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Distributed control problems for a class of degenerate semilinear evolution equations

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ABSTRACT

The unique solvability in the sense of strong solution of initial problems for a class of semilinear first order differential equations in Banach spaces with degenerate operator at the derivative is studied. It allows to prove the existence of a solution for the optimal control problem to systems, described by these initial problems. Abstract results are illustrated by the examples of degenerate systems of partial differential equations not solvable with respect to the time derivatives and of optimal control problems for them.

1. Introduction

The article presents a study of control problems for distributed systems, described by degenerate semilinear evolution equations that are not resolved with respect to the time derivative. Interest to nonlinear control problems based on their practical importance (see recent papers [1,2] and others). In Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}$ the statement of a problem for the operator equation

$$L\dot{x}(t) = Mx(t) + N(t, x(t)) + Bu(t), \tag{1.1}$$

$$x(t_0) = x_0, \tag{1.2}$$

$$u \in \mathcal{U}_\partial, \tag{1.3}$$

$$J(x, u) = \|x - \tilde{x}\|^2_{W_2^1(t_0,T;\mathcal{X})} + \|u - \tilde{u}\|^2_{L_2(t_0,T;\mathcal{U})}, \tag{1.4}$$

where the set of admissible controls $\mathcal{U}_\partial$ is a nonempty closed convex subset of controls space $\mathcal{U}$, functions $\tilde{x}$, $\tilde{u}$ are given, operators $B \in \mathcal{L}(\mathcal{U}; \mathcal{Y})$, $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, i. e. are linear and continuous. The operator L has a nontrivial kernel $\ker L \neq \{0\}$. Also, we assume that the operator M is a linear, closed, densely defined in $\mathcal{X}$ ($M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$) and the operator of N is a nonlinear,

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defined on an open set $Y \subset \mathbf{R} \times \mathcal{X}$. Such problem is an abstract framework for studying of control problems for various real systems, describing by the systems of partial differential equations not solved with respect to time derivatives [3–5].

Firstly the existence of a unique strong solution of problem (1.1), (1.2) was proved in the case $\mathcal{X} = \mathcal{Y}$, $L = I$ with Caratheodory mapping N. Then these results and methods of degenerate operator semigroups theory [5] were used for investigation of problem (1.1), (1.2) solvability with degenerate operator L. Similar results were obtained in [6] but in the case of a smooth operator N. The research of initial problem (1.1), (1.2) allows to study the optimal control problem (1.1)–(1.4). Examples in the last section illustrate general results.

This work is a continuation of optimal control problems research for degenerate distributed systems in [7–9], where linear degenerate distributed control systems are considered.

2. The Cauchy problem for nondegenerate semilinear equation

Let $\mathcal{Z}$ be Banach space, $A \in \mathcal{L}(\mathcal{Z})$. Suppose that an operator $B : (t_0, T) \times \mathcal{Z} \to \mathcal{Z}$ is Caratheodory mapping, i. e. for every $z \in \mathcal{Z}$ it defines measurable mapping on (t_0, T) and for almost all $t \in (t_0, T)$ it is continuous in $z \in \mathcal{Z}$. Consider Cauchy problem

$$z(t_0) = z_0, \tag{2.1}$$

for the semilinear equation

$$\dot{z}(t) = Az(t) + B(t, z(t)). \tag{2.2}$$

A strong solution of (2.1), (2.2) on (t_0, T) is a function $z \in W_q^1(t_0, T; \mathcal{Z})$, $q \in (1, \infty)$, for which condition (2.1) and almost everywhere on (t_0, T) equality (2.2) hold.

Lemma 2.1. *Let $A \in \mathcal{L}(\mathcal{Z})$, operator $B : (t_0, T) \times \mathcal{Z} \to \mathcal{Z}$ be Caratheodory mapping, for all $z \in \mathcal{Z}$ and almost all $t \in (t_0, T)$ the estimate*

$$\|B(t, z)\|_{\mathcal{Z}} \le a(t) + c\|z\|_{\mathcal{Z}} \tag{2.3}$$

be satisfied with some $a \in L_q(t_0, T; R)$, $c > 0$. Then for every $z_0 \in \mathcal{Z}$ the function $z \in W_q^1(t_0, T; \mathcal{Z})$ is a strong solution of problem (2.1), (2.2) if and only if $z \in L_q(t_0, T; \mathcal{Z})$ and almost everywhere on (t_0, T)

$$z(t) = e^{(t-t_0)A}z_0 + \int_{t_0}^{t} e^{(t-s)A}B(s, z(s))ds. \tag{2.4}$$

Proof. Let z be a strong solution of problem (2.1), (2.2), then by condition (2.3) operator B is bounded from $L_q(t_0, T; \mathcal{Z})$ to $L_q(t_0, T; \mathcal{Z})$. Integrating equality (2.2) on the interval (t_0, t), we obtain (2.4).

Let $z \in L_q(t_0, T; \mathcal{Z})$ almost everywhere on (t_0, T) satisfy (2.4), then the function $B(\cdot, z(\cdot)) \in L_q(t_0, T; \mathcal{Z})$ and it can be checked directly that z is a strong solution of (2.1), (2.2). •

Call a mapping $B : (t_0, T) \times \mathcal{Z} \to \mathcal{Z}$ uniformly Lipschitz continuous in z, if there exists such $l > 0$, that for all (t, y), (t, z) from $(t_0, T) \times \mathcal{Z}$ the inequality $\|B(t, y) - B(t, z)\|_{\mathcal{Z}} \le l\|y - z\|_{\mathcal{Z}}$ holds. Put $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$.

Theorem 2.1. *Let $A \in \mathcal{L}(\mathcal{Z})$, an operator $B : (t_0, T) \times \mathcal{Z} \to \mathcal{Z}$ be Caratheodory mapping, uniformly Lipschitz continuous in z, and $B(\cdot, v) \in L_q(t_0, T; \mathcal{Z})$ for some $v \in \mathcal{Z}$. Then for every $z_0 \in \mathcal{Z}$ problem (2.1), (2.2) has a unique strong solution on (t_0, T).*

Proof. It follows from the uniform Lipschitz continuity that $\|B(t, z)\|_{\mathcal{Z}} \le \|B(t, v)\|_{\mathcal{Z}} + l\|v\|_{\mathcal{Z}} + l\|z\|_{\mathcal{Z}}$ for all $z \in \mathcal{Z}$, a. e. on (t_0, T), therefore, condition (2.3) is performed with $a(t) = \|B(t, v)\|_{\mathcal{Z}} + l\|v\|_{\mathcal{Z}}$, $c = l$. By Lemma 2.1 it is enough to show that Eq. (2.2) has a unique solution $z \in L_q(t_0, T; \mathcal{Z})$.

In the Banach space $L_q(t_0, T; \mathcal{Z})$ we define an operator G by the equality

$$G(z)(t) = e^{(t-t_0)A}z_0 + \int_{t_0}^{t} e^{(t-s)A}B(s, z(s))ds.$$

By condition (2.3) $G : L_q(t_0, T; \mathcal{Z}) \to L_q(t_0, T; \mathcal{Z})$. As the G^r we denote the rth power of the operator G, $r \in \mathbf{N}$. If $T - t_0 < 1$ in the subsequent discussion we will replace $T - t_0$ by the unit. For almost all $t \in (t_0, T)$, $r \in \mathbf{N}$, $y, z \in L_q(t_0, T; \mathcal{Z})$ we will prove by the induction the inequality

$$\|G^r(y)(t) - G^r(z)(t)\|_{\mathcal{Z}} \le \frac{K^r(t - t_0)^{r-1/q}}{(r-1)!}\|y - z\|_{L_q(t_0,T;\mathcal{Z})}, \tag{2.5}$$

where $K = le^{(T-t_0)\|A\|_{\mathcal{L}(\mathcal{Z})}}$. While $r = 1$ with help of a Hölder's inequality we have almost everywhere on (t_0, T)

$$\|G(y)(t) - G(z)(t)\|_{\mathcal{Z}} \le le^{(T-t_0)\|A\|_{\mathcal{L}(\mathcal{Z})}}(t - t_0)^{1-1/q}\|y - z\|_{L_q(t_0,T;\mathcal{Z})}.$$

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Assuming that for $r-1$ inequality (2.5) runs, we get

$$\|G^r(y)(t)-G^r(z)(t)\|_{\mathcal{Z}} \le K\int_{t_0}^{t}\|G^{r-1}(y)(s)-G^{r-1}(z)(s)\|_{\mathcal{Z}}ds$$
$$\le K^r\int_{t_0}^{t}\frac{(s-t_0)^{r-1-1/q}}{(r-2)!}\|y-z\|_{L_q(t_0,T;\mathcal{Z})}ds \le \frac{K^r(t-t_0)^{r-1/q}}{(r-1)!}\|y-z\|_{L_q(t_0,T;\mathcal{Z})}.$$

From (2.5) it follows that for $r\in\mathbf{N}$

$$\|G^r(y)-G^r(z)\|_{L_q(t_0,T;\mathcal{Z})} \le \frac{K^r(T-t_0)^r}{(r-1)!(rq)^{1/q}}\|y-z\|_{L_q(t_0,T;\mathcal{Z})}.$$

Therefore, if r is sufficiently large, then G^r is a strict contraction in $L_q(t_0,T;\mathcal{Z})$, so it has a unique fixed point. Fixed point of G is a fixed point for mapping G^r also. It is a unique solution of Eq. (2.4) in the space $L_q(t_0,T;\mathcal{Z})$, hence it is a unique strong solution of problem (2.1), (2.2) on the interval (t_0,T). •

Further we will consider the equation

$$\dot{z}(t)=Az(t)+B(t,z(t))+f(t) \tag{2.6}$$

with a nonlinear mapping B which is smooth in both variables and with a function f, such that its smoothness will be the minimum necessary.

Theorem 2.2. *Let $A\in\mathcal{L}(\mathcal{Z})$, $n\in\mathbf{N}_0$, $B\in C^n([t_0,T]\times\mathcal{Z};\mathcal{Z})$ be uniformly Lipschitz continuous in z, $f\in W_q^n(t_0,T;\mathcal{Z})$. Then for any $z_0\in\mathcal{Z}$ there exists a unique strong solution $z\in W_q^{n+1}(t_0,T;\mathcal{Z})$ of problem (2.1), (2.6).*

Proof. It is obvious that the conditions of Theorem 2.2 are satisfied. Then there exists a solution $z\in W_q^1(t_0,T;\mathcal{Z})$ of problem (2.1), (2.6). Because of $B\in C^1([t_0,T]\times\mathcal{Z};\mathcal{Z})$ in the case $n\ge 1$, the function $t\to B(t,z(t))$ has a derivative a. e. on (t_0,T), hence $\ddot{z}(t)=A\dot{z}(t)+D_tB(t,z(t))+\dot{f}(t)$. Here D_tB is the total derivative in t. Therefore $z\in W_q^2(t_0,T;\mathcal{Z})\subset C^1([t_0,T];\mathcal{Z})$ by the Sobolev embedding theorem. As D_t^r for $r\in\mathbf{N}$ we will denote the rth total derivative. Continuing the process, we obtain

$$z^{(r)}(t)=A^rz(t)+\sum_{k=0}^{r-1}A^k\left(D_t^{(r-1-k)}B(t,z(t))+f^{(r-1-k)}(t)\right).$$

Whenever the order of z derivative on the left-hand side one more than the order of the highest derivative of z on the right-hand side. It allows you to continue a chain of reasoning, differentiating the right-hand side of the last equation. The expressions from the right will contain continuous derivatives of B, the continuous derivatives of z by embedding theorem, and a first power of one derivative of order, one less than on the left-hand side. By the previous step of the proof it is already in $L_q(t_0,T;\mathcal{Z})$. The chain of reasoning breaks off when the derivative of B on the right-hand side will be of the order n. On the left-hand side we will obtain the derivative of the function z of the order $n+1$. •

3. Degenerate semilinear equation

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, $L\in\mathcal{L}(\mathcal{X};\mathcal{Y})$, $M\in\mathcal{C}l(\mathcal{X};\mathcal{Y})$. Let us denote $\rho^L(M)=\{\mu\in C:(\mu L-M)^{-1}\in\mathcal{L}(\mathcal{Y};\mathcal{X})\}$, $R_\mu^L(M)=(\mu L-M)^{-1}L$, $L_\mu^L(M)=L(\mu L-M)^{-1}$. An operator M is called (L,σ)-bounded if

$$\exists a>0\ \forall\mu\in C\quad(|\mu|>a)\Rightarrow(\mu\in\rho^L(M)).$$

In a case of (L,σ)-bounded operator M and contour $\gamma=\{\mu\in C:|\mu|=r>a\}$, the operators

$$P=\frac{1}{2\pi i}\int_\gamma R_\mu^L(M)\,d\mu\in\mathcal{L}(\mathcal{X}),\qquad Q=\frac{1}{2\pi i}\int_\gamma L_\mu^L(M)\,d\mu\in\mathcal{L}(\mathcal{Y})$$

are projections. Put $\mathcal{X}^0=\ker P$, $\mathcal{Y}^0=\ker Q$; $\mathcal{X}^1=\mathrm{im}P$, $\mathcal{Y}^1=\mathrm{im}Q$. Denote by L_k (M_k) the restriction of L (M) to $\mathcal{X}^k$ ($D_{M_k}=D_M\cap\mathcal{X}^k$), $k=0,1$.

Theorem 3.1 ([5]). *Let an operator M be (L,σ)-bounded. Then*

(i) $\mathcal{X}=\mathcal{X}^0\oplus\mathcal{X}^1$, $\mathcal{Y}=\mathcal{Y}^0\oplus\mathcal{Y}^1$;
(ii) $M_1\in\mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$, $M_0\in\mathcal{C}l(\mathcal{X}^0;\mathcal{Y}^0)$, $L_k\in\mathcal{L}(\mathcal{X}^k;\mathcal{Y}^k)$, $k=0,1$;
(iii) *there exist operators* $M_0^{-1}\in\mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$, $L_1^{-1}\in\mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$.

Put $H=M_0^{-1}L_0$. For $p\in\mathbf{N}_0$ an operator M is called (L,p)-bounded if it is (L,σ)-bounded, $H^p\ne 0$, $H^{p+1}=0$.

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Consider a semilinear evolution equation

$$\frac{d}{dt}Lx(t) = Mx(t) + N(t, x(t)) + f(t) \tag{3.1}$$

with operators $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $\ker L \neq \{0\}$, $M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$, nonlinear operator $N : (t_0, T) \times \mathcal{X} \to \mathcal{Y}$ and function $f : (t_0, T) \to \mathcal{Y}$. A function $x \in L_q(t_0, T; \mathcal{X})$, $q \in (1, \infty)$, is called a strong solution of Eq. (3.1) on (t_0, T), if $Lx \in W_q^1(t_0, T; \mathcal{Y})$, almost everywhere on $(t_0, T) x(t) \in D_M$ and equality (3.1) holds. A strong solution of the generalized Showalter–Sidorov problem (see [8–10])

$$Px(t_0) = x_0 \tag{3.2}$$

to Eq. (3.1) on (t_0, T) is a strong solution $x \in W_q^1(t_0, T; \mathcal{X})$ of the equation that satisfies condition (3.2).

Theorem 3.2. *Let $p \in \mathbf{N}_0$, an operator M be (L, p)-bounded, a mapping $N : [t_0, T] \times \mathcal{X} \to \mathcal{Y}$ such that $QN \in C^{\max\{0,p-1\}}([t_0, T] \times \mathcal{X}; \mathcal{Y})$ be uniformly Lipschitz continuous in x, $H^k M_0^{-1}(I - Q)N \in C^k([t_0, T] \times \mathcal{X}; \mathcal{X})$ while $k = 0, 1, \dots, p$, for all $(t, x) \in [t_0, T] \times \mathcal{X}$ the equality*

$$N(t, x) = N(t, Px) \tag{3.3}$$

be valid, $Qf \in W_q^{\max\{0,p-1\}}(t_0, T; \mathcal{Y})$, $H^k M_0^{-1}(I - Q)f \in W_q^k(t_0, T; \mathcal{X})$ for $k = 0, 1, \dots, p$. Then for every $x_0 \in \mathcal{X}^1$ problem (3.1), (3.2) has a unique strong solution on (t_0, T).

Proof. Alternately we multiply (3.1) from the left by the continuous operators $L_1^{-1}Q$ and $M_0^{-1}(I-Q)$. With help of condition (3.3) we obtain the problem

$$\begin{aligned} &\dot{v}(t) = S_1 v(t) + L_1^{-1} Q\left[N(t, v(t)) + f(t)\right], \\ &v(t_0) = x_0, \end{aligned} \tag{3.4}$$

$$\frac{d}{dt}Hw(t) = w(t) + M_0^{-1}(I - Q)\left[N(t, v(t)) + f(t)\right] \tag{3.5}$$

for the pair of functions $v(t) = Px(t)$, $w(t) = (I - P)x(t)$. Here, as above, we use the notation $S_1 = L_1^{-1}M_1$, $H = M_0^{-1}L_0$. By Theorem 2.2 with the power $n = \max\{0, p - 1\}$ problem (3.4) has a unique strong solution $v \in W_q^{\max\{1,p\}}(t_0, T; \mathcal{X}^1)$, because of the operator S_1 is bounded by Theorem 3.1.

Knowing the v and using a nilpotency of the operator H, find the solution

$$w(t) = -\sum_{k=0}^{p} \frac{d^k}{dt^k} H^k M_0^{-1}(I - Q)\left[N(t, v(t)) + f(t)\right] \tag{3.6}$$

of (3.5), as in [5], where $\frac{d^k}{dt^k} H^k M_0^{-1}(I - Q)N(t, v(t))$ is a total derivative in t of the order k. As in the proof of Theorem 2.2 here we use the continuity of the operators $H^k M_0^{-1}(I - Q)N$ derivatives, the Sobolev embedding theorem for all derivatives of v, except the oldest, which in this case presents only in the first power. It is worth noting that $L\frac{d^p}{dt^p} H^p M_0^{-1}(I - Q)[N(t, v(t)) + f(t)] = M_0 H \frac{d^p}{dt^p} H^p M_0^{-1}(I - Q)[N(t, v(t)) + f(t)] \equiv 0$, hence $Lx \in W_q^1(t_0, T; \mathcal{X})$. •

A strong solution of Cauchy problem

$$x(t_0) = x_0 \tag{3.7}$$

for degenerate Eq. (3.1) on the interval (t_0, T) is defined analogously to the previous problem. In contrast to the generalized Showalter–Sidorov problem, the Cauchy problem for degenerate equations requires concordance condition between the initial data and the right-hand side of the equation.

Theorem 3.3. *Let $p \in \mathbf{N}_0$, an operator M be (L, p)-bounded, a mapping $N : [t_0, T] \times \mathcal{X} \to \mathcal{Y}$ such that $QN \in C^{\max\{0,p-1\}}([t_0, T] \times \mathcal{X}; \mathcal{Y})$ be uniformly Lipschitz continuous in x, $H^k M_0^{-1}(I - Q)N \in C^k([t_0, T] \times \mathcal{X}; \mathcal{X})$ while $k = 0, 1, \dots, p$, for all $(t, x) \in [t_0, T] \times \mathcal{X}$ the equality $N(t, x) = N(t, Px)$ holds, $Qf \in W_q^{\max\{0,p-1\}}(t_0, T; \mathcal{Y})$, $H^k M_0^{-1}(I - Q)f \in W_q^k(t_0, T; \mathcal{X})$ while $k = 0, 1, \dots, p$, for $x_0 \in \mathcal{X}$ the equality*

$$(I - P)x_0 = -\sum_{k=0}^{p} \left.\frac{d^k}{dt^k}\right|_{t=t_0} H^k M_0^{-1}(I - Q)\left[N(t, v(t)) + f(t)\right] \tag{3.8}$$

be valid, where $v \in W_q^p(t_0, T; \mathcal{X}^1)$ is a solution of problem (3.4). Then problem (3.1), (3.7) has a unique solution on (t_0, T).

Proof. We note only that concordance condition (3.8) means satisfying of condition (3.7) for solution (3.6) of Eq. (3.5). •

Theorem 3.4. *Let $p \in \mathbf{N}_0$, an operator M be (L,p)-bounded, an operator $N : (t_0,T) \times \mathcal{X} \to \mathcal{Y}$ be Caratheodory mapping, uniformly Lipschitz continuous in x, for some $z \in \mathcal{X}$ $N(\cdot,z) \in L_q(t_0,T;\mathcal{Y})$, $\mathrm{im}N \subset \mathcal{Y}^1$, $Qf \in L_q(t_0,T;\mathcal{Y})$, $H^kM_0^{-1}(I-Q)f \in W_q^k(t_0,T;\mathcal{X})$ while $k = 0,1,\ldots,p$. Then, for any $x_0 \in \mathcal{X}^1$ problem (3.1), (3.2) has a unique strong solution on (t_0,T).*

Proof. If $\mathrm{im}N \subset \mathcal{Y}^1$, then $(I-Q)N \equiv 0$. Eq. (3.1) after multiplying by the operator $M_0^{-1}(I-Q)$ has a form $\frac{d}{dt}Hw(t) = w(t) + M_0^{-1}(I-Q)f$. By virtue of the nilpotency of the operator H this equation has a unique solution $w(t) = -\sum_{k=0}^{p} \frac{d^k}{dt^k}H^kM_0^{-1}(I-Q)f(t)$.

It remains to use Theorem 2.1 to show the unique solvability of the problem $v(t_0) = Px_0$ to the equation $\dot{v}(t) = S_1v(t) + L_1^{-1}Qf(t) + L_1^{-1}N(t, v(t) + w(t))$, that is obtained from the problem (3.1), (3.2) with help of the action of the operator $L_1^{-1}Q$. Indeed, the nonlinear operator $B(t,v(t)) = L_1^{-1}N(t,v(t)+w(t)) + L_1^{-1}Qf(t)$ with already known w satisfies the conditions of Theorem 2.1. •

Theorem 3.5. *Let $p \in \mathbf{N}_0$, an operator M be (L,p)-bounded, a mapping $N : (t_0,T) \times \mathcal{X} \to \mathcal{Y}$ be Caratheodory, uniformly Lipschitz continuous in x, for some $z \in \mathcal{X}$ $N(\cdot,z) \in L_q(t_0,T;\mathcal{Y})$, $\mathrm{im}N \subset \mathcal{Y}^1$, $Qf \in L_q(t_0,T;\mathcal{Y})$, $H^kM_0^{-1}(I-Q)f \in W_q^k(t_0,T;\mathcal{X})$ for $k = 0,1,\ldots,p$, $x_0 \in \mathcal{X}$ satisfy the equality*

$$(I-P)x_0 = -\sum_{k=0}^{p} \left.\frac{d^k}{dt^k}\right|_{t=t_0} H^kM_0^{-1}(I-Q)f(t).$$

Then (3.1), (3.2) have a unique strong solution on (t_0,T).

Remark 3.1. Instead of Eq. (3.1) consider the equation

$$L\frac{d}{dt}x(t) = Mx(t) + N(t,x(t)) + f(t), \tag{3.9}$$

when operators L and d/dt are written in the reverse order. A strong solution of (3.9) is corresponding function x from $W_q^1(t_0,T;\mathcal{X})$. What is the difference between properties of Eqs. (3.1) and (3.9)? As it follows from the considerations above, for the proof of the strong solvability of Eq. (3.9) by the proposed methods we need a slightly more strong conditions on the smoothness of the problem data. Namely, in Theorems 3.2, 3.3 they have the form $QN \in C^p([t_0,T] \times \mathcal{X};\mathcal{Y})$, $Qf \in W_q^p(t_0,T;\mathcal{Y})$, $H^kM_0^{-1}(I-Q)N \in C^{k+1}([t_0,T] \times \mathcal{X};\mathcal{X})$, $H^kM_0^{-1}(I-Q)f \in W_q^{k+1}(t_0,T;\mathcal{X})$ for $k = 0,1,\ldots,p$. In Theorems 3.4, 3.5 we need $H^kM_0^{-1}(I-Q)f \in W_q^{k+1}(t_0,T;\mathcal{X})$ for $k = 0,1,\ldots,p$.

4. Distributed control for semilinear degenerate equation

Now let $\mathcal{X}, \mathcal{Y}, \mathcal{U}$ be Hilbert spaces, $L \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $\ker L \neq \{0\}$, $B \in \mathcal{L}(\mathcal{U};\mathcal{Y})$, $M \in \mathcal{C}l(\mathcal{X};\mathcal{Y})$, $N : [t_0,T] \times \mathcal{X} \to \mathcal{Y}$. Consider control problem

$$L\frac{d}{dt}x(t) = Mx(t) + N(t,x(t)) + Bu(t), \tag{4.1}$$

$$x(t_0) = x_0, \tag{4.2}$$

$$u \in \mathcal{U}_\partial, \tag{4.3}$$

$$J(x,u) = \frac{1}{2}\|x-\tilde{x}\|^2_{W_2^1(t_0,T;\mathcal{X})} + \frac{C}{2}\|u-\tilde{u}\|^2_{L_2(t_0,T;\mathcal{U})} \to \inf, \tag{4.4}$$

where $\tilde{x} \in W_2^1(t_0,T;\mathcal{X})$, $\tilde{u} \in L_2(t_0,T;\mathcal{U})$ are given functions, $C > 0$, *a set of admissible controls* $\mathcal{U}_\partial$ is a nonempty closed convex subset of the space $L_2(t_0,T;\mathcal{U})$.

In the study of the optimal control problem (4.1)–(4.4) we will use the concept of a strong solution of Cauchy problem (4.1), (4.2). Taking into account the form of Eq. (4.1), its strong solutions we will seek in a Hilbert space

$$\mathcal{Z} = \{z \in W_2^1(t_0,T;\mathcal{X}) : z(t) \in \mathrm{dom}M \text{ a. e. on } (t_0,T),\ L\dot{z} - Mz \in L_2(t_0,T;\mathcal{Y})\}$$

with the norm $\|z\|_{\mathcal{Z}} = \|z\|_{W_2^1(t_0,T;\mathcal{X})} + \|L\dot{z} - Mz\|_{L_2(t_0,T;\mathcal{Y})}$. Its completeness is proved, for example, in [8].

Introduce the operator $\gamma_0 : W_2^1(t_0,T;\mathcal{X}) \to \mathcal{X}$, $\gamma_0x = x(t_0)$. By Sobolev embedding theorem it is continuous operator.

Set of pairs (x,u) will be called as *admissible pairs set* $\mathcal{W}$ of problem (4.1)–(4.4) if $u \in \mathcal{U}_\partial$, $x \in W_2^1(t_0,T;\mathcal{X})$ is a strong solution of (4.1), (4.2). Problem (4.1)–(4.4) is a problem of finding pairs $(\hat{x},\hat{u}) \in \mathcal{W}$, which minimizes the cost functional $J(x,u)$:

$$J(\hat{x},\hat{u}) = \inf_{(x,u)\in\mathcal{W}} J(x,u).$$

Theorem 4.1. *Let $p \in \mathbf{N}_0$, an operator M be (L,p)-bounded, an operator $N : [t_0,T] \times \mathcal{X} \to \mathcal{Y}$ be Caratheodory mapping, uniformly Lipschitz continuous in x, for some $z \in \mathcal{X}$ $N(\cdot,z) \in L_2(t_0,T;\mathcal{Y})$, $\mathrm{im}N \subset \mathcal{Y}^1$, $B[\mathcal{U}_\partial] \cap \mathcal{Y}^1 \neq \emptyset$, $\mathcal{U}_\partial$ be a nonempty closed convex subset of the space $L_2(t_0,T;\mathcal{U})$, $x_0 \in \mathcal{X}^1$. Then there exists a solution $(\hat{x},\hat{u}) \in \mathcal{Z} \times \mathcal{U}_\partial$ of problem (4.1)–(4.4).*

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Proof. In the context of this section, the space $\mathcal{X}$ is a Hilbert space, and thus the reflexive Banach. For a fixed $u \in L_2(t_0, T; \mathcal{U})$ we introduce the operator $N_u : (t_0, T) \times \mathcal{X} \to \mathcal{Y}$ defined by the equality $N_u(t, x) = N(t, x) + Bu(t)$. It is obvious that the operator N_u is uniformly Lipschitz continuous in x, Caratheodory mapping and

$$\|N(t, x(t)) + Bu(t)\|_{\mathcal{Y}} \le a(t) + c\|x\|_{\mathcal{X}} + \|B\|_{\mathcal{L}(\mathcal{U};\mathcal{Y})}\|u\|_{L_2(t_0,T;\mathcal{U})} = \tilde{a}(t) + c\|x\|_{\mathcal{X}}.$$

In addition, by the condition of the theorem, there exists $u \in \mathcal{U}_\partial$, such that $Bu \in \mathcal{Y}^1$. It corresponds to the operator N_u such that $\mathrm{im}N_u \subset \mathcal{Y}^1$. Hence, Theorem 3.5 and Remark 3.1 imply the existence of a strong solution of Cauchy problem (4.1), (4.2) for the pair $(x_0, u) \in \left(\mathrm{dom}M \cap \mathcal{X}^1\right) \times \mathcal{U}_\partial$ with the corresponding u. So, the set of admissible pairs $\mathcal{W}$ is nonempty.

Further we will use Theorem 1.2.4 [11]. Put $Y = W_2^1(t_0, T; \mathcal{X}), Y_1 = \mathcal{Z}, U = L_2(t_0, T; \mathcal{U}), V = L_2(t_0, T; \mathcal{Y}) \times \mathcal{X}, \mathcal{F}(x(\cdot)) = (-N(\cdot, x(\cdot)), x_0), \mathcal{L}(x, u) = (L\dot{x} - Mx - Bu, \gamma_0 x)$. The continuity of the linear operator $\mathcal{L} : Y_1 \times U \to V$ follows from the inequalities

$$\|(L\dot{x} - Mx - Bu, \gamma_0 x)\|^2_{L_2(t_0,T;\mathcal{Y})\times\mathcal{X}} \le C_1 \left(\|L\dot{x} - Mx\|^2_{L_2(t_0,T;\mathcal{Y})} + \|x\|^2_{W_2^1(t_0,T;\mathcal{X})} + \|u\|^2_{L_2(t_0,T;\mathcal{U})}\right) = C_1\|(x, u)\|^2_{\mathcal{Z}\times\mathcal{U}}.$$

From the uniform Lipschitz continuity in x of the operator N for an arbitrary $t \in [t_0, T], x \in \mathcal{X}$ obtain $\|N(t, x)\|_{\mathcal{Y}} \le l\|x\|_{\mathcal{X}} + l\|z\|_{\mathcal{X}} + \|N(t, z)\|_{\mathcal{Y}} = a(t) + l\|x\|_{\mathcal{X}}$, where $a(\cdot) = \|z\|_{\mathcal{X}} + \|N(\cdot, z)\|_{\mathcal{Y}} \in L_2(t_0, T; R)$. Therefore, the operator N acts to the space $L_2(t_0, T; \mathcal{Y})$.

Let us verify the coercivity of J. We have the inequalities

$$\begin{aligned}\|x\|^2_{\mathcal{Z}} + \|u\|^2_{L_2(t_0,T;\mathcal{U})} &= \|x\|^2_{W_2^1(t_0,T;\mathcal{X})} + \|N(\cdot, x(\cdot)) + Bu\|^2_{L_2(t_0,T;\mathcal{Y})} + \|u\|^2_{L_2(t_0,T;\mathcal{U})} \\ &\le (1 + 4l^2)\|x\|^2_{W_2^1(t_0,T;\mathcal{X})} + 4\|a(\cdot)\|^2_{L_2(t_0,T;\mathbf{R})} + (2\|B\|^2_{\mathcal{L}(\mathcal{U};\mathcal{Y})} + 1)\|u\|^2_{L_2(t_0,T;\mathcal{U})} \\ &\le K_1 J(x, u) + K_2.\end{aligned}$$

From the relation $\|x_n - x\|_{\mathcal{Z}} \to 0$ for $n \to \infty$ it follows that

$$\|N(\cdot, x_n(\cdot)) - N(\cdot, x(\cdot))\|^2_{L_2(t_0,T;\mathcal{Y})} = \int_{t_0}^{T} \|N(t, x_n(t)) - N(t, x(t))\|^2_{\mathcal{Y}}\, dt \le l^2\|x_n - x\|^2_{L_2(t_0,T;\mathcal{X})} \to 0,$$

thus we proved the continuity of the operator $\mathcal{F}$.

After choosing $\mathcal{Y}_{-1} = L_2(t_0, T; \mathcal{X})$, check the remaining conditions of Theorem 1.2.4 [11]. Condition (1) of the compactness follows from Rellich–Kondrashov Theorem. To check condition (2) as $S \subset L_2(t_0, T; \mathcal{Y})$ we choose a dense lineal $C([t_0, T]; \mathcal{Y})$. Then for $v \in C([t_0, T]; \mathcal{Y})$ the uniform Lipschitz continuity of the operator N implies the inequality

$$\langle N(t, x_n(t)) - N(t, x(t)), v(t)\rangle_{L_2(t_0,T;\mathcal{Y})} \le l\|v\|_{L_2(t_0,T;\mathcal{Y})}\|x_n - x\|_{L_2(t_0,T;\mathcal{X})}.$$

It gives the continuous extension of the functional $f(\cdot) = \langle\mathcal{F}(\cdot), v\rangle$ from $\mathcal{Z}$ to $L_2(t_0, T; \mathcal{X})$. •

As $H_\partial(x_0)$ denote the set of functions $u \in L_2(t_0, T; \mathcal{U})$ such that $QBu \in W_2^p(t_0, T; \mathcal{Y}), H^k M_0^{-1}(I - Q)Bu \in W_2^{k+1}(t_0, T; \mathcal{X})$, $k = 0, 1, \ldots, p$, and for the given $x_0 \in \mathcal{X}$

$$(I - P)x_0 + \sum_{k=0}^{p} \frac{d^k}{dt^k} H^k M_0^{-1}(I - Q)N(t, v(t))|_{t=t_0} = -\sum_{k=0}^{p} \frac{d^k}{dt^k} H^k M_0^{-1}(I - Q)Bu(t)|_{t=t_0},$$

if there exists a solution of problem (3.4) with $f = Bu$.

Theorem 4.2. *Suppose that an operator M is (L, p)-bounded, $p \in \mathbf{N}_0, N : [t_0, T] \times \mathcal{X} \to \mathcal{Y}$, the mapping $QN \in C^p([t_0, T] \times \mathcal{X}; \mathcal{Y})$ is uniformly Lipschitz continuous in x, while $k = 0, 1, \ldots, p, H^k M_0^{-1}(I - Q)N \in C^{k+1}([t_0, T] \times \mathcal{X}; \mathcal{X})$, for all $(t, x) \in [t_0, T] \times \mathcal{X}$ the equality $N(t, x) = N(t, Px)$ holds, $\mathcal{U}_\partial$ is a nonempty closed convex subset of the space $L_2(t_0, T; \mathcal{U}), x_0 \in \mathcal{X}$, $\mathcal{U}_\partial \cap H_\partial(x_0) \ne \emptyset$. Then there exists a solution $(\hat{x}, \hat{u}) \in \mathcal{Z} \times \mathcal{U}_\partial$ of problem (4.1)–(4.4).*

Proof. As in the proof of Theorem 4.1, the operator $N_u : (t_0, T) \times \mathcal{X} \to \mathcal{Y}$ will be used. The condition $\mathcal{U}_\partial \cap H_\partial(x_0) \ne \emptyset$ implies (3.8) satisfying for at least one control from the set $\mathcal{U}_\partial$. Thus, nonemptiness of the set $\mathcal{W}$ follows from Theorem 3.3 and Remark 3.1. Remaining arguments were made in the proof of Theorem 4.1. •

For problem (4.1), (4.3), (4.4) with the condition

$$Px(t_0) = x_0 \tag{4.5}$$

by virtue of Theorems 3.2, 3.4 and Remark 3.1 obtains the following results.

Theorem 4.3. *Let $p \in \mathbf{N}_0$, an operator M be (L, p)-bounded, a mapping $N : [t_0, T] \times \mathcal{X} \to \mathcal{Y}$ such that $QN \in C^p([t_0, T] \times \mathcal{X}; \mathcal{Y})$ be uniformly Lipschitz continuous in x, $H^k M_0^{-1}(I - Q)N \in C^{k+1}([t_0, T] \times \mathcal{X}; \mathcal{X})$ while $k = 0, 1, \ldots, p$, for all $(t, x) \in [t_0, T] \times \mathcal{X}$ the equality $N(t, x) = N(t, Px)$ holds, $\mathcal{U}_\partial$ be a nonempty closed convex subset of the space $L_2(t_0, T; \mathcal{U}), x_0 \in \mathcal{X}^1, \mathcal{U}_\partial \cap W_2^{p+1}(t_0, T; \mathcal{U}) \ne \emptyset$. Then there exists a solution $(\hat{x}, \hat{u}) \in \mathcal{Z} \times \mathcal{U}_\partial$ of problem (4.1), (4.3)–(4.5).*

Theorem 4.4. *Let $p \in \mathbf{N}_0$, an operator M be (L, p)-bounded, an operator $N : (t_0, T) \times \mathfrak{X} \to \mathfrak{Y}$ be Caratheodory mapping and uniformly Lipschitz continuous in x, for some $z \in \mathfrak{X}$ $N(\cdot, z) \in L_2(t_0, T; \mathfrak{Y})$, $\operatorname{im}N \subset \mathfrak{Y}^1$, $\mathfrak{U}_\partial$ be a nonempty closed convex subset of the space $L_2(t_0, T; \mathfrak{U})$, $B[\mathfrak{U}_\partial] \cap \mathfrak{Y}^1 \neq \emptyset$, $x_0 \in \mathfrak{X}^1$. Then there exists a solution $(\hat{x}, \hat{u}) \in \mathfrak{Z} \times \mathfrak{U}_\partial$ of problem (4.1), (4.3)–(4.5).*

5. Example

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Consider the initial–boundary value problem

$$x_1(s, t_0) = x_{10}(s), \quad s \in \Omega, \tag{5.1}$$

$$x_i(s, t) = 0, \quad (s, t) \in \partial\Omega \times (t_0, T), \; i = 1, 2, 3, \tag{5.2}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\triangle x_1 = x_1 + g_1(s, t, x_1, x_2, x_3), \quad (s, t) \in \Omega \times (t_0, T), \\ &\frac{\partial}{\partial t}\triangle x_3 = x_2 + g_2(s, t, x_1, x_2, x_3), \quad (s, t) \in \Omega \times (t_0, T), \\ &0 = \triangle x_3 + g_3(s, t, x_1, x_2, x_3), \quad (s, t) \in \Omega \times (t_0, T). \end{aligned} \tag{5.3}$$

Let A be Laplace operator with domain $W_{2,0}^2(\Omega) = \{z \in W_2^2(\Omega) : z(s) = 0, \; s \in \partial\Omega\} \subset L_2(\Omega)$, $\{\varphi_k\}$ be the orthonormal in $L_2(\Omega)$ system of its eigenfunctions corresponding to the system $\{\lambda_k\}$ of operator A eigenvalues, indexed in nonincreasing order taking into account their multiplicities.

For reducing problem (5.1)–(5.3) to problem (3.1), (3.2) choose spaces $\mathfrak{X} = W_{2,0}^2(\Omega) \times L_2(\Omega) \times W_{2,0}^2(\Omega)$, $\mathfrak{Y} = (L_2(\Omega))^3$ and operators

$$L = \begin{pmatrix} \triangle & 0 & 0 \\ 0 & 0 & \triangle \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y}), \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \triangle \end{pmatrix} \in \mathcal{L}(\mathfrak{X}; \mathfrak{Y}).$$

It is easy to verify that $(\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{Y}; \mathfrak{X})$ where $|\mu| > |\lambda_1|^{-1}$ and projections have a form

$$P = Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

that is why $\mathfrak{X}^1 = W_{2,0}^2(\Omega) \times \{0\} \times \{0\}$, $\mathfrak{X}^0 = \{0\} \times L_2(\Omega) \times W_{2,0}^2(\Omega)$, $\mathfrak{Y}^1 = L_2(\Omega) \times \{0\} \times \{0\}$, $\mathfrak{Y}^0 = \{0\} \times L_2(\Omega) \times L_2(\Omega)$, $H = \begin{pmatrix} 0 & \triangle \\ 0 & 0 \end{pmatrix}$.Therefore $H^2 = 0$ and the operator M is $(L, 1)$-bounded.

It is clear that initial condition (5.1) in this case is the generalized Showalter–Sidorov condition (3.2). Theorem 3.2 in this case can be used when considering the problem (5.1), (5.2) for the system of equations

$$\begin{aligned} &\frac{\partial}{\partial t}\triangle x_1 = x_1 + g_1(s, t, x_1), \quad (s, t) \in \Omega \times (t_0, T), \\ &\frac{\partial}{\partial t}\triangle x_3 = x_2 + g_2(s, t, x_1), \quad (s, t) \in \Omega \times (t_0, T), \\ &0 = \triangle x_3 + g_3(s, t, x_1), \quad (s, t) \in \Omega \times (t_0, T), \end{aligned}$$

where g_i depends only on x_1. Theorem 3.4 concerns to the system

$$\begin{aligned} &\frac{\partial}{\partial t}\triangle x_1 = x_1 + g_1(s, t, x_1, x_2, x_3), \quad (s, t) \in \Omega \times (t_0, T), \\ &\frac{\partial}{\partial t}\triangle x_3 = x_2 + f_2(s, t), \quad (s, t) \in \Omega \times (t_0, T), \\ &0 = \triangle x_3 + f_3(s, t), \quad (s, t) \in \Omega \times (t_0, T), \end{aligned}$$

with a nonlinear part only in the first equation.

For the last system consider the control problem with following statement

$$\begin{aligned} &\frac{\partial}{\partial t}\triangle x_1 = x_1 + g_1(s, t, x_1, x_2, x_3) + u_1(s, t), \quad (s, t) \in \Omega \times (t_0, T), \\ &\frac{\partial}{\partial t}\triangle x_3 = x_2 + f_2(s, t) + u_2(s, t), \quad (s, t) \in \Omega \times (t_0, T), \\ &0 = \triangle x_3 + f_3(s, t) + u_3(s, t), \quad (s, t) \in \Omega \times (t_0, T), \end{aligned} \tag{5.4}$$

$$\|u_1\|_{L_2(t_0,T;L_2(\Omega))}^2 + \|u_2\|_{L_2(t_0,T;L_2(\Omega))}^2 + \|u_3\|_{L_2(t_0,T;L_2(\Omega))}^2 \leq R^2, \tag{5.5}$$

$$J(x_1, x_2, x_3) = \frac{1}{2}\sum_{k=1,3} \|x_i - \tilde{x}_i\|_{W_2^1(t_0,T;W_2^2(\Omega))}^2 + \frac{1}{2}\|x_2 - \tilde{x}_2\|_{W_2^1(t_0,T;L_2(\Omega))}^2 + \frac{C}{2}\sum_{k=1}^{3} \|u_i\|_{L_2(t_0,T;L_2(\Omega))}^2 \to \inf, \tag{5.6}$$

where $\tilde{x}_i \in W_2^1(t_0, T; W_2^2(\Omega))$ for $k = 1, 3$ and $\tilde{x}_2 \in W_2^1(t_0, T; L_2(\Omega))$ are given. Here $\mathfrak{Z} = W_2^1(t_0, T; \mathfrak{X})$, $\mathfrak{U} = L_2(t_0, T; \mathfrak{Y})$.

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Theorem 5.1. *Suppose that g_1 is Caratheodory function, that uniformly Lipschitz continuous in x_1, x_2, x_3 and for some $(z_1, z_2, z_3) \in \mathcal{X}$ $g_1(\cdot, z_1, z_2, z_3) \in L_2(t_0, T; L_2(\Omega))$, a function $x_{10} \in W^2_{2,0}(\Omega)$. Then the problem (5.4)–(5.6) has a solution $(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{u}_1, \hat{u}_2, \hat{u}_3)$ in $W^1_2(t_0, T; \mathcal{X}) \times \mathcal{U}_\partial$.*

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