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P. Maroju, Ramandeep Behl, S.S. Motsa

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Some novel and optimal families of King's method with eighth and sixteenth-order of convergence

P. Maroju, Ramandeep Behl [1] and S. S. Motsa

School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, Pietermaritzburg, South Africa

Abstract

In this study, our principle aim is to provide some novel eighth and sixteenth-order families of King's method for solving nonlinear equations which should be superior than the existing schemes of same order. The relevant optimal orders of the proposed families satisfy the classical Kung-Traub conjecture which was made in 1974. The derivations of the proposed schemes are based on the weight function and rational approximation approaches, respectively. In addition, convergence properties of the proposed families are fully investigated along with one lemma and two main theorems describing their order of convergence. We consider a concrete variety of real life problems e.g. the trajectory of an electron in the air gap between two parallel plates, chemical engineering problem, Van der Waal's equation which explains the behavior of a real gas by introducing in the ideal gas equations and fractional conversion in a chemical reactor, in order to check the validity, applicability and effectiveness of our proposed methods. Further, it is found from the numerical results that our proposed methods perform better than the existing ones of the same order when the accuracy is checked in the multi precision digits.

keywords: Order of convergence, Newton's method, King's method, Simple roots, Iterative methods.

1 Introduction

Finding the solution techniques to solve the nonlinear equations, have always been a paramount importance in the field of numerical analysis which provide the accurate and efficient approximate solution α of a nonlinear equation of the form

$$f(x) = 0. \tag{1.1}$$

One of the main reason of paramount importance of this topic is the applicability in the applied science and the four major disciples of engineering: chemical, electrical, civil and mechanical (for the detailed explanation please see the Chapra and Canale [1]). For example, the location of the extremal points of a function describing some system requires finding the zeros of the derivatives of that function, many problems which involve critical paths also require the solution of algebraic equations, such as determining all the ray paths

[1]Corresponding author: Ramandeep Behl
E-mail: ramanbehl87@yahoo.in

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that are possible in a complex optical system. Those problems can be modeled by different mathematical equations.

Analytic methods for finding the solutions of such problems are almost non-existent. So, we have to turn towards the iterative method which can provide the approximate solution corrected up to any specific degree of accuracy. Multi-point iterative methods belong to the class of most powerful methods that overcome from the theoretical limitations of one-point iterative methods regarding their order of convergence and efficiency (for the details please see [2, 3]).

Due to the advancement of digital computer, advanced computer arithmetics and symbolic computation, the construction of higher-order multi-point methods become more vital and popular because they provide more accurate and efficient approximated root with in a very small number of iterations and their efficiency index [2] is better than the classical Newton's method. Therefore, in the last two decades, a variety of optimal eighth-order multi-point methods, without memory have been proposed in [4–17]. Most of them are the extension of Newton's method or Newton like method at the expense of additional functional evaluations or increase the substep of the original methods.

It is often desirable to obtain higher-order and more accurate root-finding techniques for obtaining the roots of nonlinear equations. In 1974, Kung and Traub [18], proposed two general classes of n-point iterative methods with first-order derivative/derivatives of the involved function and without any derivative. After some years later, Neta [19], given an optimal sixteenth-order family of multi-point iterative methods. However, Neta did not present an explicit form of the error equation and more recently it was given by Guem and Kim [20]. In the recent years, scholars like Guem and Kim [20, 21], Sharma et al. [22], Ullah et al. [23], have also presented optimal sixteenth-order extension of iterative methods. In addition to this, Li et al. [24] also have proposed a sixteenth-order scheme but not optimal. Nowadays, obtaining new four-step optimal methods of order sixteen is very interesting and challenging task in the field of numerical analysis. One of the reason behind the attention of sixteenth-order iterative methods is the efficiency indices of these methods $E = \sqrt[5]{16} \approx 1.741$, which is far better than the classical Newton's method $E \approx 1.414$.

The principle aim of this manuscript is to propose a more accurate and efficient solution technique of order sixteen as compared to the existing ones. In addition, we have also discussed some important cases which were not mentioned in earlier study proposed by Artidiello et al. [25]. Then, we extended this family from eighth-order convergence to sixteenth-order with the help of rational functional approximations approach. The proposed scheme satisfy the classical Kung-Traub conjecture. The beauty of the proposed scheme is that we can develop several new optimal methods of order sixteen by considering different types of weight functions. The efficiency of the proposed methods is tested on a variety of real life problems which can be seen in the numerical section 4. The numerical experiments demonstrate that our proposed methods perform better than existing optimal methods of order sixteen in terms of absolute residual errors, difference between the two consecutive iterations and asymptotic error constants.

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2 An optimal family of eighth-order King's method

In this section, we will present an optimal eighth-order family of King's methods [26]. Therefore, we consider the following three-step scheme:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)} \frac{f(y_n)}{f'(x_n)}, \quad \beta \in \mathbb{R}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} G(u,\ v), \end{aligned} \tag{2.1}$$

where the above weight function $G : \mathbb{C}^2 \to \mathbb{C}$, is an analytic function in the neighborhood of $(0,\ 0)$ and

$$u = \frac{f(z_n)}{f(y_n)}, \quad v = \frac{f(y_n)}{f(x_n)}. \tag{2.2}$$

Since, the above scheme (2.1) uses only four functional evaluations. Then, according to Kung-Traub conjecture its maximum order can be eight. In the following results, we will discuss the conditions on the weight function so that we will reach at an optimal eighth-order of convergence.

No doubts, this scheme is also independently derived by Artidiello et al. [25] using weight functions at second and third step. However, we discussed some special cases of iterative methods which were not mentioned in the earlier study. In addition of this, we did the convergence analysis in the complex analysis $\mathbb{C}$ in stead of $\mathbb{R}$ and also shown the applicability of these methods to the real life problems (for the details please see the numerical section 4). Moreover, Artidiello et al. [25] expanded the weight function of second step only up to second term and they did not talk about the third term. However, some member from third term are also involve in the final error equation of eighth-order method which can be seen in the expression (2.20).

Lemma 2.1 *Let us assume that α be a simple zero of the involved function f. Further, we also assume that the function $f : \mathbb{C} \to \mathbb{C}$, is an analytic function in a region enclosing the required zero α. Then, the quotients defined in* (2.2) *will satisfy the following error equations*

$$u = \frac{f(z_n)}{f(y_n)} = O(e_n^2), \quad v = \frac{f(y_n)}{f(x_n)} = O(e_n). \tag{2.3}$$

Proof Let us assume that $e_n = x_n - \alpha$ be the error in the n^{th} iteration. Now, we can expand the function $f(x_n)$ around the point $x = \alpha$ with the help of the Taylor's series expansion which will leads us:

$$f(x_n) = f'(\alpha) \left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9) \right), \tag{2.4}$$

where $f'(\alpha) \neq 0$ and $c_k = \frac{1}{k!} \frac{f^k(\alpha)}{f'(\alpha)},\ \ k = 2,\ 3,\ \ldots, 8$.
Similarly, we will obtain

$$f'(x_n) = f'(\alpha) \left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + 9c_9 e_n^8 + O(e_n^9) \right). \tag{2.5}$$

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With the help of the above equations (2.4), (2.5) and the Taylor's series expansion, we will further have

$$f(y_n) = f'(\alpha)\left(c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + \sum_{j=0}^{4} A_j e_n^{j+4} + O(e_n^9)\right), \tag{2.6}$$

where $A_0 = (5c_2^3 - 7c_2c_3 + 3c_4)$, $A_1 = -2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)$, $A_2 = 28c_2^5 - 73c_2^3c_3 + 34c_2^2c_4 - 17c_3c_4 + c_2(37c_3^2 - 13c_5) + 5c_6$, $A_3 = -2\{32c_2^6 - 103c_2^4c_3 - 9c_3^3 + 52c_2^3c_4 + 6c_4^2 + c_2^2(80c_3^2 - 22c_5) + 11c_3c_5 + c_2(-52c_3c_4 + 8c_6) - 3c_7\}$ and $A_4 = (37 + 334\beta + 172\beta^2 + 70\beta^3 + 18\beta^4 + 2\beta^5)c_2^7 - 2(90 + 582\beta + 271\beta^2 + 86\beta^3 + 12\beta^4)c_2^5c_3 + (101 + 490\beta + 178\beta^2 + 30\beta^3)c_2^4c_4 + 2(25 + 36\beta)c_3^2c_4 - 17c_4c_5 + c_2^3\{(253 + 1098\beta + 432\beta^2 + 80\beta^3)c_3^2 - (51 + 156\beta + 32\beta^2)c_5\} - 13c_3c_6 + c_2^2\{10(2 + 3\beta)c_6 - \left(209 + 612\beta + 144\beta^2\right)c_3c_4\} + c_2\{4(17 + 24\beta)c_3c_5 - (91 + 240\beta + 64\beta^2)c_3^3 + (37 + 54\beta)c_4^2 - 5c_7\}$.

Further, with the help of previous equations (2.4) and (2.6), we will further obtain

$$\begin{aligned} v = \frac{f(y_n)}{f(x_n)} &= c_2 e_n + (2c_3 - 3c_2^2)e_n^2 + (8c_2^3 - 10c_3c_2 + 3c_4)e_n^3 + (4c_5 - 20c_2^4 + 37c_2^2c_3 - 8c_3^2 - 14c_2c_4)e_n^4 \\ &+ \{48c_2^5 - 118c_2^3c_3 + 51c_2^2c_4 - 22c_3c_4 + c_2(55c_3^2 - 18c_5) + 5c_6\}e_n^5 + \{344c_2^4c_3 - 112c_2^6 + 26c_3^3 \\ &- 163c_2^3c_4 - 15c_4^2 - 28c_3c_5 + c_2^2(65c_5 - 252c_3^2) + 2c_2(75c_3c_4 - 11c_6) + 6c_7\}e_n^6 + O(e_n^7). \end{aligned} \tag{2.7}$$

Now, by inserting the equations (2.4)–(2.6), in the second substep of (2.1), we have

$$z_n - \alpha = \{(2\beta + 1)c_2^3 - c_2c_3\}e_n^4 + \sum_{j=0}^{3} B_j e_n^{j+5} + O(e_n^9), \tag{2.8}$$

where $B_j = B_j(\beta,\ c_2,\ c_3,\ \dots\ c_8)$ are constant functions of $\beta,\ c_2,\ c_3,\ \dots\ c_8$ and two of them are $B_0 = -2\{(2 + 6\beta + \beta^2)c_2^4 - 2(2 + 3\beta)c_2^2c_3 + c_3^2 + c_2c_4\}$ and $B_1 = 2(5 + 22\beta + 7\beta^2 + \beta^3)c_2^5 - 2(15 + 42\beta + 8\beta^2)c_2^3c_3 + 6(2 + 3\beta)c_2^2c_4 - 7c_3c_4 + 3c_2\{(6 + 8\beta)c_3^2 - c_5\}$, etc.

In the similar fashion as we did in the previous equation (2.4), we can expand the function $f(z_n)$ about a point $x = \alpha$, which is given by

$$f(z_n) = f'(\xi)\left[\{(2\beta+1)c_2^3 - c_2c_3\}e_n^4 + B_0e_n^5 + B_1e_n^6 + B_2e_n^7 + [\{(2\beta+1)c_2^3 - c_2c_3\}^2c_2 + B_3]e_n^8 + O(e_n^9)\right]. \tag{2.9}$$

From the equations (2.6) and (2.9), we have

$$\begin{aligned} u = \frac{f(z_n)}{f(y_n)} &= \left((2\beta + 1)c_2^2 - c_3\right)e_n^2 - 2\{(1 + 4\beta + \beta^2)c_2^3 - 2(1 + 2\beta)c_2c_3 + c_4\}e_n^3 + \{(1 + 18\beta + 10\beta^2 + 2\beta^3)c_2^4 \\ &- 2(3 + 19\beta + 6\beta^2)c_2^2c_3 + (3 + 8\beta)c_3^2 + (5 + 12\beta)c_2c_4 - 3c_5\}e_n^4 - 2\{(14\beta + 13\beta^2 + 6\beta^3 + \beta^4 - 2)c_2^5 \\ &- (48\beta + 33\beta^2 + 8\beta^3 - 2)c_2^3c_3 + (2 + 26\beta + 9\beta^2)c_2^2c_4 - 3(1 + 4\beta)c_3c_4 + c_2\big(2(1 + 14\beta + 6\beta^2)c_3^2 \\ &- (3 + 8\beta)c_5\big) + 2c_6\}e_n^5 + O(e_n^6). \end{aligned} \tag{2.10}$$

This complete the proof of lemma 2.1. □

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Theorem 2.2 *Let us assume that an initial guess $x = x_0$ is sufficiently close to α for the guaranteed convergence. Then, the iterative scheme (2.1) will reach an optimal eighth-order convergence only if it satisfies the following conditions on the weight function*

$$G_{00} = 1,\ G_{01} = 2,\ G_{10} = 1,\ G_{02} = 10 - 4\beta,\ G_{11} = 4,\ G_{03} = 12(\beta^2 - 6\beta + 6), \tag{2.11}$$

where $G_{ij} = \frac{\partial^{i+j}}{\partial u^i \partial v^j} G(u,\ v)|_{(u=0,\ v=0)}$ *for* $i, j = 0, 1, 2, 3$.

Proof Since, it is clear from the above lemma 2.1 that $u = O(e_n^2)$ and $v = O(e_n)$. Therefore, we can expand the weight function $G(u,\ v)$ in the neighborhood of $(0,\ 0)$ with the help Taylor series expansion which leads to us:

$$\begin{aligned} G(u,\ v) &= G_{00} + G_{10}u + G_{01}v + \frac{1}{2!}\left(G_{20}u^2 + 2G_{11}uv + G_{02}v^2\right) \\ &+ \frac{1}{3!}\left(G_{30}u^3 + 3G_{21}u^2v + 3G_{12}uv^2 + G_{03}v^3\right). \end{aligned} \tag{2.12}$$

By using the equations (2.4)–(2.10) and (2.12), in the last substep of the proposed scheme, we will get

$$e_{n+1} = -(G_{00} - 1)c_2\left((2\beta + 1)c_2^2 - c_3\right)e_n^4 + \sum_{j=1}^{4} H_j e_n^{j+4} + O(e_n^9), \tag{2.13}$$

where $H_j = H_j(\beta,\ c_2,\ c_3,\ \ldots,\ c_8)$ are the constant functions in term of $\beta, c_2, c_3, \ldots, c_8$, e.g. $H_1 = -\{4 + G_{01} + 12\beta + 2G_{01}\beta + 2\beta^2 - 2G_{00}(3 + 8\beta + \beta^2)\}c_2^4 + \{8 + G_{01} + 12\beta - 2G_{00}(5 + 6\beta)\}c_2^2c_3 + 2(G_{00} - 1)c_3^2 + 2(G_{00} - 1)c_2c_4$. It is straightforward to say that the above error equation (2.13) will reach at least fifth-order of convergence if we choose the following value of G_{00}

$$G_{00} = 1. \tag{2.14}$$

Now, by inserting the above value of $G_{00} = 1$ in $H_1 = 0$, we will further obtain

$$G_{01} = 2. \tag{2.15}$$

Again, by using the above values of G_{00} and G_{01} in $H_2 = 0$, we will obtain the following two independent relations

$$G_{10} - 1 = 0, \quad G_{02} + 2(2\beta G_{10} + G_{10} - 6) = 0, \tag{2.16}$$

which further yield

$$G_{10} = 1, \qquad G_{02} = 10 - 4\beta. \tag{2.17}$$

In order to obtain an optimal eighth-order of convergence, we have to use the above values of $G_{00},\ G_{01}, G_{10}$ and G_{02} in $H_3 = 0$. Then, we obtain

$$G_{11} - 4 = 0,\ \ G_{03} + 6\left(-2\beta^2 + 4\beta + 2\beta G_{11} + G_{11} - 16\right) = 0, \tag{2.18}$$

which will further leads to us

$$G_{11} = 4,\ G_{03} = 12(\beta^2 - 6\beta + 6). \tag{2.19}$$

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Finally, we will obtain the following error equation by using the equations (2.14), (2.15), (2.17) and (2.19) in (2.13), which is given by

$$\begin{aligned} e_{n+1} = -\frac{c_2((2\beta+1)c_2^2-c_3)}{2}\Big[&c_2^4\left(4\beta^3-32\beta^2+44\beta+2\beta G_{12}+G_{12}+(2\beta+1)^2G_{20}-82\right) \\ &+c_3^2(G_{20}-2)-2c_4c_2-c_3c_2^2(-4\beta+G_{12}+4\beta G_{20}+2G_{20}-30)\Big]e_n^8+O(e_n^9). \end{aligned} \tag{2.20}$$

The above error equation demonstrate that our proposed scheme (2.1) reaches at eighth-order convergence by consuming only four functional evaluations per iteration. Therefore, the scheme (2.1) have also reached the optimal order of convergence in the sense of Kung-Traub conjecture. This completes the proof. □

2.1 Special cases

In this section, we will discuss some of the important cases of the proposed scheme which were not discussed in the earlier study. We can choose weight functions by two different approaches. In the first way, we can directly use the same weight function which is used for the construction of the proposed scheme. In the second way, we can pick any two variable function which satisfies the conditions of Theorem 2.2. Both approaches are defined as follows:

Approach-1:

In this case, we will use directly the conditions on G_{ij} (which are defined in (2.11)) in the weight function (2.12). This is the first way of obtaining the new weight functions, which is given as follows:

$$\begin{aligned} G(u,v) = 1+u+2v&+\frac{1}{2}\left(G_{20}u^2+8uv+(10-4\beta)v^2\right) \\ &+\frac{1}{6}\left(G_{30}u^3+3G_{12}uv^2+3G_{21}u^2v+12\left(\beta^2-6\beta+6\right)v^3\right), \end{aligned} \tag{2.21}$$

where G_{20}, G_{12}, G_{21} and G_{30} are free disposable parameters. By using the above weight function and the values of the disposable parameters in the scheme (2.1), we will obtain several optimal eighth-order families of King's method.

Approach-2:

(a) Let us consider the following weight function

$$G(u,v) = \frac{2\beta+u\left(2\beta+2(\beta^2-2\beta-4)v-5\right)-(4\beta+1)v^2+2(\beta^2-4\beta+1)v-5}{2\beta+2\left(\beta^2-6\beta+6\right)v-5}. \tag{2.22}$$

Clearly, the above function satisfies the conditions of Theorem 2.2, which will further yield

$$\begin{aligned} y_n &= x_n-\frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n-\frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n-\frac{f(z_n)}{f'(x_n)}\left(\frac{2\beta+u\left(2\beta+2(\beta^2-2\beta-4)v-5\right)-(4\beta+1)v^2+2(\beta^2-4\beta+1)v-5}{2\beta+2\left(\beta^2-6\beta+6\right)v-5}\right). \end{aligned} \tag{2.23}$$

In this way, we obtain an optimal eighth-order family of King's method.

(b) In order to obtain another optimal eighth-order family of King's method, we consider the following weight function

$$G(u,v) = 1 + u + 4uv - \frac{(4\beta+1)v}{2\left(\beta^2-6\beta+6\right)} + \frac{a_2 v}{a_1 v+1}, \tag{2.24}$$

where $a_1 = \frac{2\left(\beta^2-6\beta+6\right)}{2\beta-5}$ and $a_2 = \frac{2\beta-5}{a_1}$.

With the aid of the above weight function, we will further obtain the following optimal scheme of eighth-order

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left(1 + 4uv + u - \frac{(4\beta+1)v}{2\left(\beta^2-6\beta+6\right)} + \frac{a_2 v}{a_1 v+1}\right). \end{aligned} \tag{2.25}$$

In the similar fashion, if we consider more two varible functions provided the conditions of Theorem 2.2 should be satisfy then we will further obtain several optimal iterative methods of order eight.

3 An optimal family of sixteenth-order King's method

This section is devoted to the main contribution of this study. We will propose a new optimal sixteenth-order family of iterative methods based on rational functional approach. The idea is to consider one additional substep to the previous family (2.1). So, we use the same notation as in the previous section and rewrite the scheme (2.1) in the following way

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ t_n &= z_n - \frac{f(z_n)}{f'(x_n)}G(u,\ v). \end{aligned} \tag{3.1}$$

In order to obtain the next approximation x_{n+1} to the required root, we consider the following rational function

$$Q(x) = \frac{(x-x_n)+\theta_1}{\theta_2(x-x_n)^3+\theta_3(x-x_n)^2+\theta_4(x-x_n)+\theta_5}, \tag{3.2}$$

where $\theta_1,\ \theta_2,\ \ \theta_3,\ \theta_4$ and θ_5 are disposable parameters. These parameters can be determined by imposing tangency conditions, which are given by

$$Q(x_n) = f(x_n),\ \ Q'(x_n) = f'(x_n),\ \ Q(y_n) = f(y_n),\ \ Q(z_n) = f(z_n),\ \ Q(t_n) = f(t_n). \tag{3.3}$$

Now, we assume that the above rational function meets the $x-$ axis at a point $x = x_{n+1}$ in order to find the next approximation, which is given by

$$Q(x_{n+1}) = 0, \tag{3.4}$$

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which further yields

$$x_{n+1} = x_n - \theta_1. \tag{3.5}$$

By imposing the first two tangency conditions, we have

$$\theta_1 = \theta_5 f(x_n), \;\; \theta_4 = \frac{1-\theta_5 f'(x_n)}{f(x_n)}. \tag{3.6}$$

From the last three tangency conditions, we obtain

$$\begin{aligned}
&f(y_n)\big[f'(x_n)\left(f'(x_n)\left(2\theta_5 f'(x_n)-1\right)+\theta_3 f(x_n)^2\right)-\theta_2 f(x_n)^3\big] = f'(x_n)^2 f(x_n)\left(\theta_5 f'(x_n)-1\right),\\
&f(z_n)\left[\frac{(1-\theta_5 f'(x_n))(z_n-x_n)}{f(x_n)}+\theta_2(z_n-x_n)^3+\theta_3(x_n-z_n)^2+\theta_5\right] = \theta_5 f(x_n)+z_n-x_n,\\
&f(t_n)\left[\frac{(1-\theta_5 f'(x_n))(t_n-x_n)}{f(x_n)}+\theta_2(t_n-x_n)^3+\theta_3(t_n-x_n)^2+\theta_5\right] = \theta_5 f(x_n)+t_n-x_n.
\end{aligned} \tag{3.7}$$

Solve the above expressions in (3.7) for θ_5. Then, we have

$$\theta_5 = \frac{a_n b_n\big(u_1 f(x_n)^2 f(y_n)+u_2 f'(x_n) f(t_n) f(z_n)\big)}{v_1 f(x_n)^3+v_2 f'(x_n) f(t_n) f(z_n)}, \tag{3.8}$$

where

$$\begin{aligned}
&u_1 = f(t_n)\big(b_n^2 f'(x_n)+b_n f(x_n)-c_n f(z_n)\big)+a_n\big(f(x_n)-a_n f'(x_n)\big)f(z_n),\\
&u_2 = a_n b_n c_n f'(x_n)\big(f(y_n)-f(x_n)\big)+c_n f(y_n) f(x_n)(a_n-b_n),\\
&v_1 = f(y_n)\big[b_n f(t_n)\left(b_n^2 f'(x_n)+b_n f(x_n)-c_n f(z_n)\right)+\big(a_n^3 f'(x_n)+c_n a_n f(t_n)-a_n^2 f(x_n)\big)f(z_n)\big],\\
&v_2 = a_n^2 b_n^2 c_n f'(x_n)^2\big(2f(y_n)-f(x_n)\big)+a_n b_n c_n(2a_n-c_n)f'(x_n)f(y_n)f(x_n)+c_n\big(a_n b_n-a_n c_n-b_n^2\big)f(y_n)f(x_n)^2,\\
&a_n = x_n-z_n, \; b_n = t_n-x_n, \; c_n = t_n-z_n.
\end{aligned}$$

Now, by using the equations (3.1), (3.5) and (3.8), we obtain

$$\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\,\frac{f(y_n)}{f'(x_n)},\\
t_n &= z_n - \frac{f(z_n)}{f'(x_n)}G(u,\; v),\\
x_{n+1} &= x_n - \theta_5 f(x_n).
\end{aligned} \tag{3.9}$$

where θ_5 is defined by (3.8). The following Theorem 3.1 demonstrates that the optimal sixteenth-order of convergence is achieved.

Theorem 3.1 *Under the assumptions of Theorem 2.2, the iterative scheme defined by* (3.9) *has an optimal sixteenth-order convergence and satisfies the following error equation*

$$\begin{aligned}
e_{n+1} = &-\frac{c_2^3((2\beta+1)c_2^2-c_3)^2}{2}\Big[c_2^4\left(4\beta^3-32\beta^2+44\beta+2\beta G_{12}+G_{12}+(2\beta+1)^2 G_{20}-82\right)\\
&+c_3^2(G_{20}-2)-2c_4c_2-c_3c_2^2(-4\beta+G_{12}+4\beta G_{20}+2G_{20}-30)\Big](c_2^4-3c_3c_2^2+2c_4c_2\\
&+c_3^2-c_5)e_n^{16}+O(e_n^{17}).
\end{aligned} \tag{3.10}$$

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Proof Let us expand the function $f(x_n)$ and its first-order derivative $f'(x_n)$ around $x = \alpha$, by using the Taylor's series expansion. Then, we have

$$f(x_n) = f'(\alpha)\left[\sum_{k=1}^{16} c_k e_n^k + O(e_n^{17})\right], \tag{3.11}$$

and

$$f'(x_n) = f'(\alpha)\left[\sum_{k=1}^{16} k c_k e_n^{k-1} + O(e_n^{17})\right], \tag{3.12}$$

respectively, where $c_k = \frac{1}{k!}\frac{f^k(\alpha)}{f'(\alpha)},\ \ k = 2,\ 3,\ \ldots, 16$.

Now, we find the following expansion of $f(y_n)$ about a point $x = \xi$ with the help of above expressions (3.11) and (3.12)

$$f(y_n) = f'(\xi)\left[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 - 7c_3c_2 + 3c_4)e_n^4 + \sum_{k=0}^{11} H_k e_n^{k+5} + O(e_n^{17})\right], \tag{3.13}$$

where $H_k = H_k(c_2, c_3, \ldots, c_{16})$ are given in the term of $c_2, c_3, \ldots, c_{16}$ and some of them are $H_0 = -2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)$, $H_1 = 28c_2^5 - 73c_2^3c_3 + 34c_2^2c_4 - 17c_3c_4 + c_2(37c_3^2 - 13c_5) + 5c_6$, $H_2 = -2\{32c_2^6 - 103c_2^4c_3 - 9c_3^3 + 52c_2^3c_4 + 6c_4^2 + c_2^2(80c_3^2 - 22c_5) + 11c_3c_5 + c_2(8c_6 - 52c_3c_4) - 3c_7\}$ and $H_3 = 144c_2^7 - 552c_2^5c_3 + 297c_2^4c_4 + 75c_3^2c_4 + 2c_2^3(291c_3^2 - 67c_5) - 31c_4c_5 - 27c_3c_6 + c_2^2(-455c_3c_4 + 54c_6) + c_2(-147c_3^3 + 73c_4^2 + 134c_3c_5 - 19c_7) + 7c_8$, etc.

By using the equations (3.11), (3.12) and (3.13), in the second-step, we obtain

$$\begin{aligned} z_n - \alpha =& \lambda_0e_n^4 + \lambda_1e_n^5 + \lambda_2e_n^6 + \lambda_3e_n^7 + \lambda_4e_n^8 + \lambda_5e_n^9 + \lambda_6e_n^{10} + \lambda_7e_n^{11} + \lambda_8e_n^{12} + \lambda_9e_n^{13} \\ &+ \lambda_{10}e_n^{14} + \lambda_{11}e_n^{15} + \lambda_{12}e_n^{16} + O(e_n^{17}), \end{aligned} \tag{3.14}$$

where $\lambda_k = \lambda_k(\beta,\ c_2,\ c_3,\ \ldots,\ c_{16})$ are the functions of constants in term of $c_2, c_3, \ldots, c_{16}$ and some of them are $\lambda_0 = (2\beta + 1)c_2^3 - c_2c_3$, $\lambda_1 = -2\{(2 + 6\beta + \beta^2)c_2^4 - 2(2 + 3\beta)c_2^2c_3 + c_3^2 + c_2c_4\}$, $\lambda_2 = 2(5 + 22\beta + 7\beta^2 + \beta^3)c_2^5 - 2(15 + 42\beta + 8\beta^2)c_2^3c_3 + 6(2 + 3\beta)c_2^2c_4 - 7c_3c_4 + 3c_2\{(6 + 8\beta)c_3^2 - c_5\}$, $\lambda_3 = 2\big[(10 + 64\beta + 28\beta^2 + 8\beta^3 + \beta^4)c_2^6 - 2(20 + 88\beta + 31\beta^2 + 5\beta^3)c_2^4c_3 - 2(3 + 4\beta)c_3^3 + 4(5 + 15\beta + 3\beta^2)c_2^3c_4 + 3c_4^2 + 5c_3c_5 + 4c_2^2\{(10 + 27\beta + 6\beta^2)c_3^2 - (2 + 3\beta)c_5\} + c_2\{-2(13 + 18\beta)c_3c_4 + 2c_6\}\big]$ and $\lambda_4 = 2(18 + 165\beta + 84\beta^2 + 35\beta^3 + 9\beta^4 + \beta^5)c_2^7 - 2(89 + 580\beta + 271\beta^2 + 86\beta^3 + 12\beta^4)c_2^5c_3 + (101 + 490\beta + 178\beta^2 + 30\beta^3)c_2^4c_4 + 2(25 + 36\beta)c_3^2c_4 - 17c_4c_5 + c_2^3\{2(126 + 549\beta + 216\beta^2 + 40\beta^3)c_3^2 - (51 + 156\beta + 32\beta^2)c_5\} - 13c_3c_6 + c_2^2\{-(209 + 612\beta + 144\beta^2)c_3c_4 + 10(2 + 3\beta)c_6\} + c_2\{-(91 + 240\beta + 64\beta^2)c_3^3 + (37 + 54\beta)c_4^2 + 4(17 + 24\beta)c_3c_5 - 5c_7\}$, etc.

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Again, expand the Taylor series expansion of function $f(z_n)$ about a point $z = \alpha$, we obtain

$$\begin{aligned} f(z_n) =& f'(\xi)\Big[\lambda_0 e_n^4 + \lambda_1 e_n^5 + \lambda_2 e_n^6 + \lambda_3 e_n^7 (\lambda_0^2 c_2 + \lambda_4) e_n^8 + (2\lambda_0\lambda_1 c_2 + \lambda_5) e_n^9 + \{(\lambda_1^2 + 2\lambda_0\lambda_2) c_2 \\ & + \lambda_6\} e_n^{10} + \{(2(\lambda_1\lambda_2 + \lambda_0\lambda_3) c_2 + \lambda_7\} e_n^{11} + (\lambda_0^3 c_3 + 2\lambda_4\lambda_0 c_2 + \lambda_2^2 c_2 + 2\lambda_1\lambda_3 c_2 + \lambda_8) e_n^{12} \\ & + (3\lambda_1\lambda_0^2 c_3 + 2\lambda_5\lambda_0 c_2 + 2\lambda_2\lambda_3 c_2 + 2\lambda_1\lambda_4 c_2 + \lambda_9) e_n^{13} + (3\lambda_2\lambda_0^2 c_3 + 2\lambda_6\lambda_0 c_2 + 3\lambda_1^2\lambda_0 c_3 \\ & + \lambda_3^2 c_2 + 2\lambda_2\lambda_4 c_2 + 2\lambda_1\lambda_5 c_2 + \lambda_{10}) e_n^{14} + \{\lambda_1^3 c_3 + 6\lambda_0\lambda_2\lambda_1 c_3 + 2(\lambda_3\lambda_4 + \lambda_2\lambda_5 \\ & + \lambda_1\lambda_6 + \lambda_0\lambda_7) c_2 + 3\lambda_0^2\lambda_3 c_3 + \lambda_{11}\} e_n^{15} + (\lambda_0^4 c_4 + 3\lambda_4\lambda_0^2 c_3 + 2\lambda_8\lambda_0 c_2 + 3\lambda_2^2\lambda_0 c_3 \\ & + 6\lambda_1\lambda_3\lambda_0 c_3 + \lambda_4^2 c_2 + 2\lambda_3\lambda_5 c_2 + 2\lambda_2\lambda_6 c_2 + 2\lambda_1\lambda_7 c_2 + 3\lambda_1^2\lambda_2 c_3 + \lambda_{12}) e_n^{16} + O(e_n^{17})\Big], \end{aligned} \tag{3.15}$$

By inserting the equations (2.11), (2.12), (3.11)–(3.15) in the third substep, we have

$$t_n - \alpha = \theta_0 e_n^8 + \theta_1 e_n^9 + \theta_2 e_n^{10} + \theta_3 e_n^{11} + \theta_4 e_n^{12} + \theta_5 e_n^{13} + \theta_6 e_n^{14} + \theta_7 e_n^{15} + \theta_8 e_n^{16} + O(e_n^{17}) \tag{3.16}$$

where θ_k are the constant function of $(\beta,\ c_2,\ c_3,\ \ldots,\ c_{16})$ and some of them are $\theta_0 = \frac{c_2((2\beta+1)c_2^2 - c_3)}{2}\big[c_2^4\{4\beta^3 - 32\beta^2 + 44\beta + 2\beta G_{12} + G_{12} + (2\beta+1)^2 G_{20} - 82\} + c_3^2(G_{20} - 2) - 2c_4c_2 - c_3c_2^2(-4\beta + G_{12} + 4\beta G_{20} + 2G_{20} - 30)\big]$ and $\theta_1 = \frac{1}{2}\big[\{-1152 - G_{21} - 1888\beta - 6G_{21}\beta + 1032\beta^2 - 12G_{21}\beta^2 - 944\beta^3 - 8G_{21}\beta^3 + 20\beta^4 + 16\beta^5 + 2G_{20}(1 + 2\beta)^2(5 + 16\beta + 3\beta^2) + 2G_{12}(7 + 32\beta + 38\beta^2 + 4\beta^3)\}c_2^8 - \{-2372 - 3G_{21} - 1640\beta - 12G_{21}\beta + 252\beta^2 - 12G_{21}\beta^2 - 832\beta^3 + 116\beta^4 + 2G_{12}(19 + 56\beta + 30\beta^2) + 2G_{20}(19 + 102\beta + 162\beta^2 + 68\beta^3)\}c_2^6c_3 + 2(-2 + G_{20})c_3^4 + 2\{-118 - 36\beta - 42\beta^2 + 4\beta^3 + 3G_{20}(1 + 2\beta)^2 + G_{12}(2 + 4\beta)\}c_2^5c_4 - 4(-38 + G_{12} + 3G_{20} - 16\beta + 6G_{20}\beta)c_2^3c_3c_4 + 2(-10 + 3G_{20})c_2c_3^2c_4 + c_2^4\big\{\big(-1292 - 3G_{21} - 216\beta - 6G_{21}\beta - 408\beta^2 + 40\beta^3 + 10G_{12}(3 + 4\beta) + 6G_{20}(8 + 28\beta + 21\beta^2)\big)c_3^2 + 4(1 + 2\beta)c_5\big\} - c_2^2\{(-184 + 6G_{12} + 22G_{20} - G_{21} - 48\beta + 36G_{20}\beta)c_3^3 + 4c_4^2 + 4c_3c_5\}\big]$, etc.

Again, with the help of Taylor series, we have

$$f(t_n) = f'(\xi)\Big[\theta_0 e_n^8 + \theta_1 e_n^9 + \theta_2 e_n^{10} + \theta_3 e_n^{11} + \theta_4 e_n^{12} + \theta_5 e_n^{13} + \theta_6 e_n^{14} + \theta_7 e_n^{15} + (\theta_0^2 c_2 + \theta_8) e_n^{16} + O(e_n^{17})\Big]. \tag{3.17}$$

Using equations (3.11) – (3.17), in the last substep of the proposed scheme (3.9) and further simplifying the equations, we get

$$e_{n+1} = \lambda_0\theta_0(c_2^4 - 3c_3c_2^2 + 2c_4c_2 + c_3^2 - c_5)e_n^{16} + O(e_n^{17}),$$

or

$$\begin{aligned} e_{n+1} =& -\frac{c_2^3((2\beta+1)c_2^2 - c_3)^2}{2}\Big[c_2^4\left(4\beta^3 - 32\beta^2 + 44\beta + 2\beta G_{12} + G_{12} + (2\beta+1)^2 G_{20} - 82\right) \\ & + c_3^2(G_{20} - 2) - 2c_4c_2 - c_3c_2^2(-4\beta + G_{12} + 4\beta G_{20} + 2G_{20} - 30)\Big](c_2^4 - 3c_3c_2^2 + 2c_4c_2 \\ & + c_3^2 - c_5)e_n^{16} + O(e_n^{17}). \end{aligned} \tag{3.18}$$

This reveals that the proposed scheme (3.9) reaches an optimal sixteenth-order convergence in the sense of Kung-Traub conjecture. It is worthy to note that only λ_0 and θ_0 play the important role in the construction of an optimal sixteenth-order scheme. This is the complete proof of the theorem. □

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4 Numerical experiments

This section is fully devoted to check the effectiveness and validity of our theoretical results which we have proposed in the earlier Sections. For this purpose, we consider four number of real life problems e.g. the trajectory of an electron in the air gap between two parallel plates, chemical engineering problem, Van der Waal's equation which explains the behavior of a real gas by introducing in the ideal gas equations and fractional conversion in a chemical reactor. The details of chosen examples and zeros to the corresponding function can be seen in the following examples 4.1–4.4. In addition, initial guesses are also displayed in the tables 1–8.

First of all, we employ the eighth-order families namely, (2.23) (for $\beta = 0,\ -\frac{1}{2},\ -\frac{1}{3}$), called by (CM_1), (CM_2) and (CM_3), respectively, to check the effectiveness and validity of the theoretical results. We will compare these methods with new families of iterative methods for solving nonlinear equations with optimal eighth-order convergence designed by Heydari et al. [4], out of them, we consider one of their best method (14) (for $\lambda = 30,\ \theta = 6,\ a = 8$) (which is claimed by them not by us), denoted by (HHL). In addition, we also compare them with three-step optimal iterative scheme of order presented by Khatri [5] and Bi et al. [6], out which we choose methods namely, method (2.1) (for $\alpha = 56,\ \beta = -1,\ \mu = 0$) and method (36) (for $\alpha = 1$), called by (KM) and (BRW), respectively. Finally, we also compared our methods with a new eighth-order iterative method (17) for solving nonlinear equations developed by Thukral [8], denoted by (TM).

In the context of sixteenth-order methods for solving nonlinear equations, we employ the same weight functions which we have considered in the eighth-order schemes CM_i, $i = 1, 2, 3$, to obtain the corresponding sixteenth-order iterative methods given by (3.9). We have called the new iterative scheme by $\widehat{CM_1}$, $\widehat{CM_2}$ and $\widehat{CM_3}$ respectively. Now, we will also consider optimal sixteenth-order methods namely, method (19) (for $a = 1$) and method (9), from the methods proposed by Sharma et al. [22] and Ullah et al. [23], denoted by (SGG) and (UFA), respectively. In addition, we will also compare them with a family of multipoint methods for non-linear equations designed by Neta [19], out of which we consider method (12) (for $A = 1$), denoted by (NM). Finally, we will also compare them with a biparametric family of optimally convergent sixteenth-order multipoint methods proposed by Geum and Kim [21], out of the proposed methods we shall choose the expression (1.7), called by GK.

For better comparison of our proposed methods with the existing iterative methods, we compare them with respect to approximated zeros (x_n), absolute residual error of the corresponding function $(|f(x_n)|)$, errors between the two consecutive iterations $|x_{n+1} - x_n|$, $\left|\frac{e_{n+1}}{e_n^p}\right|$ where ($p = 8$ or $p = 16$), the asymptotic error constant $\eta = \lim\limits_{n\to\infty}\left|\dfrac{e_{n+1}}{e_n^p}\right|$ and computational order of convergence $\rho \approx \frac{\ln|\check{e}_{n+1}/\check{e}_n|}{\ln|\check{e}_n/\check{e}_{n-1}|}$, where $\check{e}_n = x_n - x_{n-1}$ (for the details of this formula for calculating ρ please see Cordero and Torregrosa [27]) which can be seen in the Tables 1 – 8.

We calculate the computational order of convergence and asymptotic error constant and other constants

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up to several number of significant digits (minimum 1000 significant digits) to minimize the round off error. Due to the page limitation, we only displayed the values of x_n up to 20 significant digits with/without exponents. In addition, the absolute residual error in the function $|f(x_n)|$ and the difference between the two consecutive iterations $|x_{n+1} - x_n|$ are up to 2 significant digits with exponent power. Moreover, the values of $\left|\frac{e_{n+1}}{e_n^p}\right|$ and η are up to 10 significant digits with/without exponent power. Finally, computational order of convergence ρ is up to 5 significant digits.

For the computer programming, all computations have been performed using the programming package *Mathematica* 10 with multiple precision arithmetic. In addition, $a(\pm b)$ stands for $a \times 10^{(\pm b)}$ in the following Tables 1–8.

Example 4.1 In the study of the multi-factor effect, the trajectory of an electron in the air gap between two parallel plates is given by

$$x(t) = x_0 + \left(v_0 + e\frac{E_0}{m\omega}\sin(\omega t_0 + \alpha)\right)(t - t_0) + e\frac{E_0}{m\omega^2}\big(\cos(\omega t + \alpha) + \sin(\omega + \alpha)\big), \tag{4.1}$$

where e and m are the charge and the mass of the electron at rest, x_0 and v_0 are the position and velocity of the electron at time t_0 and $E_0\sin(\omega t + \alpha)$ is the RF electric field between the plates. We choose the particulars parameters in the expression (4.1) in order to deal with a simpler expression, which is defined as follows:

$$f_1(x) = x - \frac{1}{2}\cos(x) + \frac{\pi}{4}. \tag{4.2}$$

The required zero of the above function $\alpha = -0.30909327154179495274198680892 4$.

Example 4.2 Let us consider a quartic equation from [28, 29], which describes the fraction of the nitrogen-hydrogen feed that gets converted to ammonia (this fraction is called fractional conversion). By considering 250 atm and 500^0 C, then the mentioned equation can be converter in to the following form

$$f_3(z) = z^4 - 7.79075z^3 + 14.7445z^2 + 2.511z - 1.674. \tag{4.3}$$

The above function has total four number of zeros and out of them two are real and other two are complex conjugate to each other. However, our desired zero is $\alpha = 3.9485424455620457727 + 0.3161235708970163733i$.

Example 4.3 Van der Waals equation of state

$$\left(P + \frac{a_1 n^2}{V^2}\right)(V - na_2) = nRT, \tag{4.4}$$

explains the behavior of a real gas by introducing in the ideal gas equations two parameters, a_1 and a_2, specific for each gas. The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V.

$$PV^3 - (na_2P + nRT)V^2 + a_1n^2V - a_1a_2n^2 = 0. \tag{4.5}$$

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Given the constants a_1 and a_2 of a particular gas, one can find values for n, P and T, such that this equation has a three simple roots. By using the particular values, we obtain the following nonlinear function

$$f_2(x) = 0.986x^3 - 5.181x^2 + 9.067x - 5.289. \tag{4.6}$$

have three zeros and out of them two are complex zeros and third one is real zero. However, our desired root is $\alpha = 1.92984624284786221848752742787$.

Example 4.4 Fractional conversion in a chemical reactor:
Let us consider the following expression (for the details of this problem please see [30])

$$f_1(x) = \frac{x}{1-x} - 5\log\left[\frac{0.4(1-x)}{0.4-0.5x}\right] + 4.45977, \tag{4.7}$$

In the above expression x represents the fractional conversion of species A in a chemical reactor. Since, there will be no physical meaning of above fractional conversion if x is less than zero or greater than one. In this sense, x is bounded in the region $0 \leq x \leq 1$. In addition, our required zero to this problem is $\alpha = 0.757396246253753879459641297929$. Moreover, it is interesting to note that the above expression will be undefined in the region $0.8 \leq x \leq 1$ which is very close to our desired zero. Furthermore, there are some other properties to this function which make the solution more difficult. The derivative of the above expression will be very close to zero in the region $0 \leq x \leq 0.5$ and there is an infeasible solution for $x = 1.098$.

4.1 Results and discussion

It is straightforward to say from the Table 2, 4, 6, 8 that our new optimal sixteenth-order methods have smaller residual error in the each corresponding test function as compared to the existing methods of same order. In addition, smaller difference error between the two consecutive iterations belongs to our methods. So, we can say that our method of sixteenth-order converge faster towards the required root as compared to the existing ones. Moreover, our methods also have simple asymptotic error constant corresponding to each test function which can be seen in the Table 2, 4, 6, 8. Similarly, optimal eighth-order methods show the same qualties as sixteenth-order method which can be seen in tables 1, 3, 5, 7 comparatively to existing optimal eighth-order methods. However, one can find different behavior of our methods when he/she consider some different nonlinear equations. Actually, the behavior of the iterative methods depend on several things like the body structure of the iterative method, considered test function, initial approximation and programming softwares, etc.

5 Conclusions

In this paper, we contributed further to the development of the theory of iteration processes and presented some novel eighth and sixteenth-order families of King's method for solving nonlinear equations which is

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Table 1: (Convergence behavior of different optimal eighth-order methods for $f_1(x)$)

I.M.	n	x_n	$\|f(x_n)\|$	$\|x_{n+1}-x_n\|$	ρ	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}$	η
	0	-1.4	$7.0(-1)$	1.1			
HHL	1	-0.31054252241788734737	$1.2(-3)$	$1.4(-3)$		$7.302327376(-4)$	$4.783792148(-4)$
	2	-0.30909327154179495274	$7.9(-27)$	$9.3(-27)$	8.0639	$4.783792148(-4)$	
	0	-1.4	$7.0(-1)$	1.2			
KM	1	-0.22621526699385031990	$7.2(-2)$	$8.3(-2)$		$2.300020113(-2)$	$8.991135622(-3)$
	2	-0.30909327152178118590	$1.7(-11)$	$2.0(-11)$	8.3544	$8.991135622(-3)$	
	0	-1.4	$7.0(-1)$	1.1			
BRW	1	-0.30810534816829296667	$8.4(-4)$	$9.9(-4)$		$4.889645234(-4)$	$2.505693707(-4)$
	2	-0.30909327154179495274	$1.9(-28)$	$2.3(-28)$	8.0954	$2.505693707(-4)$	
	0	-1.4	$7.0(-1)$	1.1			
TM	1	-0.31346329528647993607	$3.7(-3)$	$4.4(-3)$		$2.249720579(-3)$	$4.961187072(-3)$
	2	-0.30909327154179495274	$5.6(-22)$	$6.6(-22)$	7.8566	$4.961187072(-3)$	
	0	-1.4	$7.0(-1)$	1.1			
CM_1	1	-0.30890418608346173211	$1.6(-4)$	$1.9(-4)$		$9.413584255(-5)$	$3.070176352(-5)$
	2	-0.30909327154179495274	$4.3(-35)$	$5.0(-35)$	8.1294	$3.070176352(-5)$	
	0	-1.4	$7.0(-1)$	1.1			
CM_2	1	-0.30873789131635075682	$3.0(-4)$	$3.6(-4)$		$1.767098072(-4)$	$1.219195717(-4)$
	2	-0.30909327154179495274	$2.6(-32)$	$3.1(-32)$	8.0462	$1.219195717(-4)$	
	0	-1.4	$7.0(-1)$	1.1			
CM_3	1	$-0.308708830668831814040 7209$	$3.3(-4)$	$3.8(-4)$		$1.911192477(-4)$	$1.296839259(-5)$
	2	$-0.30909327154179495274 19868$	$5.2(-33)$	$6.2(-33)$	8.3384	$1.296839259(-5)$	

(I.M. stands for iterative method.)

Table 2: (Convergence behavior of different optimal sixteenth-order methods for $f_1(x)$)

I.M.	n	x_n	$\|f(x_n)\|$	$\|x_{n+1}-x_n\|$	ρ	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^{16}}$	η
	0	-1.4	$7.0(-1)$	1.1			
SGG	1	-0.30909308285600657847	$1.6(-7)$	$1.9(-7)$		$4.689602825(-8)$	$5.574538270(-8)$
	2	-0.30909327154179495274	$1.2(-115)$	$1.4(-115)$	15.989	$5.574538270(-8)$	
	0	-1.4	$7.0(-1)$	$1.1(-)$			
UFA	1	-0.31988758845307883793	$9.1(-3)$	$1.1(-2)$		$3.145515715(-3)$	$1.879138194(-6)$
	2	-0.30909327154179495274	$5.4(-38)$	$6.4(-38)$	17.612	$1.879138194(-6)$	
	0	-1.4	$7.0(-1)$	1.1			
NM	1	-0.30909226304255541811	$8.6(-7)$	$1.0(-6)$		$2.506497207(-7)$	$2.816558964(-8)$
	2	-0.30909327154179495274	$2.7(-104)$	$3.2(-104)$	16.157	$2.816558964(-8)$	
	0	-1.4	$7.0(-1)$	1.2			
GK	1	-0.22488634461964713535	$7.3(-2)$	$8.4(-2)$		$6.368979489(-3)$	$1.836934337(-7)$
	2	-0.30909327154179495274	$1.0(-24)$	$1.2(-24)$	19.966	$1.836934337(-7)$	
	0	-1.4	$7.0(-1)$	1.1			
$\widehat{CM_1}$	1	-0.30909326391515776445	$6.5(-9)$	$7.6(-9)$		$1.895531963(-9)$	$1.380398523(-9)$
	2	-0.30909327154179495274	$1.5(-139)$	$1.8(-139)$	16.017	$1.380398523(-9)$	
	0	-1.4	$7.0(-1)$	1.1			
$\widehat{CM_2}$	1	-0.30909322152103935082	$4.2(-8)$	$5.0(-8)$		$1.243220042(-8)$	$3.343337786(-9)$
	2	-0.30909327154179495274	$4.4(-126)$	$5.1(-126)$	16.078	$3.343337786(-9)$	
	0	-1.4	$7.0(-1)$	1.1			
$\widehat{CM_3}$	1	-0.30909322944174186550	$3.6(-8)$	$4.2(-8)$		$1.046358361(-8)$	$4.310973773(-11)$
	2	-0.30909327154179495274	$3.6(-129)$	$4.2(-129)$	16.322	$4.310973773(-11)$	

Table 3: (Convergence behavior of different optimal eighth-order methods for $f_2(x)$)

I.M.	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}$	η
	0	$3.7+0.25i$	2.2	2.6(−1)			
HHL	1	$3.9461582106838576735+0.3244500764547603881i$	8.9(−2)	8.7(−3)		452.6946045	108.4285690
	2	$3.9485424455620427064+0.3161235708970179056i$	3.5(−14)	3.4(−15)	8.4215	108.4285690	
	0	$3.7+0.25i$	2.2	1.4(+1)			
KM	1	$14.173169989904450595+8.972128120579796566i$	5.2(+4)	9.9		8.331271374(−9)	3.341767653(−8)*
	2	$6.3365181967521583250+2.8886252101364599707i$	6.9(+2)	3.1	3.6264	3.341767653(−8)	
	0	$3.7+0.25i$	2.2	2.6(−1)			
BRW	1	$3.9489549971040609806+0.3158097221175631175i$	5.2(−3)	5.2(−4)		26.81207461	38.13353148
	2	$3.9485424455620457811+0.3161235708970163774i$	2.0(−24)	2.0(−25)	7.9433	38.13353148	
	0	$3.7+0.25i$	2.2	3.4(−1)			
TM	1	$4.0362974538615480508+0.3122463813278141890i$	9.3(−1)	8.8(−2)		469.2294516	1813.071595
	2	$3.9485428513760347267+0.3161171575747692166i$	6.5(−4)	6.4(−6)	7.0056	1813.071595	
	0	$3.7+0.25i$	2.2	2.5(−1)			
CM_1	1	$3.9451578944579097603+0.3188770196103508534i$	4.4(−2)	4.4(−3)		246.7464818	70.82996656
	2	$3.9485424455620457727+0.3161235708970163733i$	9.4(−17)	9.3(−18)	8.3069		
	0	$3.7+0.25i$	2.2	2.6(−1)			
CM_2	1	$3.9487782373203197330+0.3159986648044963396I$	2.7(−3)	2.7(−4)		13.85440138	12.79093893
	2	$3.9485424455620457811+0.3161235708970163774I$	3.3(−27)	3.3(−28)	8.0116	12.79093893	
	0	$3.7+0.25i$	2.2	2.6(−1)			
CM_3	1	$3.9482032052129934710+0.3156349392190159856i$	6.0(−3)	5.9(−4)		31.51578066	9.425998318
	2	$3.9485424455620457811+0.3161235708970163774i$	1.5(−24)	1.5(−25)	8.1989	9.425998318	

* stands for the lowest asymptotic error constant. But, you can see that the method KM is far way from the required zero.

Table 4: (Convergence behavior of different optimal sixteenth-order methods for $f_2(x)$)

I.M.	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^{16}}$	η
	0	$3.7+0.25i$	2.2	2.6(−1)			
SSG	1	$3.9485462355554471971+0.3161264649882498978i$	4.8(−5)	4.8(−6)		13009.56675	13811.45870
	2	$3.9485424455620457811+0.3161235708970163774i$	1.0(−80)	9.9(−82)	15.995	13811.45870	
	0	$3.7+0.25i$	2.2	2.6(−1)			
UFA	1	$3.9484626775672587214+0.3160075863965059338i$	1.4(−3)	1.4(−4)		386703.1697	8934.008788
	2	$3.9485424455620457811+0.3161235708970163774i$	2.1(−57)	2.1(−58)	16.502	8934.008788	
	0	$3.7+0.25i$	2.2	2.6(−1)			
NM	1	$3.9489774984650989428+0.3161558563556219178i$	4.4(−3)	4.4(−4)		1159174.966	11307198.79
	2	$3.9485424455620457811+0.3161235708970163774i$	2.0(−47)	1.9(−48)	16.004	1130719.879	
	0	$3.7+0.25i$	2.2	2.6(−1)			
GK	1	$3.9487161094780216385+0.3159856662724177312i$	2.2(−3)	2.2(−4)		600193.9616	603111.2675
	2	$3.9485424455620457811+0.3161235708970163774i$	2.1(−52)	2.1(−53)	15.999	603111.2675	
	0	$3.7+0.25i$	5.5(−1)	5.4(−2)			
$\widehat{CM_1}$	1	$3.9485424455620458064-0.3161235708970163861i$	2.7(−16)	2.7(−17)		5231.165298	3312.965593
	2	$3.9485424455620457811-0.3161235708970163774i$	2.3(−261)	2.3(−262)	16.013	3312.965593	
	0	$3.7+0.25i$	2.2	2.6(−1)			
$\widehat{CM_2}$	1	$3.9485424894115482310+0.3161236372093710606i$	8.0(−7)	7.9(−8)		216.9447057	183.7100071
	2	$3.9485424455620457811+0.3161235708970163774i$	4.7(−111)	4.7(−112)	16.011	183.7100071	
	0	$3.7+0.25i$	2.2	2.6(−1)			
$\widehat{CM_3}$	1	$3.9485424691967845225+0.3161232918008315069i$	2.8(−6)	2.8(−7)		764.3553603	141.2285802
	2	$3.9485424455620457811+0.3161235708970163774i$	2.1(−102)	2.0(−103)	16.123	141.2285802	

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Table 5: (Convergence behavior of different optimal eighth-order methods for $f_3(x)$)

I.M.	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\left\lvert\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}\right\rvert$	η
HHL	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360190299327	$2.3(-12)$	$1.6(-13)$		$9.179659358(-3)$	$1.139245679(-2)$
	2	-0.45486388360206112538	$6.4(-104)$	$4.5(-105)$	7.9918	$1.139245679(-2)$	
KM	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360632983604	$6.1(-11)$	$4.3(-12)$		0.2478010878	0.1532598525
	2	-0.45486388360206112538	$2.4(-91)$	$1.7(-92)$	8.0208	0.1532598525	
BRW	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360212890660	$9.8(-13)$	$6.8(-14)$		$3.934738594(-3)$	$4.816376176(-3)$
	2	-0.45486388360206112538	$3.1(-107)$	$2.1(-108)$	7.9926	$4.816376176(-3)$	
TM	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360440583158	$3.4(-11)$	$2.3(-12)$		0.1361115319	0.1624400952
	2	-0.45486388360206112538	$2.1(-93)$	$1.5(-94)$	7.9925	0.1624400952	
CM_1	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360210171401	$5.8(-13)$	$4.5(-14)$		$2.356193116(-3)$	$2.397926607(-3)$
	2	-0.45486388360206112538	$2.5(-109)$	$1.8(-110)$	7.9994	$2.397926607(-3)$	
CM_2	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360208461155	$3.4(-13)$	$2.3(-14)$		$1.363385276(-3)$	$1.623626931(-3)$
	2	-0.45486388360206112538	$2.2(-111)$	$1.5(-112)$	7.9938	$1.623626931(-3)$	
CM_3	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360206016159	$1.4(-14)$	$9.6(-16)$		$5.594847795(-5)$	$6.417013796(-7)$
	2	-0.45486388360206112538	$6.9(-126)$	$4.8(-127)$	8.1419	$6.417013796(-7)$	

Table 6: (Convergence behavior of different optimal sixteenth-order methods for $f_3(x)$)

I.M.	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\left\lvert\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^{16}}\right\rvert$	η
SGG	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360206112538	$6.2(-27)$	$4.3(-28)$		$1.462246081(-6)$	$1.866092671(-6)$
	2	-0.45486388360206112538	$4.2(-443)$	$2.9(-444)$	15.996	$1.866092671(-6)$	
UFA	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360206112538	$3.6(-25)$	$2.5(-26)$		$8.315288407(-5)$	$8.867741641(-6)$
	2	-0.45486388360206112538	$2.4(-437)$	$1.7(-438)$	16.988	$8.867741641(-6)$	
NM	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360206112538	$3.8(-24)$	$2.6(-25)$		$8.789478072(-4)$	$1.226655221(-3)$
	2	-0.45486388360206112538	$8.1(-396)$	$5.6(-397)$	15.994	$1.226655221(-3)$	
GK	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360206112538	$3.8(-24)$	$2.7(-25)$		$8.938142323(-4)$	$4.250332533(-5)$
	2	-0.45486388360206112538	$3.7(-397)$	$2.6(-398)$	16.057	$4.250332533(-5)$	
$\widehat{CM_1}$	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360206112538	$1.2(-27)$	$8.6(-29)$		$2.906305046(-7)$	$3.188722638(-7)$
	2	-0.45486388360206112538	$4.3(-455)$	$3.0(-456)$	15.998	$3.188722638(-7)$	
$\widehat{CM_2}$	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360206112538	$3.3(-28)$	$2.3(-29)$		$7.816949998(-8)$	$1.078677960(-7)$
	2	-0.45486388360206112538	$1.1(-464)$	$7.6(-466)$	15.995	$1.078677960(-7)$	
$\widehat{CM_3}$	0	-0.5	$6.6(-1)$	$4.5(-2)$			
	1	-0.45486388360206112538	$7.9(-31)$	$5.5(-32)$		$1.848183196(-10)$	$2.260671818(-14)$
	2	-0.45486388360206112538	$2.2(-513)$	$1.5(-514)$	16.131	$2.260671818(-14)$	

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Table 7: (Convergence behavior of different optimal eighth-order methods for $f_4(x)$)

I.M.	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\left\lvert\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^8}\right\rvert$	η
	0	0.76	2.2(−1)	2.6(−3)			
HHL	1	0.75739624625394371490	1.5(−11)	1.9(−13)		8.986240332(+7)	8.260494485(+7)
	2	0.75739624625375387946	1.1(−92)	1.4(−94)	8.0036	8.260494485(+7)	
	0	0.76	2.2(−1)	2.6(−3)			
KM	1	0.75739624637402723293	9.6(−9)	1.2(−10)		5.693381924(+10)	3.338940315(+10)
	2	0.75739624625375387946	1.2(−67)	1.5(−69)	8.0316	3.338940315(+10)	
	0	0.76	2.2(−1)	2.6(−3)			
BRW	1	0.75739624625619642800	1.9(−10)	2.4(−12)		1.156229226(+9)	1.042496594(+9)
	2	0.75739624625375387946	1.1(−82)	1.3(−84)	8.0050	1.042496594(+9)	
	0	0.76	2.2(−1)	2.6(−3)			
TM	1	0.75739624625936593295	4.5(−10)	5.6(−12)		2.656577845(+9)	2.637991832(+9)
	2	0.75739624625375387946	2.1(−79)	2.6(−81)	8.0004	2.637991832(+9)	
	0	0.76	2.2(−1)	2.6(−3)			
CM_1	1	0.75739624625378120221	2.2(−12)	2.7(−14)		1.293377275(+7)	1.087518132(+7)
	2	0.75739624625375387946	2.7(−100)	3.4(−102)	8.0069	1.087518132(+7)	
	0	0.76	2.2(−1)	2.6(−3)			
CM_2	1	0.75739624625392423067	1.4(−11)	1.7(−13)		8.063915443(+7)	6.628788778(+7)
	2	0.75739624625375387946	3.7(−93)	4.7(−95)	8.008	6.628788778(+7)	
	0	0.76	2.2(−1)	2.6(−3)			
CM_3	1	0.75739624625410961573	2.8(−11)	3.6(−13)		1.683948825(+8)	1.475120052(+8)
	2	0.75739624625375387946	3.0(−90)	3.8(−92)	8.0058	1.475120052(+8)	

Table 8: (Convergence behavior of different optimal sixteenth-order methods for $f_4(x)$)

I.M.	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	ρ	$\left\lvert\frac{x_{n+1}-x_n}{(x_n-x_{n-1})^{16}}\right\rvert$	η
	0	0.76	2.2(−1)	2.6(−3)			
SGG	1	0.75739624625375387946	1.1(−23)	1.3(−25)		3.014905005(+16)	2.625622084(+16)
	2	0.75739624625375387946	2.4(−380)	3.0(−382)	16.003	2.625622084(+16)	
	0	0.76	2.2(−1)	2.6(−3)			
UFA	1	0.75739624625375387947	6.6(−19)	8.3(−21)		1.855260159(+21)	2.261301253(+21)
	2	0.75739624625375387946	8.8(−299)	1.1(−300)	15.995	2.261301253(+21)	
	0	0.76	2.2(−1)	2.6(−3)			
NM	1	0.75739624625375387946	1.1(−21)	1.4(−23)		3.026115120(+18)	3.185978259(+18)
	2	0.75739624625375387946	3.1(−346)	3.9(−348)	15.999	3.185978259(+18)	
	0	0.76	2.2(−1)	2.6(−3)			
GK	1	0.75739624625375387946	1.1(−21)	1.4(−23)		3.085852902(+18)	4.005025354(+18)
	2	0.75739624625375387946	5.3(−346)	6.7(−348)	15.994	4.005025354(+18)	
	0	0.76	2.2(−1)	2.6(−3)			
$\widehat{CM_1}$	1	0.75739624625375387946	1.1(−25)	1.4(−27)		3.063390532(+14)	2.048207850(+14)
	2	0.75739624625375387946	2.4(−414)	3.0(−416)	16.007	2.048207850(+14)	
	0	0.76	2.2(−1)	2.6(−3)			
$\widehat{CM_2}$	1	0.75739624625375387946	1.2(−23)	1.5(−25)		3.402606716(+16)	2.422970554(+16)
	2	0.75739624625375387946	1.5(−379)	1.9(−381)	16.007	2.422970554(+16)	
	0	0.76	2.2(−1)	2.6(−3)			
$\widehat{CM_3}$	1	0.75739624625375387946	1.7(−23)	2.2(−25)		4.837223430(+16)	3.687202764(+16)
	2	0.75739624625375387946	6.5(−377)	8.2(−379)	16.005	3.687202764(+16)	

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based on the weight function and rational functional approximation approaches. Analysis of convergence demonstrate that the order of convergence of the proposed families are eight and sixteen. Further, the proposed families are optimal in the sense of the classical Kung-Traub conjecture. The computational efficiency index is defined as $E = p^{1/\theta}$, where p is the order of convergence and θ is the number of functional evaluations per iteration. Thus, the efficiency indices of the proposed families are $E = \sqrt[4]{8} \approx 1.682$ and $E = \sqrt[5]{16} \approx 1.741$ which are better than the classical Newton's method $E \approx 1.414$. Moreover, the beauty of the proposed sixteenth-order family is that we can easily obtain several new optimal methods of order sixteen by considering different types of weight functions. Finally, on accounts of the results obtained, it can be concluded that our proposed methods are highly efficient and perform better than the existing methods.

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