1 \\ u - \sum_{j=1}^{\sigma_u-1} (X_j - W_j), & \sigma_u > 1. \end{cases} \quad (19)$$

Using [Corollary 4](#) and

$$\phi(X_1, W_1, \dots, X_{\sigma_u}, W_{\sigma_u}) = e^{-\delta \sum_{j=1}^{\sigma_u} W_j} w_{1234} \left(X_{\sigma_u} - \sum_{j=1}^{\sigma_u} (X_j - W_j) + u, \sum_{j=1}^{\sigma_u} (X_j - W_j) - u, A_{1, \sigma_u}, A_{2, \sigma_u} \right),$$

(16) becomes

$$\chi_{\delta, 1234}(u) = e^{-\rho_\delta u} E^{(\rho_\delta)} \left[w_{1234}(U(\tau_u^-), |U(\tau_u)|, A_{1, \sigma_u}, A_{2, \sigma_u}) e^{-\rho_\delta (V_{\sigma_u} - u)} \right]. \quad \square$$

[Corollary 4](#) can also be used to study other ruin-related quantities defined in function of $(X_1, W_1, \dots, X_{\sigma_u}, W_{\sigma_u})$. For instance, we could consider the rv $A_{3, \sigma_u} = \max(X_1, \dots, X_{\sigma_u})$ which corresponds to the maximal claim amount up to ruin or the rv $A_{4, \sigma_u} = \sum_{j=1}^{\sigma_u} X_j$ which corresponds to the sum of claims up to ruin. Then, we have

$$\chi_{0,5}(u) = E[w_5(A_{3, \sigma_u}) 1_{\{\tau_u < \infty\}} | U(0) = u] = e^{-\rho_0 u} E^{(\rho_0)} \left[w_5(\max(X_1, \dots, X_{\sigma_u})) e^{-\rho_0 (V_{\sigma_u} - u)} \right]$$

or

$$\chi_{0,6}(u) = E[w_6(A_{4, \sigma_u}) 1_{\{\tau_u < \infty\}} | U(0) = u] = e^{-\rho_0 u} E^{(\rho_0)} \left[w_6\left(\sum_{j=1}^{\sigma_u} X_j\right) e^{-\rho_0 (V_{\sigma_u} - u)} \right].$$

As for $m_\delta(u)$, the expressions of $\chi_{\delta, 123}(u)$, $\chi_{\delta, 1234}(u)$, $\chi_{0,5}(u)$, and $\chi_{0,6}(u)$ under $\mathbb{P}^{(\rho_\delta)}$ enable the use of importance sampling to compute them.

3. Importance sampling via change of measure

As already mentioned, an important advantage of the expressions derived in [Proposition 1](#) is that they can be easily simulated. Given that the drift of the surplus process is negative under $\mathbb{P}^{(\rho_\delta)}$, and hence $\Pr(\tau_u < \infty) = 1$, we can simulate a sample path of the process until ruin occurs which is not the case for a crude Monte Carlo simulation method (see details in [\[15\]](#)). Interestingly, we show that the important sampling method provides unbiased approximations for the Gerber–Shiu function and bounded relative errors.

Algorithm 7. The following steps are repeated m times, i.e. for $j = 1, \dots, m$.

1. Generate $\{(X_k^{(j)}, W_k^{(j)}), k \in \mathbb{N}^+\}$ under $\mathbb{P}^{(\rho_\delta)}$ and $\{U^{(j)}(T_k^{(j)}), k \in \mathbb{N}^+\}$, where $T_k^{(j)} = W_1^{(j)} + \dots + W_k^{(j)}$, until ruin occurs.
2. Denote by $\sigma_u^{(j)}$ the claim number at which ruin occurs, by $\tau_u^{(j)} = T_{\sigma_u^{(j)}}^{(j)}$ the time of ruin, and by $U^{(j)}(\tau_u^{(j)}) = u - V_{\sigma_u^{(j)}}^{(j)}$ the deficit at ruin.

The approximation of $m_\delta(u)$, defined by

$$m_\delta^{IS}(u) = \frac{1}{m} \sum_{j=1}^m e^{-\rho_\delta u} w \left(X_{\sigma_u^{(j)}}^{(j)} - V_{\sigma_u^{(j)}}^{(j)} + u, V_{\sigma_u^{(j)}}^{(j)} - u \right) e^{-\rho_\delta (V_{\sigma_u^{(j)}}^{(j)} - u)},$$

is unbiased. Moreover, the relative error under $\mathbb{P}^{(\rho_\delta)}$, defined by

$$\frac{\text{Var}(m_\delta^{\text{IS}}(u))}{E[m_\delta^{\text{IS}}(u)]^2}, \quad (20)$$

is bounded, as shown in the following proposition.

Proposition 8. We consider two cases for $w(x, y)$.

1. Let $w(x, y) = 1$. Then, $m_\delta^{\text{IS}}(u)$ computed under $\mathbb{P}^{(\rho_\delta)}$ has a bounded relative error.
2. Assume $w(x, y)$ is bounded. Then, $m_\delta^{\text{IS}}(u)$ computed under $\mathbb{P}^{(\rho_\delta)}$ has a bounded relative error.

Proof. 1. As suggested in [7, Theorem XV.3.1], we just need to prove that $E[(m_\delta^{\text{IS}}(u))^2]$ is bounded. Indeed, we have

$$E[(m_\delta^{\text{IS}}(u))^2] \leq e^{-2\rho_\delta u}. \text{ By Proposition 3, the result follows from}$$

$$E[(\tilde{m}_\delta^{\text{IS}}(u))^2] \leq e^{-2\rho_\delta u} \sim \frac{m_\delta(u)^2}{C_\delta^2}.$$

2. Similarly, we have

$$E[(\tilde{m}_\delta^{\text{IS}}(u))^2] \leq \left(\sup_{x,y \geq 0} (w(x, y)) \right)^2 e^{-2\rho_\delta u} \sim \left(\sup_{x,y \geq 0} (w(x, y)) \right)^2 \frac{m_\delta(u)^2}{C_\delta^2}. \quad \blacksquare$$

4. Bivariate distributions for (X, W)

In this section, we consider specific bivariate distributions for (X, W) and derive their corresponding distributions under $\mathbb{P}^{(\rho_\delta)}$. We provide without proof the following lemma which will be helpful to find the joint distribution of (X, W) under $\mathbb{P}^{(\rho_\delta)}$.

Lemma 9. Under $\mathbb{P}^{(\rho_\delta)}$, the joint mgf of (X, W) is given by

$$M_{X,W}^{(\rho_\delta)}(r_1, r_2) = M_{X,W}(r_1 + \rho_\delta, r_2 - (\rho_\delta + \delta)).$$

In the numerical examples that follow, we illustrate the importance sampling method by computing various ruin quantities and, in some cases, we also compare the obtained results by importance sampling to the exact values in the aim to validate the quality of the approximation. We mention that, except for the Moran–Downton bivariate exponential distribution, numerical optimization needs to be used to find the adjustment coefficient ρ_δ . Importance sampling is always performed with 10 000 simulations.

4.1. Bivariate mixed distributions

Let us consider bivariate mixed distributions whose joint pdf and mgf are respectively of the form

$$f_{X,W}(x, t) = \sum_{i=1}^m \sum_{j=1}^m p_{i,j} f_i(x) g_j(t),$$

$$M_{X,W}(r_1, r_2) = \sum_{i=1}^m \sum_{j=1}^m p_{i,j} \tilde{f}_i(r_1) \tilde{g}_j(r_2),$$

with $p_{i,j} \geq 0$ and $\sum_{i=1}^m \sum_{j=1}^m p_{i,j} = 1$. Also, \tilde{f}_i and \tilde{g}_j are the mgf associated to the pdf f_i and g_j , respectively. Since $p_{i,j} \geq 0$ and $\sum_{i=1}^m \sum_{j=1}^m p_{i,j} = 1$ and since $f_i(x)$ and $g_j(t)$ are pdfs for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$, then it ensures that $f_{X,W}(x, t)$ is a proper bivariate pdf. Similarly, the existence of the joint $M_{X,W}(r_1, r_2)$ is also guaranteed. For details on such bivariate distributions, see e.g. [17].

Lemma 10. For bivariate mixed distributions, the pdf of (X, W) under $\mathbb{P}^{(\rho_\delta)}$ is given by

$$f_{X,W}^{(\rho_\delta)}(x, t) = \sum_{i=1}^m \sum_{j=1}^m p_{i,j}^{(\rho_\delta)} f_i^{(\rho_\delta)}(x) g_j^{(\rho_\delta)}(t),$$

where $p_{i,j}^{(\rho_\delta)} = p_{i,j} \tilde{f}_i(\rho_\delta) \tilde{g}_j(-\rho_\delta - \delta)$, $f_i^{(\rho_\delta)}(x) = \frac{e^{\rho_\delta x}}{f_i(\rho_\delta)} f_i(x)$, and $g_j^{(\rho_\delta)}(t) = \frac{e^{-\rho_\delta t - \delta t}}{g_j(-\rho_\delta - \delta)} g_j(t)$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$.

Table 1

Values of parameters of the bivariate mixed gamma distribution for Cases 1, 2, and 3.

Case	1	2 (Independence)	3
$p_{1,1}$	0.6	0.64	0.8
$p_{1,2}$	0.2	0.16	0
$p_{2,1}$	0.2	0.16	0
$p_{2,2}$	0	0.04	0.2

Table 2

Values of ruin probabilities obtained by importance sampling for Cases 1, 2, and 3 assuming the bivariate mixed gamma distribution.

Case	1	2	3
$\rho_P(X, W)$	-0.124981	0	0.499924
ρ_0	0.004910	0.005515	0.010799
$\psi(0)$	0.919547	0.912392	0.857763
$\psi(50)$	0.703977	0.674885	0.457544
$\psi(100)$	0.549941	0.510741	0.265902
$\psi(200)$	0.337440	0.294404	0.090493
$\psi(500)$	0.077243	0.056383	0.003537
$\psi(1000)$	0.006635	0.003571	0.000016

Proof. The result follows directly from the application of (4). ■

As an example, we assume here that (X, W) follows a bivariate mixed gamma distribution with joint pdf given by

$$f_{X,W}(x, t) = \sum_{i=1}^m \sum_{j=1}^m p_{i,j} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i-1} e^{-\beta_i x} \frac{\lambda_j^{\gamma_j}}{\Gamma(\gamma_j)} t^{\gamma_j-1} e^{-\lambda_j t}.$$

This bivariate distribution is a generalization of a slightly simpler one that is suggested and applied by Jones et al. [18]. A special case of this distribution, the bivariate mixed exponential distribution is used in [19] in the context of a compound renewal risk model with dependence and diffusion. In [20, Theorem 1.1.2], it is shown that a joint distribution with a completely monotone pdf, such as the bivariate Pareto distribution and the bivariate Weibull distribution, can be approximated by a finite mixture of bivariate exponential distributions. With Lemma 10, the joint pdf of (X, W) under $\mathbb{P}^{(\rho_\delta)}$ is

$$f_{X,W}^{(\rho_\delta)}(x, t) = \sum_{i=1}^m \sum_{j=1}^m p_{i,j}^{(\rho_\delta)} \frac{(\beta_i - \rho_\delta)^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i-1} e^{-(\beta_i - \rho_\delta)x} \frac{(\lambda_j + \rho_\delta + \delta)^{\gamma_j}}{\Gamma(\gamma_j)} t^{\gamma_j-1} e^{-(\lambda_j + \rho_\delta + \delta)t}, \quad \text{for } x, t \geq 0,$$

where $p_{i,j}^{(\rho_\delta)} = p_{i,j} \left(\frac{\beta_i}{\beta_i - \rho_\delta} \right)^{\alpha_i} \left(\frac{\lambda_j}{\lambda_j + \rho_\delta + \delta} \right)^{\gamma_j}$, for $i, j \in \{1, 2, \dots, m\}$. We observe that, under $\mathbb{P}^{(\rho_\delta)}$, the joint distribution of (X, W) remains a bivariate mixed gamma distribution with both modified probabilities and scale parameters. The shape parameters remain unchanged. In the following example, we illustrate the interest of using importance sampling for non-integer valued parameters α_i .

Example 11. Assume that (X, W) follows a bivariate mixed gamma distribution with $m = 2$, $\alpha_1 = 0.5$, $\beta_1 = 1$, $\gamma_1 = 0.8$, $\lambda_1 = 0.8$, $\alpha_2 = 1.2$, $\beta_2 = \frac{1.2}{23}$, $\gamma_2 = 1.5$, and $\lambda_2 = \frac{1.5}{26}$. In Table 1, we indicate the three cases for the $p_{i,j}$'s which are fixed such that the marginal distributions for X and W are identical in all three cases.

Values obtained for $\psi(u)$ by importance sampling are provided in Table 2, where $\rho_P(X, W)$ is the Pearson correlation coefficient. Notice that Case 1 corresponds to negatively correlated X and W and Case 3 to positively correlated X and W .

As expected, we observe that $\psi(u)$ decreases as ρ_P increases whatever the initial capital u considered. The dependence acts as an hedging mechanism between premium incomes and claim amounts. □

In [15] other examples of bivariate mixed distributions are considered.

4.2. Bivariate gamma Cheriyan–Ramabhadran–Mathai–Moschopoulos (CRMM) distribution

We now consider the gamma CRMM bivariate distribution, which is constructed as follows. Let Y_0, Y_1 and Y_2 be three independent rvs where $Y_0 \sim \text{Gamma}(\gamma_0, \beta_0)$, $Y_1 \sim \text{Gamma}(\alpha_1 - \gamma_0, \beta)$ and $Y_2 \sim \text{Gamma}(\alpha_2 - \gamma_0, \lambda)$, with $0 \leq \gamma_0 \leq \min(\alpha_1; \alpha_2)$, $\alpha_1, \alpha_2 \in \mathbb{R}^+$. We define the rvs X and W by $X = \frac{\beta_0}{\beta} Y_0 + Y_1$ and $W = \frac{\beta_0}{\lambda} Y_0 + Y_2$. Such a couple (X, W) is said to follow a gamma CRMM bivariate distribution with $X \sim \text{Gamma}(\alpha_1, \beta)$, $W \sim \text{Gamma}(\alpha_2, \lambda)$, and Pearson's correlation

coefficient $\rho_P(X, W) = \frac{\gamma_0}{\sqrt{\alpha_1 \alpha_2}}$, with $0 \leq \rho_P(X, W) \leq \frac{\min(\alpha_1, \alpha_2)}{\sqrt{\alpha_1 \alpha_2}}$. The parameter γ_0 corresponds to the dependence parameter. The expression of the mgf of (X, W) is given by

$$\begin{aligned} M_{X,W}(r_1, r_2) &= E[e^{r_1 X} e^{r_2 W}] \\ &= E[e^{r_1 Y_1}] E[e^{r_2 Y_2}] E\left[e^{\left(\frac{\beta_0}{\beta} r_1 + \frac{\beta_0}{\lambda} r_2\right) Y_0}\right] \\ &= \left(1 - \frac{r_1}{\beta}\right)^{-(\alpha_1 - \gamma_0)} \left(1 - \frac{r_2}{\lambda}\right)^{-(\alpha_2 - \gamma_0)} \times \left(1 - \frac{r_1}{\beta} - \frac{r_2}{\lambda}\right)^{-\gamma_0}. \end{aligned} \quad (21)$$

The gamma CRMM distribution has been proposed independently by Cheriyan [21], Ramabhadran [22], and Mathai and Moschopoulos [23]. See also e.g. [24] for a review on this bivariate distribution. Ambagaspitiya [10] has found the explicit expression for $\psi(u)$, when $\alpha_1, \alpha_2 \in \mathbb{N}^+$. It is worth mentioning that importance sampling allows us to compute numerically $m_\delta(u)$ in addition to $\psi(u)$ for $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $\delta \geq 0$.

Lemma 12. Under $\mathbb{P}^{(\rho_\delta)}$, the rvs X and W can be written as follows

$$\begin{aligned} X &= C_1 + \frac{C_0}{\beta - \rho_\delta + \frac{\beta}{\lambda}(\rho_\delta + \delta)}, \\ W &= C_2 + \frac{C_0}{\lambda - \frac{\lambda}{\beta}\rho_\delta + (\rho_\delta + \delta)}, \end{aligned}$$

where C_0, C_1 , and C_2 are independent rvs with $C_0 \sim \text{Gamma}(\gamma_0, 1)$, $C_1 \sim \text{Gamma}(\alpha_1 - \gamma_0, \beta - \rho_\delta)$ and $C_2 \sim \text{Gamma}(\alpha_2 - \gamma_0, \lambda + \rho_\delta + \delta)$.

Proof. Given Lemma 9, we have

$$\begin{aligned} M_{X,W}^{(\rho_\delta)}(r_1, r_2) &= \left(1 - \frac{r_1 + \rho_\delta}{\beta}\right)^{-(\alpha_1 - \gamma_0)} \left(1 - \frac{r_2 - \rho_\delta - \delta}{\lambda}\right)^{-(\alpha_2 - \gamma_0)} \left(1 - \frac{r_1 + \rho_\delta}{\beta} - \frac{r_2 - \rho_\delta - \delta}{\lambda}\right)^{-\gamma_0} \\ &= \left(\frac{\beta}{\beta - \rho_\delta}\right)^{\alpha_1 - \gamma_0} \left(\frac{\beta - \rho_\delta}{\beta - r_1 - \rho_\delta}\right)^{\alpha_1 - \gamma_0} \left(\frac{\lambda}{\lambda + \rho_\delta + \delta}\right)^{\alpha_2 - \gamma_0} \left(\frac{\lambda + \rho_\delta + \delta}{\lambda + \rho_\delta + \delta - r_2}\right)^{\alpha_2 - \gamma_0} \\ &\quad \times \left(\frac{\beta\lambda}{\beta\lambda - \lambda\rho_\delta + \beta(\rho_\delta + \delta)}\right)^{\gamma_0} \left(\frac{\beta\lambda - \lambda\rho_\delta + \beta(\rho_\delta + \delta)}{\beta\lambda - \lambda\rho_\delta + \beta(\rho_\delta + \delta) - \lambda r_1 - \beta r_2}\right)^{\gamma_0}. \end{aligned}$$

Since

$$\left(\frac{\beta}{\beta - \rho_\delta}\right)^{\alpha_1 - \gamma_0} \left(\frac{\lambda}{\lambda + \rho_\delta + \delta}\right)^{\alpha_2 - \gamma_0} \left(\frac{\beta\lambda}{\beta\lambda - \lambda\rho_\delta + \beta(\rho_\delta + \delta)}\right)^{\gamma_0} = 1,$$

then

$$M_{X,W}^{(\rho_\delta)}(r_1, r_2) = \left(\frac{1}{1 - \frac{r_1}{\beta - \rho_\delta}}\right)^{\alpha_1 - \gamma_0} \left(\frac{1}{1 - \frac{r_2}{\lambda + \rho_\delta + \delta}}\right)^{\alpha_2 - \gamma_0} \left(\frac{1}{1 - \frac{r_1}{\beta - \rho_\delta + \frac{\beta}{\lambda}(\rho_\delta + \delta)} - \frac{r_2}{\lambda - \frac{\lambda}{\beta}\rho_\delta + (\rho_\delta + \delta)}}\right)^{\gamma_0},$$

which completes the proof. ■

Note from Lemma 12 that under $\mathbb{P}^{(\rho_\delta)}$, the marginal distributions of the rvs X and W are no longer gamma but rather a sum of independent gamma rvs with different scale parameters (see [25] for details on sums of independent gamma rvs).

In the following example, we compare the results obtained by importance sampling to exact values.

Example 13. Let us consider Example 4 of Ambagaspitiya [10], in which the parameters are $\alpha_1 = 2$, $\alpha_2 = 2$, $\gamma_0 = 1$, $\beta = 2$, and $\lambda = \frac{2}{1.1}$. In such a case, with $\delta = 0$, he obtains

$$\psi(u) = 0.8198e^{-0.3604u}. \quad (22)$$

In Table 3, we provide the exact values (computed with (22)) and those computed by importance sampling for $\psi(u)$ with $\rho_0 = 0.3604196$. We also provide in Table 3 the values computed by importance sampling for $m_{0.2}(u)$.

As anticipated, we observe that the importance sampling results for $\psi(u)$ are very close to the exact values. □

Let us consider below a second example in which the parameters α_1 and α_2 are no longer integers, and hence cannot be treated by Ambagaspitiya [10].

Example 14. Let $\alpha_1 = 0.6$, $\alpha_2 = 1.2$, $\beta = 0.5$, $\lambda = 0.8$, and $\delta = 0.02$. In Table 4, we provide the values computed by importance sampling for $\bar{G}_{0.02}(u)$ and $m_{0.02,2}(u)$ for Case 1 ($\gamma_0 = 0.1$) and Case 2 ($\gamma_0 = 0.4$):

For a fixed initial surplus u , the values $\bar{G}_{0.02}(u)$ and $m_{0.02,2}(u)$ decrease as the dependence parameter α_0 increases. □

Table 3

Exact values and values obtained by importance sampling of ruin measures assuming bivariate gamma CRMM distribution.

u	$\psi(u)$ (exact)	$\psi(u)$ (IS)	$m_{0,2}(u)$ (IS)
0	0.8198	0.8202	0.4095
2	0.3987	0.3987	0.1995
5	0.1352	0.1359	0.0665
10	0.0223	0.0229	0.0101

Table 4

Values computed by importance sampling of ruin measures assuming bivariate gamma CRMM distribution.

u	Case $1-\gamma_0 = 0.1$		Case $1-\gamma_0 = 0.4$	
	$\rho_P(X, W) = 0.117851$		$\rho_P(X, W) = 0.471405$	
	$\rho_{0,02} = 0.197693$		$\rho_{0,02} = 0.294809$	
	$\bar{G}_{0,02}(u)$ (IS)	$m_{0,02,2}(u)$ (IS)	$\bar{G}_{0,02}(u)$ (IS)	$m_{0,02,2}(u)$ (IS)
0	0.711491	0.989035	0.697644	0.629724
10	0.087144	0.163357	0.024404	0.043776
20	0.011845	0.022842	0.001209	0.002313

4.3. Raftery bivariate exponential distribution

The Raftery bivariate exponential distribution is constructed as follows. Let Y_0, Y_1 , and Y_2 be independent exponentially distributed rvs with mean 1 and (I_1, I_2) be a couple of rvs with bivariate Bernoulli distribution with $p_{i_1 i_2} = \Pr(I_1 = i_1, I_2 = i_2)$, for $i_1, i_2 \in \{0, 1\}$. Then, we define (X, W) in terms of the rvs Y_0, Y_1 , and Y_2 as

$$X = \frac{(1 - p_{10} - p_{11}) Y_1 + I_1 Y_0}{\beta},$$

$$W = \frac{(1 - p_{01} - p_{11}) Y_2 + I_2 Y_0}{\lambda}.$$

It can be shown that the marginal distributions of X and W are exponentials with means $\frac{1}{\beta}$ and $\frac{1}{\lambda}$ respectively. Also, the Pearson correlation coefficient is given by

$$\rho_P(X, W) = 2p_{11} - (p_{10} + p_{11})(p_{01} + p_{11}),$$

where $\rho_P(X, W) \in [-0.25, 1]$. When $p_{01} = p_{11} = 0$ or $p_{10} = p_{11} = 0$, it corresponds to the independence case while $p_{00} = p_{01} = p_{10} = 0, p_{11} = 1$ corresponds to the comonotonicity case.

By conditioning on (I_1, I_2) , the mgf of (X, W) is

$$\begin{aligned} M_{X,W}(r_1, r_2) &= p_{00} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}}}{\frac{\beta}{1-p_{10}-p_{11}} - r_1} \right) \left(\frac{\frac{\lambda}{1-p_{01}-p_{11}}}{\frac{\lambda}{1-p_{01}-p_{11}} - r_2} \right) + p_{10} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}}}{\frac{\beta}{1-p_{10}-p_{11}} - r_1} \right) \left(\frac{\beta}{\beta - r_1} \right) \left(\frac{\frac{\lambda}{1-p_{01}-p_{11}}}{\frac{\lambda}{1-p_{01}-p_{11}} - r_2} \right) \\ &\quad + p_{01} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}}}{\frac{\beta}{1-p_{10}-p_{11}} - r_1} \right) \left(\frac{\frac{\lambda}{1-p_{01}-p_{11}}}{\frac{\lambda}{1-p_{01}-p_{11}} - r_2} \right) \left(\frac{\lambda}{\lambda - r_2} \right) \\ &\quad + p_{11} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}}}{\frac{\beta}{1-p_{10}-p_{11}} - r_1} \right) \left(\frac{\frac{\lambda}{1-p_{01}-p_{11}}}{\frac{\lambda}{1-p_{01}-p_{11}} - r_2} \right) \left(\frac{1}{1 - \left(\frac{r_1}{\beta} + \frac{r_2}{\lambda} \right)} \right). \end{aligned} \quad (23)$$

Details on the Raftery bivariate exponential distribution can be found in e.g. [26] or [24].

Lemma 15. Under $\mathbb{P}^{(\rho_\delta)}$, the couple (X, W) is constructed as follows:

$$X = \frac{1}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta} C_1 + \left(\frac{J_1(1-J_2)}{\beta - \rho_\delta} + \frac{J_1 J_2}{\beta - \rho_\delta + \frac{\beta}{\lambda} \rho_\delta + \frac{\beta}{\lambda} \delta} \right) C_0,$$

$$W = \frac{1}{\frac{\lambda}{1-p_{01}-p_{11}} + \rho_\delta + \delta} C_2 + \left(\frac{(1-J_1)J_2}{\lambda + \rho_\delta + \delta} + \frac{J_1 J_2}{\lambda - \frac{\lambda}{\beta} \rho_\delta + \rho_\delta + \delta} \right) C_0,$$

where the rvs C_0, C_1 and C_2 are independent with $C_0 \sim \text{Exp}(1)$, $C_1 \sim \text{Exp}(1)$ and $C_2 \sim \text{Exp}(1)$. They are also independent of (J_1, J_2) , whose joint probability mass function is denoted by

$$\Pr(J_1 = j_1, J_2 = j_2) = p_{j_1 j_2}^{(\rho_\delta)}$$

Table 5

Values of ruin probabilities obtained by importance sampling assuming Raftery bivariate exponential distribution (Cases 1 and 2).

Case	1	2
ρ_0	0.0400952	0.0929024
$\rho_P(X, W)$	-0.25	0.51
$\psi(0)$	0.8313375	0.6851976
$\psi(10)$	0.5667875	0.2525691
$\psi(20)$	0.3798958	0.1008767
$\psi(50)$	0.1134398	0.0061751
$\psi(100)$	0.0153967	0.000588
$\psi(500)$	0.0002776	0

for $j_1, j_2 \in \{0, 1\}$, where

$$p_{00}^{(\rho_\delta)} = p_{00} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}}}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta} \right) \left(\frac{\frac{\lambda}{1-p_{01}-p_{11}}}{\frac{\lambda}{1-p_{01}-p_{11}} + \rho_\delta + \delta} \right), \quad (24)$$

$$p_{10}^{(\rho_\delta)} = p_{10} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}}}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta} \right) \left(\frac{\beta}{\beta - \rho_\delta} \right) \left(\frac{\frac{\lambda}{1-p_{01}-p_{11}}}{\frac{\lambda}{1-p_{01}-p_{11}} + \rho_\delta + \delta} \right), \quad (25)$$

$$p_{01}^{(\rho_\delta)} = p_{01} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}}}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta} \right) \left(\frac{\frac{\lambda}{1-p_{01}-p_{11}}}{\frac{\lambda}{1-p_{01}-p_{11}} + \rho_\delta + \delta} \right) \left(\frac{\lambda}{\lambda + \rho_\delta + \delta} \right), \quad (26)$$

$$p_{11}^{(\rho_\delta)} = p_{11} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}}}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta} \right) \left(\frac{\frac{\lambda}{1-p_{01}-p_{11}}}{\frac{\lambda}{1-p_{01}-p_{11}} + \rho_\delta + \delta} \right) \left(\frac{\beta\lambda}{\beta\lambda - \lambda\rho_\delta - \beta(-\rho_\delta - \delta)} \right). \quad (27)$$

Proof. From (23), we find

$$\begin{aligned} M_{X,W}^{(\rho_\delta)}(r_1, r_2) &= p_{00}^{(\rho_\delta)} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta - r_1} \right) \left(\frac{\frac{\lambda}{1-p_{10}-p_{11}} + \rho_\delta + \delta}{\frac{\lambda}{1-p_{10}-p_{11}} + \rho_\delta + \delta - r_2} \right) \\ &+ p_{10}^{(\rho_\delta)} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta - r_1} \right) \left(\frac{\beta - \rho_\delta}{\beta - \rho_\delta - r_1} \right) \left(\frac{\frac{\lambda}{1-p_{10}-p_{11}} + \rho_\delta + \delta}{\frac{\lambda}{1-p_{10}-p_{11}} + \rho_\delta + \delta - r_2} \right) \\ &+ p_{01}^{(\rho_\delta)} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta - r_1} \right) \left(\frac{\frac{\lambda}{1-p_{10}-p_{11}} + \rho_\delta + \delta}{\frac{\lambda}{1-p_{10}-p_{11}} + \rho_\delta + \delta - r_2} \right) \left(\frac{\lambda + \rho_\delta + \delta}{\lambda + \rho_\delta + \delta - r_2} \right) \\ &+ p_{11}^{(\rho_\delta)} \left(\frac{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta}{\frac{\beta}{1-p_{10}-p_{11}} - \rho_\delta - r_1} \right) \left(\frac{\frac{\lambda}{1-p_{10}-p_{11}} + \rho_\delta + \delta}{\frac{\lambda}{1-p_{10}-p_{11}} + \rho_\delta + \delta - r_2} \right) \left(\frac{1}{1 - \frac{r_1}{\beta - \rho_\delta + \frac{\beta}{\lambda}\rho_\delta + \frac{\beta}{\lambda}\delta} - \frac{r_2}{\lambda - \frac{\lambda}{\beta}\rho_\delta + \rho_\delta + \delta}} \right), \end{aligned} \quad (28)$$

with $p_{00}^{(\rho_\delta)}$, $p_{10}^{(\rho_\delta)}$, $p_{01}^{(\rho_\delta)}$ and $p_{11}^{(\rho_\delta)}$ given by (24)–(27) respectively.

Now, we define the couple of Bernoulli rvs (J_1, J_2) whose joint probability mass function is given by

$$\Pr(J_1 = j_1, J_2 = j_2) = p_{j_1 j_2}^{(\rho_\delta)},$$

for $j_1, j_2 \in \{0, 1\}$. Also, based on (28), we notice that, if $J_1 = 1$ and $J_2 = 0$ (resp. $J_1 = 1$ and $J_2 = 1$), the scale parameter within the mgf associated to the second component in the definition of the rv X is $\beta - \rho_\delta$ (resp. $\beta - \rho_\delta + \frac{\beta}{\lambda}\rho_\delta + \frac{\beta}{\lambda}\delta$). Similarly, if $J_2 = 1$ and $J_1 = 0$ (resp. $J_2 = 1$ and $J_1 = 1$), the scale parameter within the mgf associated to the second component in the definition of the rv W is $\lambda + \rho_\delta + \delta$ (resp. $\lambda - \frac{\lambda}{\beta}\rho_\delta + \rho_\delta + \delta$). The desired result is then obtained from this observation and by inverting (28). ■

Note from Lemma 15 that, under $\mathbb{P}^{(\rho_\delta)}$, the marginals of (X, W) are no longer exponentials but rather a mixture of an exponential distribution and two generalized Erlang distributions. In the following example, we compute several ruin-related measures.

Example 16. Let $\beta = \frac{1}{4}$, $\lambda = \frac{1}{5}$ and $c = 1$ ($\eta = 25\%$). We consider two cases, namely $p_{00} = p_{11} = 0$, $p_{01} = p_{10} = 0.5$ (Case 1) and $p_{00} = 0.1$, $p_{01} = 0.2$, $p_{10} = 0.2$, $p_{11} = 0.5$ (Case 2). Table 5 shows the values obtained for $\psi(u)$ by importance sampling with the values for ρ_0 and for $\rho_P(X, W)$.

Table 6

Information obtained by importance sampling about the conditional distribution of the claim causing the ruin assuming Raftery bivariate exponential distribution (Cases 1 and 2).

Case	1	1	1	2	2	2
u	0	100	500	0	100	500
$E[X_{\sigma_u} \tau_u < \infty]$	8.32635	12.81813	12.63947	6.50317	13.13048	12.96192
$\text{Var}(X_{\sigma_u} \tau_u < \infty)$	29.94235	52.88868	50.68590	25.58575	69.53349	67.16921
$F_{X_{\sigma_u}}^{-1}(\tau_u < \infty) (0.001)$	0.48883	1.00315	1.04906	0.16034	0.80821	0.67447
$F_{X_{\sigma_u}}^{-1}(\tau_u < \infty) (0.01)$	1.04898	2.24367	2.19802	0.51526	1.73110	1.61518
$F_{X_{\sigma_u}}^{-1}(\tau_u < \infty) (0.1)$	2.71304	5.04596	4.89487	1.63129	4.32240	4.17643
$F_{X_{\sigma_u}}^{-1}(\tau_u < \infty) (0.5)$	7.05495	11.37632	11.35924	5.11525	11.34054	11.21650
$F_{X_{\sigma_u}}^{-1}(\tau_u < \infty) (0.9)$	15.53064	22.67247	22.12477	13.21539	24.46190	24.28467
$F_{X_{\sigma_u}}^{-1}(\tau_u < \infty) (0.99)$	26.56121	35.38690	35.03689	24.39351	39.73495	38.14429
$F_{X_{\sigma_u}}^{-1}(\tau_u < \infty) (0.999)$	37.96562	48.86099	47.06605	34.67310	56.85226	50.52593

Table 7

Values obtained by importance sampling about the conditional distribution of the minimal surplus prior to ruin assuming Raftery bivariate exponential distribution (Cases 1 and 2).

Case	1	2
$E[A_{1,\sigma_{100}} \tau_u < \infty]$	5.00378	5.99546
$\text{Var}(A_{1,\sigma_{100}} \tau_u < \infty)$	21.69617	40.83723
$F_{A_{1,\sigma_{100}}}^{-1}(\tau_u < \infty) (0.001)$	0.00641	0.00950
$F_{A_{1,\sigma_{100}}}^{-1}(\tau_u < \infty) (0.01)$	0.06134	0.05743
$F_{A_{1,\sigma_{100}}}^{-1}(\tau_u < \infty) (0.1)$	0.59971	0.55455
$F_{A_{1,\sigma_{100}}}^{-1}(\tau_u < \infty) (0.5)$	3.65406	3.89515
$F_{A_{1,\sigma_{100}}}^{-1}(\tau_u < \infty) (0.9)$	11.20908	14.15393
$F_{A_{1,\sigma_{100}}}^{-1}(\tau_u < \infty) (0.99)$	21.09926	30.11496
$F_{A_{1,\sigma_{100}}}^{-1}(\tau_u < \infty) (0.999)$	30.04665	45.43755

Table 8

Values obtained by importance sampling about the sum of claims up to ruin assuming Raftery bivariate exponential distribution (Cases 1 and 2).

Case	1	1	2	2
u	0	100	0	100
$E[A_{4,\sigma_u} \tau_u < \infty]$	27.48469	521.68111	19.25464	454.99306
$\text{Var}(A_{4,\sigma_u} \tau_u < \infty)$	5284.33807	101513.59675	1607.94599	38247.49805
$F_{A_{4,\sigma_u}}^{-1}(\tau_u < \infty) (0.001)$	0.32502	134.99361	0.18132	137.87116
$F_{A_{4,\sigma_u}}^{-1}(\tau_u < \infty) (0.01)$	0.95398	158.87008	0.49555	168.15236
$F_{A_{4,\sigma_u}}^{-1}(\tau_u < \infty) (0.1)$	2.75328	228.66904	1.76256	248.42026
$F_{A_{4,\sigma_u}}^{-1}(\tau_u < \infty) (0.5)$	8.97310	437.38795	7.59184	416.66486
$F_{A_{4,\sigma_u}}^{-1}(\tau_u < \infty) (0.9)$	52.42119	918.86292	43.89826	715.69402
$F_{A_{4,\sigma_u}}^{-1}(\tau_u < \infty) (0.99)$	360.63096	1675.50733	189.35263	1090.68607
$F_{A_{4,\sigma_u}}^{-1}(\tau_u < \infty) (0.999)$	934.53610	2606.48323	514.34348	1485.32623

Again, we observe that $\psi(u)$ decreases with ρ_p . In Table 6, we also provide information about the conditional distribution of X_{σ_u} for $u = 0, 100, 500$ in both cases.

From Table 6, we can reasonably expect that the conditional distribution of X_{σ_u} converges when u gets larger. This statement is confirmed by graphs of the conditional cdf's of X_{σ_u} which we do not provide here to lighten the presentation. Then, in Table 7, we give information about the conditional distribution of the minimal surplus prior to ruin A_{1,σ_u} for $u = 100$ in both cases.

The variance observed in Case 1 is twice smaller than in Case 2. This might be explained by a smaller $\psi(u)$ in Case 2 leading to a larger average time for ruin to occur. Finally, in Table 8, we also provide information about the conditional distribution of A_{4,σ_u} for $u = 0, 100$ in both cases.

Obviously, in both cases, a larger u leads to a larger $F_{A_{4,\sigma_u}}^{-1}(\tau_u < \infty)$. Also, for a given u , we notice that Case 1 yields larger figures, which is line with the intuition as $\psi(u)$ is larger in Case 1. \square

4.4. Bivariate distribution with FGM copula and exponential marginals

The FGM copula, given by

$$C_\theta(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2), \quad (29)$$

(see e.g. [27]), has been used notably by Cossette et al. [3] within a compound Poisson risk model with dependence and Chadjiconstantinidis and Vrontos [11] in the context of a compound renewal risk model with dependence. Here, we assume that $F_{X,W}(x, t) = C_\theta(F_X(x), F_W(t))$, with (29) and with exponential marginals ($X \sim \text{Exp}(\beta)$ and $W \sim \text{Exp}(\lambda)$). It implies that the expression for the joint pdf of (X, W) is given by

$$f_{X,W}(x_1, x_2) = \beta e^{-\beta x_1} \lambda e^{-\lambda x_2} + \theta \sum_{i=1}^2 \sum_{j=1}^2 (-1)^{i+j} \times i\beta e^{-i\beta x_1} \times j\lambda e^{-j\lambda x_2}.$$

The Pearson correlation coefficient is given by $\rho_P(X, W) = \frac{\theta}{4}$, where $\rho_P(X, W) \in [-\frac{1}{4}, \frac{1}{4}]$.

We note that the bivariate distribution defined with the FGM copula and exponential marginals can be seen as a combination of bivariate exponential distributions. Combinations of univariate exponential distributions are a subset within the family of matrix exponential distributions (see e.g. [28,29] for details). Similarly, combinations of bivariate exponential distributions are a subset within the family of bivariate matrix exponential distributions (see [30]).

Lemma 17. The joint pdf of (X, W) under $\mathbb{P}^{(\rho_\delta)}$ is

$$\begin{aligned} f_{X,W}^{(\rho_\delta)}(x, t) &= c_{1,1}^{(\rho_\delta)} e^{-(\beta-\rho_\delta)x} (\lambda + \rho_\delta + \delta) e^{-(\lambda+\rho_\delta+\delta)t} \\ &\quad + \theta \sum_{i=1}^2 \sum_{j=1}^2 (-1)^{i+j} \times c_{i,j}^{(\rho_\delta)} (i\beta - \rho_\delta) e^{-(i\beta-\rho_\delta)x} (j\lambda + \rho_\delta + \delta) e^{-(j\lambda+\rho_\delta+\delta)t}, \end{aligned}$$

where $c_{i,j}^{(\rho_\delta)} = \frac{i\beta}{i\beta-\rho_\delta} \times \frac{j\lambda}{j\lambda+\rho_\delta+\delta}$, for $i, j \in \{1, 2\}$.

Proof. The proof follows from an adaptation of Lemma 10 in the case of combinations of bivariate exponentials. ■

According to Lemma 17, the joint pdf of (X, W) under $\mathbb{P}^{(\rho_\delta)}$ cannot be defined by an FGM copula and exponential marginals. The marginals are now combinations of exponential distributions, i.e.

$$f_X^{(\rho_\delta)}(x) = c_{1,1}^{(\rho_\delta)} (\beta - \rho_\delta) e^{-(\beta-\rho_\delta)x} + \theta \sum_{i=1}^2 \sum_{j=1}^2 (-1)^{i+j} \times c_{i,j}^{(\rho_\delta)} (i\beta - \rho_\delta) e^{-(i\beta-\rho_\delta)x}$$

and

$$f_W^{(\rho_\delta)}(t) = c_{1,1}^{(\rho_\delta)} (\lambda + \rho_\delta + \delta) e^{-(\lambda+\rho_\delta+\delta)t} + \theta \sum_{i=1}^2 \sum_{j=1}^2 (-1)^{i+j} \times c_{i,j}^{(\rho_\delta)} (j\lambda + \rho_\delta + \delta) e^{-(j\lambda+\rho_\delta+\delta)t}.$$

However, the bivariate distribution remains a combination of bivariate exponential distributions whose marginals are combinations of exponential distributions (which can be seen as extensions of the exponential distribution).

In the following example, we compare the values obtained by simulation with the exact values given in [3].

Example 18. Let $\beta = 1$, $\lambda = \frac{1}{1.5}$, $\theta = 0.5$ and $\delta = \frac{0.05}{1.5}$, which are equivalent to parameters used in Example 8.1 of Cossette, et al. [3]. Numerical optimization leads to $\rho_0 = 0.378826$ and $\rho_\delta = 0.432150$. In Table 9, we provide the results for $\psi(u)$ and $m_{\delta,2}(u)$ with $w_2(y) = y$.

Even with ten times less simulations, the IS method is the closest to the exact values for both $\psi(u)$ and $m_{\delta,2}(u)$. An interesting ruin measure inspired from the ruin probability and the expected deficit at ruin has been investigated by Mitric and Trufin [31]. This measure is computed in two steps. First, for some specified probability level κ , we denote by u_κ the smallest amount of initial capital needed such that the infinite-time ruin probability is at most equal to κ , that is $u_\kappa = \inf\{u \geq 0 \mid \psi(u) \leq \kappa\}$. In the second step, we compute

$$\xi_{\kappa,\delta} = u_\kappa + E[e^{-\delta\tau_{u_\kappa}} \mid U(\tau_{u_\kappa}) \mid \tau_{u_\kappa} < \infty, U(0) = u_\kappa],$$

i.e. τ_{u_κ} is the time of ruin assuming that the initial capital is u_κ (i.e. the surplus process $\bar{U} = \{u_\kappa + ct - S(t), t \geq 0\}$). In other words, $\xi_{\kappa,\delta}$ represents the smallest amount of capital needed to ensure that the infinite-time ruin probability is smaller than κ and to cope in expectation with the first occurrence of a ruin event. In the context on this example, the values of the ruin measure $\xi_{\kappa,\delta}$ (with u_κ) are provided in Table 10. □

Let us mention that we could also consider the bivariate distribution defined with the FGM copula and mixed Erlang distributions as marginals (see [32]). It could be also of interest to consider the bivariate distribution defined with the Ali–Mikhail–Haq copula and exponential marginals but since, its treatment is similar to the one for the bivariate distribution defined with the FGM copula and exponential marginals, we refer the reader to Cossette et al. [15] for the details.

Table 9

Exact values, values obtained by crude Monte Carlo, and values obtained by importance sampling of ruin probabilities of the conditional distribution of the deficit at ruin assuming bivariate FGM exponential distribution.

u	$\psi(u)$ (exact)	$\psi(u)$ (CMC)	$\psi(u)$ (IS)	$m_{\delta,2}(u)$ (exact)	$m_{\delta,2}(u)$ (CMC)	$m_{\delta,2}(u)$ (IS)
0	0.64512	0.64454	0.64569	0.54568	0.54659	0.54523
5	0.09495	0.09548	0.09482	0.06443	0.06499	0.06445
10	0.01429	0.01358	0.01428	0.00743	0.00715	0.00742
20	0.00032	0.00030	0.00032	0.00010	0.00008	0.00010

Table 10

Exact values and values obtained by importance sampling for the ruin measure proposed by Mitric and Trufin [31] assuming the bivariate FGM exponential distribution.

κ	u_κ (exact)	u_κ (IS)	$\xi_{\kappa,\delta}$ (exact)	$\xi_{\kappa,\delta}$ (IS)
0.95	6.69297	6.67682	6.72397	6.70821
0.995	12.77120	12.76674	12.77344	12.76898

4.5. Moran–Downton's bivariate exponential distribution

We assume that (X, W) has a Moran–Downton bivariate exponential distribution with joint pdf given by

$$f_{X,W}(x, t) = \sum_{k=0}^{\infty} (1-\gamma) \gamma^k h\left(x; k+1, \frac{\beta}{1-\gamma}\right) h\left(t; k+1, \frac{\lambda}{1-\gamma}\right),$$

for $x, w \geq 0$ and $\gamma \in [0, 1)$. The Pearson correlation coefficient is $\rho_P(X, W) = \gamma$. Hence, $\gamma = 0$ corresponds to the independence case while $\gamma \rightarrow 1$ corresponds to the comonotonicity case. For further information on the Moran–Downton bivariate exponential distribution, see e.g. [33,24], or [34]. In their Example 5.5, Albrecher and Teugels [1] find the expression of the adjustment coefficient ρ_0 . Also, Ambagaspitiya and Thompson [35] proposed an extension to this model in a ruin theory context for which marginals are not restricted to exponential distributions. The interest of the Moran–Downton bivariate exponential distribution notably lies in its mathematical tractability. Also, it constitutes a special case of Kibble and Moran's bivariate gamma distribution, for which Ambagaspitiya [10] obtains an analytical expression for $\psi(u)$. The joint mgf of (X, W) is

$$\begin{aligned} M_{X,W}(r_1, r_2) &= \sum_{k=0}^{\infty} (1-\gamma) \gamma^k \left(1 - \frac{1-\gamma}{\beta} r_1\right)^{-(k+1)} \left(1 - \frac{1-\gamma}{\lambda} r_2\right)^{-(k+1)} \\ &= \frac{(1-\gamma)}{\left(1 - \frac{1-\gamma}{\beta} r_1\right) \left(1 - \frac{1-\gamma}{\lambda} r_2\right) - \gamma}. \end{aligned}$$

We obtain below a nice analytical expression for ρ_δ .

Lemma 19. The analytical expression for ρ_δ is

$$\rho_\delta = \frac{(\eta - \bar{\gamma} \frac{\delta}{\lambda}) + \sqrt{(\bar{\gamma} \frac{\delta}{\lambda} - \eta)^2 + 4\bar{\gamma}(1+\eta) \frac{\delta}{\lambda}}}{2\bar{\gamma}(1+\eta)} \beta,$$

where $\bar{\gamma} = 1 - \gamma$.

Remark 20. Note that $\rho_0 = \frac{1}{1-\gamma} \frac{\eta}{(1+\eta)} \beta$, which can be found also in Example 5.5 of Albrecher and Teugels [1]. Also, ρ_0 does not depend on the parameter λ of the distribution of the interclaim time rv W . Moreover, as the dependence parameter γ increases (i.e. as the positive dependence relation between the components of (X, W) increases), we observe that ρ_0 also increases, which implies that the riskiness of the risk process decreases. Moreover, when $\gamma = 0$, we obtain the expression of the adjustment coefficient within the classical compound Poisson risk model with exponentially distributed claims.

Applying Lemma 9, we obtain the expression for $f_{X,W}^{(\rho_\delta)}$.

Lemma 21. The joint pdf of (X, W) under $\mathbb{P}^{(\rho_\delta)}$ is

$$f_{X,W}^{(\rho_\delta)}(x, t) = \sum_{k=0}^{\infty} p_k^{(\rho_\delta)} h\left(x; k+1, \frac{\beta}{1-\gamma} - \rho_\delta\right) h\left(t; k+1, \frac{\lambda}{1-\gamma} + \rho_\delta + \delta\right),$$

where

$$p_k^{(\rho_\delta)} = (1-\gamma) \gamma^k \left(\frac{\beta}{\beta - (1-\gamma) \rho_\delta}\right)^{k+1} \left(\frac{\lambda}{\lambda + (1-\gamma) (\rho_\delta + \delta)}\right)^{k+1}, \quad k \in \mathbb{N}.$$

Table 11

Exact values and values obtained by importance sampling for ruin probabilities assuming Moran–Downton’s bivariate exponential distribution.

u	$\psi(u)$ (exact)	$\psi(u)$ (IS)
0	0.7034648	0.7034776
5	0.0577439	0.0576902
10	0.0047399	0.0047324
20	0.0000319	0.0000321

Table 12

Exact values and values obtained by importance sampling for Laplace transform of the time of ruin assuming Moran–Downton’s bivariate exponential distribution.

u	$\bar{G}_{0.03}(u)$ (IS)	
	$\gamma = 0.6$	$\gamma = 0.2$
0	0.6683739	0.7224677
5	0.0339860	0.1377348
10	0.0017340	0.0259881
20	0.0000044	0.0009380

Note that the joint distribution of (X, W) under $\mathbb{P}^{(\rho_\delta)}$ is no longer a Moran–Downton bivariate exponential distribution but rather a bivariate mixed Erlang distribution. Its marginals are mixed Erlang distributions with

$$f_X^{(\rho_\delta)}(x) = \sum_{k=0}^{\infty} p_k^{(\rho_\delta)} h\left(x; k+1, \frac{\beta}{1-\gamma} - \rho_\delta\right),$$

$$f_W^{(\rho_\delta)}(t) = \sum_{k=0}^{\infty} p_k^{(\rho_\delta)} h\left(t; k+1, \frac{\lambda}{1-\gamma} + \rho_\delta + \delta\right).$$

Example 22. Let $\beta = 1$, $\lambda = \frac{1}{1.25}$, and $\gamma = 0.6$. When $\delta = 0$, we find $\rho_0 = 0.5$ using Lemma 19. The expression of $\psi(u)$ (only) is obtained in [10, Example 1] from which we have computed the exact values for different initial capitals u . In Table 11, we provide the results for the ruin probability $\psi(u)$.

Again, as for Example 18, the IS method performs very well. For $\delta = 3\%$, we obtain $\rho_{0.03} = 0.595867$ with Lemma 19. The values of $\bar{G}_{0.03}(u)$, given in Table 12, are obtained by importance sampling and computed for $\gamma = 0.6$ and $\gamma = 0.2$.

We note that $\bar{G}_{0.03}(u)$ decreases with γ which is in line with our intuition as larger dependence leads to a lower ruin probability and hence to a larger average ruin time. \square

Acknowledgments

This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (research grants CG053993 and CG053934) and by the Chaire en actuariat de l’Université Laval (research grant F0502323). The authors are grateful to two anonymous referees for their valuable comments. Julien Trufin also thanks the financial support of Risk Dynamics under grant “New approaches in risk measurement within advanced risk models in actuarial science” (research grant ON105572). Etienne Larrivée-Hardy is grateful for the Intact Grant (AXA 9005).

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