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Steklov approximations of harmonic boundary value problems on planar regions

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ABSTRACT

Error estimates for approximations of solutions of Laplace's equation with Dirichlet, Robin or Neumann boundary value conditions are described. The solutions are represented by orthogonal series using the harmonic Steklov eigenfunctions. Error bounds for partial sums involving the lowest eigenfunctions are found. When the region is a rectangle, explicit formulae for the Steklov eigenfunctions and eigenvalues are known. These were used to find approximations for problems with known explicit solutions. Results about the accuracy of these solutions, as a function of the number of eigenfunctions used, are given.

1. Introduction

This paper treats the approximation of solutions of Laplace's equations using harmonic Steklov eigenfunctions. The problems are posed on bounded planar regions Ω and the functions should satisfy either Dirichlet, Robin or Neumann boundary conditions

$$u = g \quad \text{or} \quad D_\nu u + b\,u = g \quad \text{on } \partial\Omega. \tag{1.1}$$

Here ν is the outward unit normal and $b \geq 0$ is a constant.

Results about orthogonal bases of the class of all finite energy harmonic functions $\mathcal{H}(\Omega) \subset H^1(\Omega)$ consisting of harmonic Steklov eigenfunctions are summarized below in Section 3. These functions have the property that they generate a basis of $\mathcal{H}(\Omega)$ and their boundary traces provide orthogonal bases of $L^2(\partial\Omega, d\sigma)$ and $H^{1/2}(\partial\Omega)$. This *spectral theory of trace spaces* is described in Auchmuty [1]. Here some results obtained in the computational approximation of harmonic functions using Steklov eigenfunctions associated with the lowest Steklov eigenvalues will be described. General results and error estimates for approximations are described in Sections 4 and 6. Computational results for some problems with exact solutions are described in Sections 5 and 7. The explicit formulae for the Steklov eigenvalues and eigenfunctions on rectangles of aspect ratio $h > 0$ are used here. For similar problems on general regions, further errors are introduced when approximations of the Steklov eigenfunctions and eigenvalues are used.

Existence–uniqueness theorems for these problems may be found in most texts that treat elliptic boundary value problems. From Weyl's lemma, the solutions are C^∞ on the region Ω. Under various assumptions on g, Ω and $\partial\Omega$ the

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solutions will be in specific Banach or Hilbert spaces of functions on Ω or $\overline{\Omega}$. For an excellent review of classical results about these problems see chapter 2 by Benilan in [2]. A function $u \in L^1(\Omega)$ is said to be an *ultraweak* solution of Laplace's equation provided it obeys

$$\int_{\Omega} u \, \Delta \varphi \; dxdy = 0 \quad \text{for all } \varphi \in C_c^{\infty}(\Omega). \tag{1.2}$$

Such an ultraweak solution is a *classical solution* of Laplace's equation provided it is equivalent to a continuous function on $\overline{\Omega}$. There are classical, and other ultraweak, harmonic functions that are not in the standard Sobolev space $H^1(\Omega)$ — even when Ω is a disk.

General results about Steklov approximations of harmonic functions are described in Sections 3, 4 and 6. An algorithm for constructing a basis of the subspace of harmonic functions in $H^1(\Omega)$ consisting of harmonic Steklov eigenfunctions. is described in Auchmuty [1,3]. It requires the solution of a sequence of constrained variational principles. The boundary traces of these eigenfunctions are L^2-orthogonal on the boundary and are proved to be bases of a scale of Hilbert spaces of functions on $\partial\Omega$. In Sections 4 and 6 various error estimates for Steklov approximations are obtained.

When the region is a planar disk, the Steklov eigenfunctions are the usual harmonic functions $r^m \cos m\theta$, $r^m \sin m\theta$ of Fourier analysis and the question of the approximation of harmonic functions on the unit disc by harmonic polynomials has a huge literature. The text of Axler, Bourdon and Ramey [4] is a recent introduction to the theory.

Here attention will be on the case where the region is a rectangle. In this case, the Steklov eigenfunctions are known explicitly see Auchmuty and Cho [5] or Girouard and Polterovich [6] where a completeness proof for this family is given. Computational results for Steklov approximations of certain harmonic functions regarded as solutions of Laplace's equations with various boundary value conditions are described in Sections 5 and 7. Dirichlet problems are considered in Sections 4 and 5 while results for Robin and Neumann problems are described in Sections 6 and 7.

For general regions, the Steklov eigenvalues and eigenfunctions are not (yet) known explicitly. However a number of authors have studied the numerical determination of these eigenfunctions including Cheng, Lin and Zhang [7], and Kloucek, Sorensen and Wightman [8]. The software FreeFem++ [9] has subroutines for the computation of Steklov eigenfunctions and eigenvalues that was used for confirmation of some of the analytical results described here.

Our general conclusion is that many harmonic functions are well-approximated by Steklov expansions with a relatively small number of Steklov eigenfunctions. They appear to provide very good approximations in the interior of the region and become quite oscillatory close to, and on, the boundary. It should be noted that this analysis extends to the solution of more general self-adjoint second order elliptic equations of the form $\mathcal{L}u = 0$ using similar general constructions as described in the paper [10].

2. Assumptions and notation

This paper treats various Laplacian boundary value problems on regions Ω in the plane $\mathbb{R}^2$. A region is a non-empty, connected, open subset of $\mathbb{R}^2$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. Some regularity of the boundary $\partial\Omega$ is required. Each component (= maximal connected closed subset) of the boundary is assumed to be a Lipschitz continuous closed curve. Let σ denote arc-length along a curve so the unit outward normal $\nu(z)$ is defined σ *a.e.*

$L^p(\Omega)$ and $L^p(\partial\Omega, d\sigma)$, $1 \le p \le \infty$ are the usual spaces with p-norm denoted by $\|u\|_p$ or $\|u\|_{p,\partial\Omega}$ respectively. When $p = 2$ these are real Hilbert spaces with inner products defined by

$$\langle u, v \rangle := \int_{\Omega} u \, v \; dxdy \quad \text{and} \quad \langle u, v \rangle_{\partial\Omega} := |\partial\Omega|^{-1} \int_{\partial\Omega} u \, v \; d\sigma.$$

$C(\overline{\Omega})$ is the space of continuous functions on the closure $\overline{\Omega}$ of Ω with the sup norm $\|u\|_b := sup_{\overline{\Omega}} |u(x, y)|$.

The weak jth derivative of u is $D_j u$ — and all derivatives will be taken in a weak sense. Then $\nabla u := (D_1 u, D_2 u)$ is the gradient of u and $H^1(\Omega)$ is the usual real Sobolev space of functions on Ω. It is a real Hilbert space under the standard H^1-inner product

$$[u, v]_1 := \int_{\Omega} [u \, v + \nabla u \cdot \nabla v] \; dxdy. \tag{2.1}$$

The corresponding norm is denoted $\|u\|_{1,2}$.

The region Ω is said to satisfy *Rellich's theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \le p < \infty$.

The boundary trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is the linear extension of the map restricting Lipschitz continuous functions on $\overline{\Omega}$ to $\partial\Omega$. The region Ω is said to satisfy a *compact trace theorem* provided the boundary trace mapping $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is compact. Theorem 1.5.1.10 of Grisvard [11] proves an inequality that implies the compact trace theorem for bounded regions in $\mathbb{R}^N$ with Lipschitz boundaries. Usually γ is omitted so u is used in place of $\gamma(u)$ for the trace of a function on $\partial\Omega$.

The *Gauss–Green* theorem holds on Ω provided

$$\int_{\Omega} u D_j v \; dxdy = \int_{\partial\Omega} \gamma(u) \, \gamma(v) \, \nu_j \; d\sigma \; - \int_{\Omega} v \, D_j u \; dxdy \quad \text{for } 1 \le j \le N \tag{2.2}$$

for all u, v in $H^1(\Omega)$. The requirements on the region will be

Condition (B1): *Ω is a bounded region in $\mathbb{R}^2$ whose boundary $\partial\Omega$ is a finite number of disjoint closed Lipschitz curves, each of finite length and such that the Gauss–Green, Rellich and compact trace theorems hold.*

We will use the equivalent inner products on $H^1(\Omega)$ defined by

$$[u, v]_\partial := \int_\Omega \nabla u \cdot \nabla v \ dxdy + \int_{\partial\Omega} u \ v \ d\sigma. \tag{2.3}$$

The corresponding norm will be denoted by $\|u\|_\partial$. The proof that this norm is equivalent to the usual $(1, 2)$-norm on $H^1(\Omega)$ when (B1) holds is Corollary 6.2 of [10] and also is part of Theorem 21A of [12].

A function $u \in C(\overline{\Omega})$ or $H^1(\Omega)$ is said to be harmonic provided it satisfies (1.2). Define $\mathcal{H}(\Omega)$ to be the space of all harmonic functions in $H^1(\Omega)$. When (B1) holds, the closure of $C_c^1(\Omega)$ in the H^1-norm is the usual Sobolev space $H_0^1(\Omega)$. Then (1.2) is equivalent to saying that $\mathcal{H}(\Omega)$ is ∂-orthogonal to $H_0^1(\Omega)$. This may be expressed as

$$H^1(\Omega) = H_0^1(\Omega) \oplus_\partial \mathcal{H}(\Omega), \tag{2.4}$$

where $\oplus_\partial$ indicates that this is a ∂-orthogonal decomposition.

The analysis to be described here is based on the construction of a ∂-orthogonal basis of the Hilbert space $\mathcal{H}(\Omega)$ consisting of harmonic Steklov eigenfunctions. In particular we shall prove results about the approximation of solutions of harmonic boundary value problems by such eigenfunctions.

3. Steklov representations of solutions of harmonic boundary value problems

Let Ω be a bounded region in $\mathbb{R}^2$ that satisfies (B1). A non-zero function $s \in H^1(\Omega)$ is said to be a *harmonic Steklov eigenfunction* on Ω corresponding to the Steklov eigenvalue δ provided s satisfies

$$\int_\Omega \nabla s \cdot \nabla v \ dxdy = \delta \ \langle s, v\rangle_{\partial\Omega} = \delta \ |\partial\Omega|^{-1} \int_{\partial\Omega} s \, v \, d\sigma. \quad \text{for all } v \in H^1(\Omega). \tag{3.1}$$

This is the weak form of the boundary value problem

$$\Delta s = 0 \quad \text{on } \Omega \text{ with } D_\nu s = \delta \ |\partial\Omega|^{-1} \, s \text{ on } \partial\Omega. \tag{3.2}$$

Here Δ is the Laplacian and $D_\nu s := \nabla s \cdot \nu$ is the unit outward normal derivative of s at a point on the boundary.

Descriptions of the analysis of these eigenproblems may be found in Auchmuty [10,1,3,13]. These eigenvalues and a corresponding family of ∂-orthonormal eigenfunctions may be found using variational principles as described in Sections 6 and 7 of Auchmuty [10]. $\delta_0 = 0$ is the least eigenvalue of this problem corresponding to the eigenfunction $s_0(x) \equiv 1$ on Ω. This eigenvalue is simple as Ω is connected. Let the first k Steklov eigenvalues be $0 = \delta_0 < \delta_1 \le \delta_2 \le \cdots \le \delta_{k-1}$ and $s_0, s_1, \ldots, s_{k-1}$ be a corresponding set of ∂-orthonormal eigenfunctions. The kth eigenfunction s_k will be a maximizer of the functional

$$\mathcal{B}(u) := \int_{\partial\Omega} |\gamma(u)|^2 \ d\sigma, \tag{3.3}$$

over the subset B_k of functions in $H^1(\Omega)$ which satisfy

$$\|u\|_\partial \le 1 \quad \text{and} \quad \langle \gamma(u), \gamma(s_l) \rangle_{\partial\Omega} = 0 \quad \text{for } 0 \le l \le k-1. \tag{3.4}$$

The existence and some properties of such eigenfunctions are described in Sections 6 and 7 of [10] for a more general system. In particular, that analysis shows that each δ_j is of finite multiplicity and $\delta_j \to \infty$ as $j \to \infty$; see Theorem 7.2 of [10]. The maximizers not only are ∂-orthonormal but they also satisfy

$$\int_\Omega \nabla s_k \cdot \nabla s_l \ dxdy = |\partial\Omega|^{-1} \int_{\partial\Omega} s_k \, s_l \ d\sigma = 0 \qquad \text{for } \ k \ne l. \tag{3.5}$$

$$\int_\Omega |\nabla s_k|^2 \ dxdy = \frac{\delta_k}{1+\delta_k} \qquad \text{and} \quad |\partial\Omega|^{-1} \int_{\partial\Omega} |\gamma(s_k)|^2 \ d\sigma = \frac{1}{1+\delta_k} \quad \text{for } k \ge 0. \tag{3.6}$$

Recently Daners [14, Corollary 4.3] has shown that, when Ω is a Lipschitz domain, then the Steklov eigenfunctions are continuous on $\overline{\Omega}$.

The analysis in this paper is based on the fact that harmonic Steklov eigenfunctions on Ω can be chosen to be orthogonal bases of both $\mathcal{H}(\Omega)$ and of $L^2(\partial\Omega, \ d\sigma)$. It should be noted that, for regions other than discs (or balls in higher dimensions), these Steklov eigenfunctions are generally not L^2-orthogonal on Ω.

Let $\mathcal{S} := \{s_j : j \ge 0\}$ be the maximal family of ∂-orthonormal eigenfunctions constructed inductively as above. For this paper, it is more convenient to use the Steklov eigenfunctions normalized by their boundary norms.

Define the functions $\tilde{s}_j := \sqrt{1+\delta_j} \quad s_j$ for $j \geq 0$. From (3.6), these satisfy

$$\int_{\partial\Omega} \tilde{s}_j\, \tilde{s}_k\, d\sigma = 0 \quad \text{when } j \neq k \quad \text{and} \quad \int_{\partial\Omega} \tilde{s}_j^{\,2}\, d\sigma = |\partial\Omega|. \tag{3.7}$$

These Steklov eigenfunctions are said to be *boundary normalized* and the associated set $\tilde{\mathcal{S}} := \{\tilde{s}_j : j \geq 0\}$ is an orthonormal basis of $L^2(\partial\Omega,\, d\sigma)$. See Theorem 4.1 of [1].

For given $g \in L^2(\partial\Omega,\, d\sigma)$, let

$$g_M(x, y) := \overline{g} + \sum_{j=1}^{M} \hat{g}_j\, \tilde{s}_j(x, y) \quad \text{with } \hat{g}_j = \langle g, \tilde{s}_j \rangle_{\partial\Omega} \tag{3.8}$$

be the Mth Steklov approximation of g on $\partial\Omega$. Here $\overline{g} := g_0$ is the mean value of g on $\partial\Omega$ and g_j is called the jth Steklov coefficient of g. This is a standard approximation of an element in an L^2 space with respect to this orthonormal basis. Note that each g_M is continuous and bounded on $\partial\Omega$ as each $\tilde{s}_j$ is, and g_M converges strongly to g in $L^2(\partial\Omega,\, d\sigma)$ from the Riesz–Fischer theorem and

$$\| g - g_M \|_{2,\partial\Omega}^2 = \| g \|_{2,\partial\Omega}^2 - \| g_M \|_{2,\partial\Omega}^2. \tag{3.9}$$

The unique solution of Laplace's equation on Ω subject to the Dirichlet boundary condition $\gamma(u) = g$ on $\partial\Omega$ is given by

$$u(x, y) = E_H g(x, y) = \overline{g} + \lim_{M\to\infty} \sum_{j=1}^{M} \hat{g}_j \tilde{s}_j(x, y) \quad \text{for } (x, y) \in \Omega. \tag{3.10}$$

See Section 6 of [3] for a proof; the limit here is in the L^2 norm on Ω when g is L^2. E_H will be called the *harmonic extension* operator and is a compact linear map from $L^2(\partial\Omega,\, d\sigma)$ to $L^2(\Omega)$. Classically this map has been represented as an integral operator with the *Poisson kernel*. Theorem 6.3 of [3] says that E_H is an isometric isomorphism of $L^2(\partial\Omega,\, d\sigma)$ with a space denoted $\mathcal{H}^{1/2}(\Omega)$ that is a proper subspace of $L^2(\Omega)$.

4. Error estimates for Steklov approximations

Let $E_H : L^2(\partial\Omega,\, d\sigma) \to L^2(\Omega)$ be the harmonic extension operator defined by (3.10). An old result of G. Fichera [15] says that there is a constant $C_2 > 0$ such that

$$\| E_H\, g \|_{2,\Omega} \leq C_2 \| g \|_{2,\partial\Omega} \quad \text{for all } g \in L^2(\partial\Omega,\, d\sigma). \tag{4.1}$$

Fichera identified C_2 as being related to the first eigenvalue of the Dirichlet Biharmonic Steklov eigenproblem on the region Ω. Recently Auchmuty [16] extended this result in a number of ways, including a description of the boundary regularity required for it to hold.

C_2 will be called the Fichera constant and (4.1) the Fichera inequality. Henceforth the region Ω is assumed **(B2):** to be sufficiently regular that (B1) and (4.1) hold.

From the Perron construction, it is known that $E_H : C(\partial\Omega) \to C(\overline{\Omega})$ is continuous and the maximum principle implies that

$$\| E_H\, g \|_{\infty,\Omega} \leq \| g \|_{\infty,\partial\Omega} \quad \text{for all } g \in C(\partial\Omega). \tag{4.2}$$

These two inequalities may be combined to yield the following

Lemma 4.1. *Assume that Ω satisfies* (B2) *and $p \in [2, \infty]$. Then $E_H : L^p(\partial\Omega,\, d\sigma) \to L^p(\Omega)$ is a continuous linear transformation and*

$$\| E_H\, g \|_{p,\Omega} \leq (C_2)^{2/p} \| g \|_{p,\partial\Omega} \quad \textit{for all } g \in L^p(\partial\Omega). \tag{4.3}$$

Proof. This inequality is a direct consequence of (4.1), (4.2) and the Riesz Thorin interpolation theorem. □

It is worth noting that these three inequalities are equivalent to coercivity inequalities for the trace operator on weakly harmonic functions. Namely if $u \in C(\overline{\Omega})$ satisfies (1.2) and $p \in [2, \infty]$, then

$$\| \gamma(u) \|_{p,\partial\Omega} \geq (C_2)^{-2/p} \| u \|_{p,\Omega}. \tag{4.4}$$

This holds as $\gamma(E_H)$ is the identity operator on $L^p(\partial\Omega)$.

In addition to these L^p bounds, there are gradient bounds for our Steklov approximations as summarized in the following result.

Theorem 4.2. *Assume* (B1) *and* $g \in H^{1/2}(\partial\Omega)$, g_M *is defined by* (3.8), $u = E_H g$ *and* $u_M = E_H g_M$. *Then* g_M *converges strongly to* g *in* $H^{1/2}(\partial\Omega)$ *and* u_M *converges uniformly to* u *on compact subsets of* Ω. *Moreover*

$$\|\nabla(u - u_M)\|_{2,\Omega}^2 = \sum_{j=M+1}^{\infty} \delta_j \hat{g}_j^2 = \|g\|_{1/2,\partial\Omega}^2 - \|g_M\|_{1/2,\partial\Omega}^2. \tag{4.5}$$

Proof. The fact that g_M converges strongly to g in $H^{1/2}(\partial\Omega)$ and $H^1(\Omega)$ follows from the fact that $\mathcal{S}$ is an orthonormal basis of $\mathcal{H}(\Omega)$. The proof of uniform convergence is standard, while (4.5) follows from the orthogonality properties of Steklov eigenfunctions. □

Also note that the Steklov eigenfunction have scaling properties. Given $\Omega_1 \subset \mathbb{R}^2$, let $\Omega_L := \{Lx : x \in \Omega_1\}$ with $L > 0$. When s is a harmonic function on Ω_1, then the function $s_L(y) := s(y/L)$ will be a harmonic function on Ω_L. If s is a harmonic Steklov eigenfunction on Ω_1 with Steklov eigenvalue δ, then s_L will be a harmonic Steklov eigenfunction on Ω_L with the Steklov eigenvalue δ/L. Thus it suffices to study problems with a normalized bounded region Ω_1; the eigenvalues and eigenfunctions for scalings of a region then follow from these formulae.

The following sections will look at some aspects of the approximation of solutions of Laplace's equation on rectangles by finite sums of the form (3.10). Rectangles are chosen since we have explicit expressions for the Steklov eigenfunctions and eigenvalues on rectangles.

5. Steklov approximations of harmonic functions on a rectangle

When $\Omega = R_h := (-1, 1) \times (-h, h)$ is a rectangle with aspect ratio h, the Steklov eigenfunctions and eigenvalues are known explicitly. See Auchmuty and Cho [5, Section 4] where eight families of eigenfunctions are described and characterized by their symmetry properties with respect to the center. Class I eigenfunctions are even in x and y, class II are odd in x and y, class III are even in x and odd in y, class IV are odd in x and even in y.

By separation of variables the explicit formulae for the Steklov eigenfunctions may be found. The first eigenfunction $s_0(x, y) \equiv 1$ is in class I and the other (unnormalized) Steklov eigenfunctions have the forms

$$s(x, y) := \cosh \nu x \cos \nu y \quad \text{when } \tan \nu h + \tanh \nu = 0, \tag{5.1}$$
$$s(x, y) := \cos \nu x \cosh \nu y \quad \text{when } \tan \nu + \tanh \nu h = 0. \tag{5.2}$$

When $h = 1$, the first eigenfunction in class II is $s_3(x, y) = xy$. Otherwise the (unnormalized) eigenfunctions and eigenvalues in this class have the forms

$$s(x, y) := \sinh \nu x \sin \nu y \quad \text{when } \cot \nu h - \coth \nu = 0, \tag{5.3}$$
$$s(x, y) := \sin \nu x \sinh \nu y \quad \text{when } \cot \nu - \coth \nu h = 0. \tag{5.4}$$

Similarly eigenfunctions in class III have the forms

$$s(x, y) := \cosh \nu x \sin \nu y \quad \text{when } \cot \nu h - \tanh \nu = 0, \tag{5.5}$$
$$s(x, y) := \cos \nu x \sinh \nu y \quad \text{when } \tan \nu + \coth \nu h = 0. \tag{5.6}$$

Finally the eigenfunctions in class IV have the forms

$$s(x, y) := \sinh \nu x \cos \nu y \quad \text{when } \tan \nu h + \coth \nu = 0, \tag{5.7}$$
$$s(x, y) := \sin \nu x \cosh \nu y \quad \text{when } \cot \nu - \tanh \nu h = 0. \tag{5.8}$$

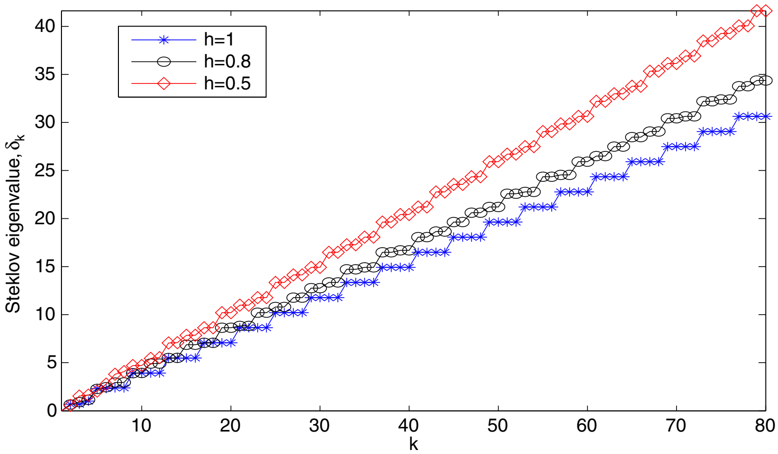


Fig. 1. First 80 Steklov eigenvalues on R_h corresponding to $h = 1$, 0.8, and 0.5.

Table 1
$g(x, y) = f_1(x, y)$ and $h = 1$.

	P_1	P_2	P_3	P_4	P_5
$M = 2$	−2.626748	0.694643	−0.844238	0.230283	−0.249859
$M = 3$	−2.625942	0.607979	−0.842944	0.225907	−0.249983
$M = 5$	−2.624712	0.607588	−0.843208	0.226837	−0.250000
$g(x, y)$	−2.624400	0.607600	−0.843200	0.226800	−0.250000
$D_2(x, y)$	0.002348	0.002957	0.001038	0.003483	0.000141
$D_3(x, y)$	0.001542	0.000379	0.000256	0.000893	0.000017
$D_5(x, y)$	0.000312	0.000012	0.000008	0.000037	0

The associated Steklov eigenvalues, δ are given by the following formulae and the first 80 eigenvalues for different aspect ratios are graphed in Fig. 1.

(i) $\delta = \nu \tanh \nu$ when ν is a solution of the equation in (5.1) or (5.5).
(ii) $\delta = \nu \tanh \nu h$ when ν is a solution of the equation in (5.2) or (5.8).
(iii) $\delta = \nu \coth \nu$ when ν is a solution of the equation in (5.3) or (5.7).
(iv) $\delta = \nu \coth \nu h$ when ν is a solution of the equation in (5.4) or (5.6).

Knowing these explicit formulae for the eigenvalues and eigenfunctions the approximations of some given harmonic functions using relatively few harmonic Steklov eigenfunctions will be computed. Since there are eight families of harmonic Steklov eigenfunctions associated with different even/odd symmetries about the center we have concentrated on approximations involving the first 8M eigenfunctions with $M = 2$, 3 and 5.

Note that the convergence results for the Steklov series expansions hold only when the coefficients are precisely the Steklov coefficients $\hat{g}_j$ defined by (3.8). The value of $\bar{u}$ is the mean value of the integral of g around $\partial\Omega$. However the approximation results of Section 4 hold quite generally for any choice of coefficients.

For the following calculations the coefficients were obtained by evaluating the boundary integrals $\hat{g}_j$ of (3.8) using the global adaptive quadrature (MATLAB's integral). The absolute and relative error tolerance are 10^{-10} and 10^{-6}, respectively. Then the Mth Steklov approximation u_M is the function defined by $u_M = E_H g_M$ with $\hat{u}_j = \hat{g}_j$.

Tables 1–3 illustrate the pointwise approximations obtained for these sums at the points $P_1 = (0.9, 0.9)$, $P_2 = (0.9, 0.1)$, $P_3 = (0.8, 0.6)$, $P_4 = (0.3, 0.9)$, $P_5 = (0.5, 0.5)$ and for M = 2,3, 5 and the exact results to 6 decimal places. Let $D_M(x, y) := |g(x, y) - g_M(x, y)|$ be the absolute error at (x, y). Also let $f_1(x, y) := x^4 - 6x^2y^2 + y^4$, $f_2(x, y) := \frac{2-x}{(2-x)^2+y^2}$, and $f_3(x, y) := \ln(\sqrt{(x-3)^2 + (y-3)^2})$. Tables 4–9 give the relative errors found for these approximations.

Let $\mathrm{rerr}_\infty(g) := \frac{\|g-g_M\|_{\infty,\partial\Omega}}{\|g\|_{\infty,\partial\Omega}}$ and $\mathrm{rerr}_2(g) := \frac{\|g-g_M\|_{2,\partial\Omega}}{\|g\|_{2,\partial\Omega}}$ be the relative error of Mth Steklov approximation of g in $L^\infty(\Omega)$ norm and $L^2(\partial\Omega, d\sigma)$, respectively.

It was observed that the above approximations were improved when some preliminary processing was performed. In particular it was worthwhile to first find the coefficients a_j for a function $g_0(x, y) = a_0 + a_1 x + a_2 y + a_3 xy$ that interpolated the boundary data at the 4 corners of the rectangle. Then the Steklov approximations of solutions of Laplace's equation subject to the reduced boundary condition $g_1(z) := g(z) - g_0(z)$ for $z \in \partial\Omega$ were observed to be better (have smaller error) than those for the boundary data g.

Table 10 shows the comparison of relative errors of Steklov approximations of f_1 and $f_1 + 4$. Note that $f_1 + 4$ is the reduced boundary condition of f_1 such that the value of the function at the 4 corners of R_1 is zero.

Table 2
$g(x, y) = f_2(x, y)$ and $h = 1$.

	P_1	P_2	P_3	P_4	P_5
$M = 2$	0.544285	0.899505	0.666815	0.455438	0.600096
$M = 3$	0.544745	0.902138	0.667202	0.460368	0.599985
$M = 5$	0.544675	0.901609	0.666636	0.459219	0.600000
$g(x, y)$	0.544554	0.901639	0.666667	0.459459	0.600000
$D_2(x, y)$	0.000269	0.002135	0.000148	0.004021	0.000096
$D_3(x, y)$	0.000191	0.000498	0.000535	0.000909	0.000015
$D_5(x, y)$	0.000121	0.000030	0.000031	0.000240	0

Table 3
$g(x, y) = f_3(x, y)$ and $h = 1$.

	P_1	P_2	P_3	P_4	P_5
$M = 2$	1.088867	1.277069	1.179619	1.230746	1.262756
$M = 3$	1.088349	1.274927	1.180394	1.229961	1.262881
$M = 5$	1.088384	1.275412	1.180439	1.229874	1.262864
$g(x, y)$	1.088511	1.275503	1.180427	1.229794	1.262864
$D_2(x, y)$	0.000356	0.001566	0.000808	0.000952	0.000108
$D_3(x, y)$	0.000162	0.000576	0.000033	0.000167	0.000017
$D_5(x, y)$	0.000127	0.000091	0.000012	0.000080	0

Table 4
Relative errors of the Steklov approximations of f_1, f_2, and f_3, respectively where $h = 1$.

	$\mathrm{rerr}_\infty(f_1)$	$\mathrm{rerr}_\infty(f_2)$	$\mathrm{rerr}_\infty(f_3)$
$M = 2$	6.59553×10^{-3}	1.82382×10^{-2}	6.48245×10^{-3}
$M = 3$	2.28748×10^{-3}	1.21554×10^{-2}	4.3219×10^{-3}
$M = 5$	5.55757×10^{-4}	7.35222×10^{-3}	2.59338×10^{-3}

Table 5
Relative errors of the Steklov approximations of f_1, f_2, and f_3, respectively where $h = 0.8$.

	$\mathrm{rerr}_\infty(f_1)$	$\mathrm{rerr}_\infty(f_2)$	$\mathrm{rerr}_\infty(f_3)$
$M = 2$	4.82556×10^{-2}	2.46749×10^{-2}	6.38229×10^{-3}
$M = 3$	4.20662×10^{-2}	1.78505×10^{-2}	4.18945×10^{-3}
$M = 5$	2.28023×10^{-2}	1.0105×10^{-2}	2.47618×10^{-3}

Table 6
Relative errors of the Steklov approximations of f_1, f_2, and f_3, respectively where $h = 0.5$.

	$\mathrm{rerr}_\infty(f_1)$	$\mathrm{rerr}_\infty(f_2)$	$\mathrm{rerr}_\infty(f_3)$
$M = 2$	2.09505×10^{-1}	3.40908×10^{-2}	5.58445×10^{-3}
$M = 3$	1.12233×10^{-1}	2.00031×10^{-2}	3.84456×10^{-3}
$M = 5$	7.66842×10^{-2}	1.29479×10^{-2}	2.24773×10^{-3}

Table 7
Relative errors of the Steklov approximations of f_1, f_2, and f_3, respectively where $h = 1$.

	$\mathrm{rerr}_2(f_1)$	$\mathrm{rerr}_2(f_2)$	$\mathrm{rerr}_2(f_3)$
$M = 2$	5.22051×10^{-3}	1.30532×10^{-2}	2.9694×10^{-3}
$M = 3$	1.57535×10^{-3}	7.2083×10^{-3}	1.62779×10^{-3}
$M = 5$	3.1167×10^{-4}	3.43748×10^{-3}	7.59478×10^{-4}

Table 8
Relative errors of the Steklov approximations of f_1, f_2, and f_3, respectively where $h = 0.8$.

	$\mathrm{rerr}_2(f_1)$	$\mathrm{rerr}_2(f_2)$	$\mathrm{rerr}_2(f_3)$
$M = 2$	5.13497×10^{-2}	1.69181×10^{-2}	2.77799×10^{-3}
$M = 3$	4.15782×10^{-2}	1.0364×10^{-2}	1.52184×10^{-3}
$M = 5$	1.78172×10^{-2}	4.58322×10^{-3}	6.98156×10^{-4}

In his thesis Cho [17, chapter 4], also investigated the approximation of harmonic functions by eigenfunctions of the Neumann Laplacian on a rectangle. Even though such eigenfunctions form an orthogonal basis of $H^1(\Omega)$, finite

Table 9
Relative errors of the Steklov approximations of f_1, f_2, and f_3, respectively where $h = 0.5$.

	$\mathrm{rerr}_2(f_1)$	$\mathrm{rerr}_2(f_2)$	$\mathrm{rerr}_2(f_3)$
$M = 2$	2.36676×10^{-1}	2.14194×10^{-2}	2.31158×10^{-3}
$M = 3$	1.00467×10^{-1}	1.04072×10^{-2}	1.3035×10^{-3}
$M = 5$	5.79567×10^{-2}	5.45324×10^{-3}	5.9589×10^{-4}

Table 10
Relative errors of Steklov approximations of f_1 and $f_1 + 4$ where $h = 1$.

	$\mathrm{rerr}_\infty(f_1)$	$\mathrm{rerr}_\infty(f_1+4)$	$\mathrm{rerr}_2(f_1)$	$\mathrm{rerr}_2(f_1+4)$
$M = 2$	6.59553×10^{-3}	5.27642×10^{-3}	5.22051×10^{-3}	2.54632×10^{-3}
$M = 3$	2.28748×10^{-3}	1.82998×10^{-3}	1.57535×10^{-3}	7.6838×10^{-4}
$M = 5$	5.55757×10^{-4}	4.46061×10^{-4}	3.1167×10^{-4}	1.52018×10^{-4}

approximations involving the first M eigenfunctions were found to provide poor approximation properties for harmonic functions in $\mathcal{H}(\Omega)$.

6. Approximations of solutions of Robin harmonic boundary value problems

When the first M harmonic Steklov eigenfunctions and eigenvalues are known, the associated Galerkin approximations of Robin or Neumann boundary value problems for Laplace's equations may be found. See Steinbach [18], chapter 8 or Zeidler [12] chapter 19 for descriptions of such constructions and their general properties. Here some specific error analyses for harmonic functions will be proved and some numerical results will be described in the next section.

A function $u \in \mathcal{H}(\Omega)$ is said to be a (finite-energy) solution of the Robin harmonic boundary value problem on Ω provided it satisfies

$$\int_\Omega \nabla u \cdot \nabla v \, dxdy + b \int_{\partial\Omega} u \, v \, d\sigma = \int_{\partial\Omega} g \, v \, d\sigma \quad \text{for all } v \in H^1(\Omega). \tag{6.1}$$

When (B1) holds, standard variational arguments guarantee the existence and uniqueness of solutions of (6.1) in $\mathcal{H}(\Omega)$. The solution is denoted $E_b \, g$ and satisfies the Robin boundary condition $D_\nu u + bu = g$ on $\partial\Omega$ in a weak sense. For $b > 0$, it is

$$\tilde{u}(x, y) = E_b g(x, y) := \lim_{M\to\infty} \sum_{j=0}^{M} \frac{\hat{g}_j \tilde{s}_j(x, y)}{b + \tilde{\delta}_j} \quad \text{for } (x, y) \in \Omega. \tag{6.2}$$

Here $\tilde{\delta}_j = \delta_j / |\partial\Omega|$. This limit exists in the H^1-norm provided $g \in H^{-1/2}(\partial\Omega)$ as described in [3, Section 10]. In particular, this holds when $g \in L^2(\partial\Omega, \, d\sigma)$; note that even for linear functions on a rectangle, the Robin or Neumann data g may be discontinuous on the boundary so a useful analysis should allow such g.

When g_M is given by (3.8), take $v = \tilde{s}_j$ in (6.1) to find that the solution is

$$u_M(x, y) := E_b \, g_M \, (x, y) = \frac{\overline{g}}{b} + \sum_{j=1}^{M} \frac{\hat{g}_j}{b + \tilde{\delta}_j} \, \tilde{s}_j(x, y) \quad \text{on } \overline{\Omega}. \tag{6.3}$$

That is, after the Steklov spectrum has been found, the Mth Galerkin approximation of $E_b g$, just requires that the Steklov coefficients $\hat{g}_j := \langle g, \tilde{s}_j \rangle_{\partial\Omega}$ be evaluated as in (3.8).

The error estimate for these approximations is the following.

Theorem 6.1. *Assume* (B1) *holds,* $b > 0, \ g \in L^2(\partial\Omega, \, d\sigma)$ *and* g_M *is defined by* (3.8)*. Then the function* u_M *of* (6.3) *is in* $\mathcal{H}(\Omega)$ *and*

$$\|E_b \, g - u_M \, \|_\partial^2 \le \frac{1 + \delta_{M+1}}{(b + \delta_{M+1})^2} \, \left[\|g\|_{2,\partial\Omega}^2 - \|g_M\|_{2,\partial\Omega}^2 \right]. \tag{6.4}$$

Moreover the functions u_M *converge uniformly to* $E_b \, g$ *on compact subsets of* Ω.

Proof. From (6.2) and (6.3) one sees that

$$E_b \, g(x, y) - E_b \, g_M(x, y) = \sum_{j=M+1}^{\infty} \frac{\hat{g}_j}{b + \tilde{\delta}_j} \, \tilde{s}_j(x, y) \quad \text{on } \overline{\Omega}.$$

Evaluating the ∂-norm of this yields, using the orthogonality of the eigenfunctions, that

$$\| E_b\, g - E_b\, g_M \|_\partial^2 = \sum_{j=M+1}^{\infty} \frac{\hat{g}_j^2\, (1+\delta_j)}{(b+\tilde{\delta}_j)^2}.$$

Thus

$$\| E_b\, g - E_b\, g_M \|_\partial^2 \le \frac{1+\delta_{M+1}}{(b+\hat{\delta}_{M+1})^2} \quad \|g - g_M\|_{2,\partial\Omega}^2. \tag{6.5}$$

Since δ_M increase to infinity, the coefficient here is bounded so $E_b\, g_M$ converges to $E_b\, g$ in $H^1(\Omega)$. This equation implies (6.4) as the Steklov eigenfunctions are L^2-orthogonal on $\partial\Omega$. Again the uniform convergence on compact subsets of Ω is a standard result for harmonic functions. □

The estimate in (6.4) shows again that H^1 error bounds for $E_b\, g$ on Ω may be found in terms of norms of $g - g_M$ on $\partial\Omega$. Some computational results for specific examples are described in the next section.

When the Neumann boundary condition ($b = 0$) holds then (6.2) holds provided $\overline{g} = 0$ and the solution is unique up to a constant. The minimum norm solution now is

$$\tilde{u}(x,y) = E_N g(x,y) := \lim_{M\to\infty} \sum_{j=1}^{M} \frac{\hat{g}_j}{\delta_j}\, \tilde{s}_j(x,y) \quad \text{for } (x,y) \in \Omega. \tag{6.6}$$

Let u_M be this Mth partial sum, then u_M converges to $E_N g$ in norm on $H^1(\Omega)$ and E_N is a continuous map of $H^{-1/2}(\partial\Omega)$ to $\mathcal{H}(\Omega)$. See Section 10 of [3] for more details.

The following error estimate for these approximations is proved using the same arguments as those for Theorem 6.1.

Theorem 6.2. *Assume* (B1) *holds,* $g \in L^2(\partial\Omega,\, d\sigma)$, $\overline{g} = 0$ *and* g_M *is defined by* (3.8)*. Then* u_M *defined by* (6.6) *is in* $\mathcal{H}(\Omega)$ *and*

$$\|E_N g - u_M\|_\partial^2 \le \frac{1+\delta_{M+1}}{\delta_{M+1}^2} \left[\|g\|_{2,\partial\Omega}^2 - \|g_M\|_{2,\partial\Omega}^2 \right]. \tag{6.7}$$

Moreover the functions u_M *converge uniformly to* $E_b\, g$ *on compact subsets of* Ω.

7. Computation of solutions of Robin harmonic boundary value problems

The results of the preceding section provide representations of the solutions of Robin and Neumann problems for the Laplacian in terms of the harmonic Steklov eigenproblems. Our observations are that approximations with relatively few (16–40) Steklov eigenfunctions compared quite well with numerical solutions obtained using finite element software such as FreeFem++ (see [9]).

Rather than comparing the results with such software, however, we will present some data about comparisons with problems with exact solutions to illustrate the phenomenology observed. In particular we observed good approximations away from the boundary and some difficulty in handling discontinuity in the data g at points of discontinuity — even when the solution is nice. There is a Gibb's type effect in this case.

Denote $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 to be the side with $x = 1, y = h, x = -1$, and $y = -h$, respectively such that $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

7.1. Neumann harmonic boundary value problem

Consider the boundary value problem on $\Omega = R_h$

$$\Delta u = 0 \quad \text{on } \mathbb{R}_h \text{ with } D_\nu\, u = g \text{ on } \partial\Omega \tag{7.1}$$

with Dirichlet data

$$g(x,y) = \begin{cases} +1 & \text{on } \Gamma_1 \text{ and } \Gamma_2 \\ -1 & \text{on } \Gamma_3 \text{ and } \Gamma_4. \end{cases} \tag{7.2}$$

We note that this example has a unique solution $u(x,y) = x + y$ with mean value zero on R_h. This solution is infinitely differentiable but the boundary data g is discontinuous at $(-1, h)$ and $(1, -h)$ because the domain R_h has corners.

A graph of the numerical solution and of the error $u - u_5$ of the solution with $M = 5$ is given in Fig. 2.

Another Neumann problem (7.1) on $\Omega = R_h$ used g

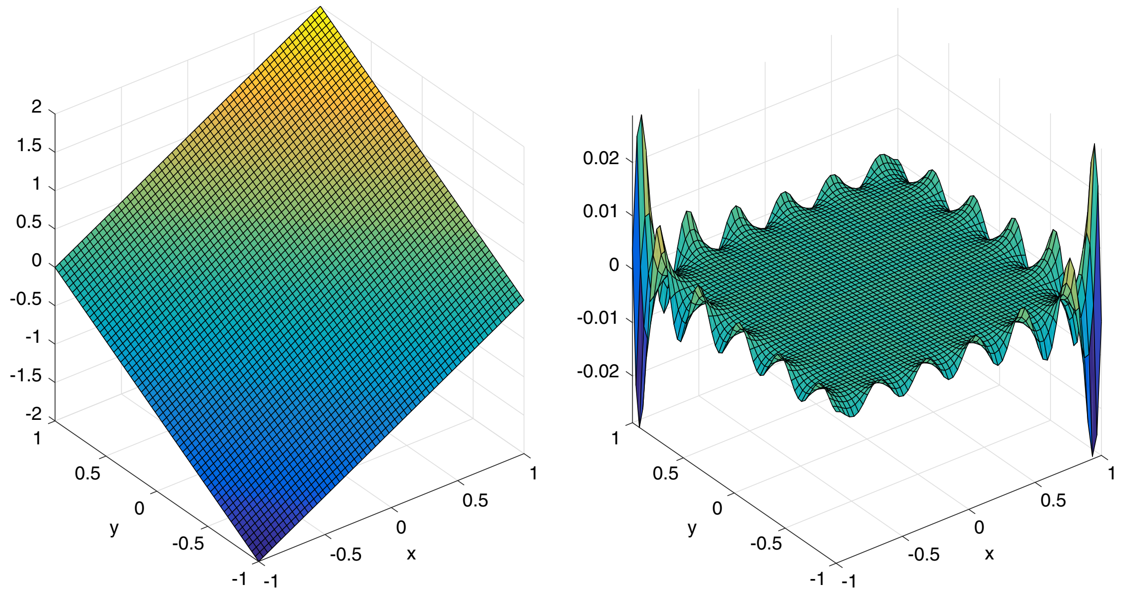
$$g(x,y) = \begin{cases} +2 & \text{on } \Gamma_1 \text{ and } \Gamma_3 \\ -2h & \text{on } \Gamma_2 \text{ and } \Gamma_4. \end{cases} \tag{7.3}$$

Table 11
Relative error of the Steklov approximation of the solution of (7.1) with the boundary condition (7.2) where $h = 1$.

	$\mathrm{rerr}_\infty(u)$	$\mathrm{rerr}_2(u)$
$M = 2$	3.44988×10^{-2}	2.17341×10^{-2}
$M = 3$	2.34853×10^{-2}	1.23794×10^{-2}
$M = 5$	1.43896×10^{-2}	5.98271×10^{-3}

Table 12
Relative error of the Steklov approximation of the solution of (7.1) with the boundary condition (7.3) where $h = 1$.

	$\mathrm{rerr}_\infty(u)$	$\mathrm{rerr}_2(u)$
$M = 2$	9.07987×10^{-2}	1.32590×10^{-1}
$M = 3$	5.34729×10^{-2}	9.20000×10^{-2}
$M = 5$	2.64002×10^{-2}	5.70258×10^{-2}



(a) Steklov approximation, u_5.

(b) Error in the solution, $u - u_5$.

Fig. 2. Numerical results of the Steklov approximation of the solution of (7.1) with the boundary condition (7.2) where $h = 1$.

This problem has a unique solution $u(x, y) = x^2 - y^2$ with mean value zero on the rectangle. This solution is a well-known saddle function but now the boundary data g is discontinuous at each corner. Tables 11 and 12 give some relative errors and graphs of the Steklov approximation with $M = 5$ and the error $u - u_5$ are provided in Fig. 3.

These simple examples show that the Steklov approximations of solutions of these problems provide quite good approximations in the interior of the region even for small choices of M. The approximations satisfy the maximum principle, so the solutions are less accurate at, or near, the boundary.

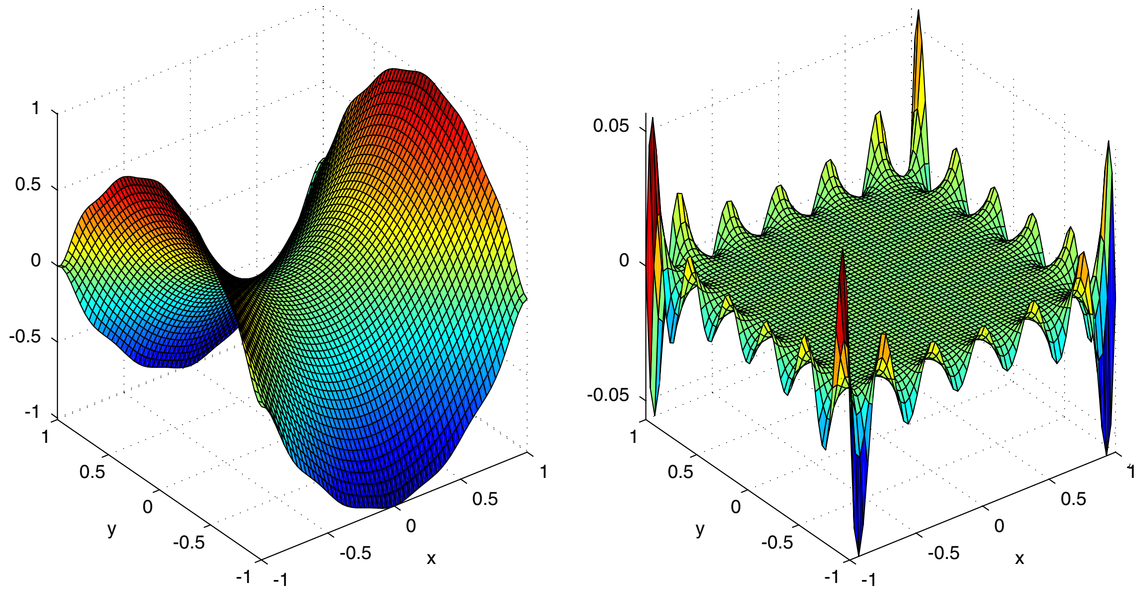
7.2. Robin harmonic boundary value problem

We consider a solution of the Robin harmonic boundary value problem with $b = 1$ on R_h,

$$\Delta u = 0 \quad \text{on } R_h \text{ with } D_\nu u + bu = g \text{ on } \partial\Omega \tag{7.4}$$

where g is given by

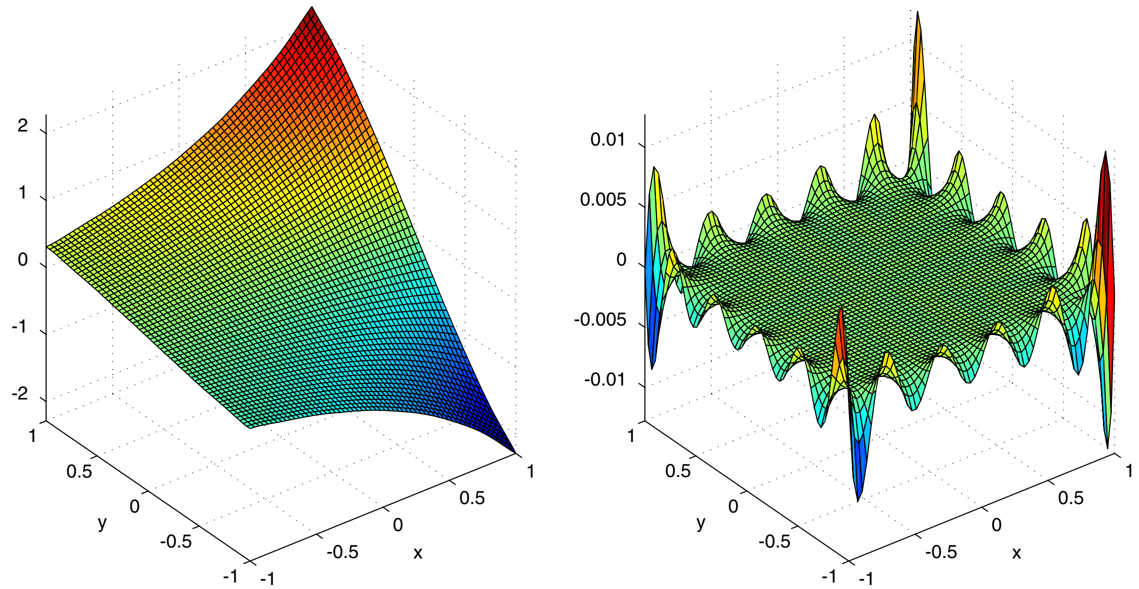
$$g(x, y) = \begin{cases} 2(e^1 \sin(y)) & \text{on } \Gamma_1 \\ e^x(\cos(h) + \sin(h)) & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \\ -e^x(\cos(h) + \sin(h)) & \text{on } \Gamma_4. \end{cases} \tag{7.5}$$



(a) Steklov approximation, u_5.

(b) Error in the solution, $u - u_5$.

Fig. 3. Numerical results of the Steklov approximation of the solution of (7.1) with the boundary condition (7.3) where $h = 1$.



(a) Steklov approximation, u_5.

(b) Error in the solution, $u - u_5$.

Fig. 4. Numerical results of the Steklov approximation of the solution of (7.4) with the boundary condition (7.5) where $h = 1$.

The unique solution of this problem is $u(x, y) = e^x \sin(y)$. The Steklov approximation with $M = 5$ is shown in Fig. 4, together with a graph of the error function $u - u_5$. Again the relative error is quite reasonable and the approximations are very accurate away from the boundary (see Table 13).

These simple examples were chosen primarily to illustrate the phenomenology observed in computing Steklov approximations. There clearly are many further questions about the efficacy of such approximations but the primary observation is that low order Steklov approximations do provide good interior approximations to solutions of harmonic boundary value problems.

Table 13
Relative error of the Steklov approximation of the solution of (7.4) with the boundary condition (7.5) where $h = 1$.

	$\mathrm{rerr}_\infty(u)$	$\mathrm{rerr}_2(u)$
$M = 2$	1.51186×10^{-2}	1.4854×10^{-2}
$M = 3$	9.64123×10^{-3}	7.84911×10^{-3}
$M = 5$	5.60122×10^{-3}	3.53263×10^{-3}

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