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Forward starting options pricing with double stochastic volatility, stochastic interest rates and double jumps

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Abstract

We present an extension of double Heston stochastic volatility model by introducing CIR stochastic interest rate and double exponential jumps in the stock price process. We derive the characteristic function and forward characteristic function of the log asset price and thereby forward starting options are well evaluated by the COS method. We also provide efficient simulation of the proposed model and Monte Carlo solutions for forward starting options based on the QE scheme. Numerical results show that the COS method is fast and efficient for pricing forward starting options.

Keywords: Forward starting options; COS method; Double exponential jumps; Stochastic interest rates; Double Heston model

1. Introduction

Forward starting options are the options which commence at some time in the future with an expiration further at some specified future date. They are the building blocks of cliquet options which are very popular in the world of equity derivatives. Stochastic volatility models fit the empirical implied volatility surface for long expirations fairly well [1]. Lucié [2], Kruse and Nögel [3] priced forward starting options under the Heston stochastic volatility model [4]. However, stochastic volatility models including the Heston model are inconsistent with the short-dated term structure of skew [5]. Two extensions have been present in some literatures. One is to add jumps

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in the stock price or (and) variance process and the other is to enrich the variance process by specifying a multifactor structure for the volatility. Representative generalizations include the Bates model [6] and the double Heston model proposed by Christoffersen et al. [7]. Broadie and ÖKaya [8] and Hong [9] considered forward starting options pricing under the Bates model. Guo and Hung [10] covered forward starting options pricing under the Heston model with simultaneous jumps in stock price and variance. Recently, Sun [11-12] confirmed the double Heston model with jumps has an improved market fitting and hedging performance compared with the double Heston model and the Bates model. As securities with forward starting features often have a long-dated maturities, stochastic interest rate is crucial for pricing forward starting options. Moreover, compare with lognormal jumps [13], the double exponential jumps, proposed by Kou [14], generate a highly skewed and leptokurtic distribution and are capable of matching key features of stock and index returns. Therefore, this paper incorporates double Heston stochastic volatility, double exponential jumps and stochastic interest rates into our model for pricing forward starting options.

Broadie and ÖKaya [8] evaluated forward starting options by exact but time-consuming Monte Carlo simulation. Hong [9] speeded up forward starting options pricing by FFT [15]. However, being affected by the dampening factor FFT is usually unstable. A more efficient pricing method was proposed by Fang and Oosterlee [16] who pioneered the use of the COS method based on Fourier cosine series expansion. The COS method achieves an exponential convergence rate for European options. Forward starting options belong to the class of path-dependent European-style contracts. Therefore it is possible that the COS method can be extended to evaluate efficiently forward starting options.

The main goal of this paper is to provide an efficient pricing method for forward starting options under the double Heston model with stochastic interest rates and double exponential jumps. The main contributions of this paper are two-fold. Firstly, this paper extends the double Heston model by introducing stochastic interest rates and double exponential jumps in the stock price. Secondly, the paper extends the COS method to forward starting options pricing with stochastic interest rates. The rest of the paper is organized as follows. Section 2 develops the underlying pricing model. Section 3 derives the characteristic function and forward characteristic function of the log asset price. Section 4 dilates the COS method for pricing forward starting options. Section 5 presents simulation of the proposed model and

some numerical results. Section 6 concludes.

2. The model

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P\}$ be a complete probability space with a filtration satisfying the usual conditions, i.e., the filtration is continuous on the right and \mathcal{F}_0 contains all P -null sets, and P is a risk-neutral probability. Suppose $W_j^S(t)$, $W_j^V(t)$ ($j = 1, 2$), and $W^r(t)$ are all standard Brownian motions which are \mathcal{F}_t -adapted. Moreover, $Cov(dW_1^S(t), dW_1^V(t)) = \rho_1 dt$, $Cov(dW_2^S(t), dW_2^V(t)) = \rho_2 dt$, $W^r(t)$ is independent of $W_j^S(t)$ and $W_j^V(t)$. Suppose that the asset price process $S(t)$ is governed by the following double stochastic volatility model with CIR stochastic interest rate and double exponential jumps (DSVSIDJ):

$$\frac{dS(t)}{S(t)} = (r(t) - \lambda\delta)dt + \sum_{j=1}^n \sqrt{V_j(t)} dW_j^S(t) + d\left(\sum_{j=1}^{N(t)} (\zeta_j - 1)\right), \quad (1)$$

$$dV_1(t) = k_1(\theta_1 - V_1(t))dt + \sigma_1 \sqrt{V_1(t)} dW_1^V(t), \quad (2)$$

$$dV_2(t) = k_2(\theta_2 - V_2(t))dt + \sigma_2 \sqrt{V_2(t)} dW_2^V(t), \quad (3)$$

$$dr(t) = k_r(\theta_r - r(t))dt + \sigma_r \sqrt{r(t)} dW^r(t), \quad (4)$$

where k_j, θ_j, σ_j ($j = 1, 2$) are the mean-reverting rates, long-term volatility and volatility of variance of variance process $V_j(t)$, respectively. We suppose $2k_j\theta_j \geq \sigma_j^2$ and $2k_r\theta_r \geq \sigma_r^2$ to make the processes $V_j(t)$ and $r(t)$ remain strictly positive. Suppose $V_1(0) = V_1$, $V_2(0) = V_2$, $S(0) = S$, $r(0) = r$, $N(t)$ is a Poisson process with constant intensity $\lambda > 0$, and $\zeta = (\zeta_j)_{j \geq 1}$ is a sequence of independent identically distribute nonnegative random variables, such that $Y = \ln \zeta$ has an asymmetric double exponential distribution with the density

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} \mathbf{1}_{y \geq 0} + q\eta_2 e^{\eta_2 y} \mathbf{1}_{y < 0}, \quad \eta_1 > 1, \eta_2 > 0, \quad (5)$$

where $\mathbf{1}$ denotes the indicator function, so $\mathbf{1}_{y \geq 0}$ equals 1 if $y \geq 0$, but 0 otherwise. $p, q \geq 0, p + q = 1$ are up-move jump and down-move jump, respectively. $\delta = E^P(\zeta - 1) = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1$. Suppose the process $W_j^S(t)$, $W_j^V(t)$, and $W^r(t)$ are independent of ζ and $N(t)$.

For the interest rate process (4), one can express the time-0 price of a zero-coupon bond maturing at time T as the following exponential affine

form:

$$P(0, T) = e^{A(T) + B(T)r}, \quad (6)$$

where

$$\begin{aligned} A(T) &= \frac{(k_r - \gamma)k_r\theta_r T}{\sigma_r^2} + \frac{2k_r\theta_r}{\sigma_r^2} \ln \frac{2\gamma}{2\gamma + (k_r - \gamma)(1 - e^{-\gamma T})}, \\ B(T) &= \frac{2(e^{-\gamma T} - 1)}{2\gamma + (k_r - \gamma)(1 - e^{-\gamma T})}, \\ \gamma &= \sqrt{k_r^2 + 2\sigma_r^2}. \end{aligned}$$

3. Deriving the characteristic function and forward characteristic function

To simplify calculation we change from the money market account numeraire measure P to the T -forward measure Q by the following Radon-Nikodym derivative

$$\frac{dQ}{dP} = \frac{\exp(-\int_0^T r(t)dt)}{P(0, T)}.$$

Lemma 3.1. Let $x(t) = \ln S(t)$, $x = \ln S$. Under the T -forward measure Q , the characteristic function of $x(t)$ is given by

$$\phi(u, x, r, V_1, V_2, t) = \exp \left(iux + A(u, t) + \sum_{j=1}^2 B_j(u, t)V_j + C(u, t)r \right), \quad (7)$$

where

$$\begin{aligned} A(u, t) &= \sum_{j=1}^2 \left[\frac{2k_j\theta_j}{\sigma_j^2} \ln \frac{2\gamma_j}{2\gamma_j + \beta_j} + \frac{k_j\theta_j(k_j - \gamma_j)}{\sigma_j^2} - \frac{iuk_j\theta_j\rho_j t}{\sigma_j} \right] \\ &\quad + \frac{k_r\theta_r}{\sigma_r^2} \left[(\gamma - \gamma_u)t + 2 \ln \frac{2\gamma_u}{2\gamma_u + (k_r - \gamma_u)(1 - e^{-\gamma_u t})} \right. \\ &\quad \left. - 2 \ln \frac{2\gamma}{2\gamma + (k_r - \gamma)(1 - e^{-\gamma t})} \right] \\ &\quad + \lambda t \left(-iu\delta + \frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1 \right), \end{aligned}$$

$$\begin{aligned}
 B_j(u, t) &= \frac{iu(iu - 1)(1 - e^{-\gamma_j t})}{2\gamma_j + \beta_j}, \\
 C(u, t) &= \frac{2(1 - e^{-\gamma t})}{2\gamma + (k_r - \gamma)(1 - e^{-\gamma t})} - \frac{2(1 - iu)(1 - e^{-\gamma_u t})}{2\gamma_u + (k_r - \gamma_u)(1 - e^{-\gamma_u t})}, \\
 \beta_j &= (k_j - \gamma_j - iu\rho_j\sigma_j)(1 - e^{-\gamma_j t}), \\
 \gamma_j &= \sqrt{(k_j - iu\rho_j\sigma_j)^2 + iu(1 - iu)\sigma_j^2}, \\
 \gamma_u &= \sqrt{k_r^2 + 2(1 - iu)\sigma_r^2}.
 \end{aligned}$$

Proof By applying multi-dimensional Feynman-Kac threorem, $\phi(\cdot)$ satisfies the following PIDE:

$$\begin{cases}
 0 = -r\phi + \phi_t + (r - \lambda\delta - \frac{V_1+V_2}{2})\phi_x + \frac{V_1+V_2}{2}\phi_{xx} \\
 \quad + \sum_{j=1}^2 [\frac{1}{2}\sigma_j^2 V_j \phi_{V_j V_j} + \rho_j \sigma_j V_j \phi_{x V_j} + k_j(\theta_j - V_j)\phi_{V_j}] \\
 \quad + k_r(\theta_r - V_r)\phi_r + \frac{1}{2}\sigma_r^2 r \phi_{rr} + \lambda \int_{-\infty}^{+\infty} [\phi(x + \zeta) - \phi(x)] \varpi(\zeta) d\zeta, \\
 \phi(u, x, r, V_1, V_2, 0) = e^{iux}.
 \end{cases} \quad (8)$$

According to Duffie et al. [17], we conjecture $\phi_X(\cdot)$ has the form as

$$\phi(u, x, r, V_1, V_2, t) = \exp \left(iux + A(u, t) + \sum_{j=1}^2 B_j(u, t)V_j + C(u, t)r \right) \quad (9)$$

with initial conditions $A(u, 0) = 0$, $B_j(u, 0) = 0$ and $C(u, 0) = 0$.

We consider the integral term in (8).

$$\begin{aligned}
 \int_{-\infty}^{+\infty} (\phi(x + \zeta) - \phi(x)) \varpi(\zeta) d\zeta &= \int_{-\infty}^{+\infty} (E^Q(e^{iu(x+\zeta)}) - E^Q(e^{iux})) \varpi(\zeta) d\zeta \\
 &= \int_{-\infty}^{+\infty} (E^Q(e^{iux})E^Q(e^{iu\zeta} - 1)) \varpi(\zeta) d\zeta \\
 &= \phi \int_{-\infty}^{+\infty} (e^{iu\zeta} - 1) \varpi(\zeta) d\zeta \\
 &= \phi \left(\frac{p}{\eta_1 - iu} + \frac{q}{\eta_2 + iu} - 1 \right).
 \end{aligned}$$

Substituting (9) into (8) yields

$$0 = -r + A_t + B_{1t}V_1 + B_{2t}V_2 + C_t r - \frac{1}{2}(V_1 + V_2)u^2$$

$$\begin{aligned}
& + \frac{1}{2}\sigma_1^2 V_1 B_1^2 + \frac{1}{2}\sigma_2^2 V_2 B_2^2 + \frac{1}{2}\sigma_r^2 r C^2 + \rho_1 \sigma_1 V_1 iu B_1 + \rho_2 \sigma_2 V_2 iu B_2 \\
& + k_1(\theta_1 - V_1)B_1 + k_2(\theta_2 - V_2)B_2 + k_r(\theta_r - r)C \\
& + \left[r - \lambda\delta - \frac{1}{2}(V_1 + V_2) \right] iu + \lambda \left(\frac{p}{\eta_1 - iu} + \frac{q}{\eta_2 + iu} - 1 \right).
\end{aligned}$$

From the above equation, we have a system of four ordinary differential equations as

$$\begin{cases}
B_{1t} + \frac{1}{2}\sigma_1^2 B_1^2 + (iu\rho_1\sigma_1 - k_1)B_1 - \frac{1}{2}u(i+u) = 0, \\
B_{2t} + \frac{1}{2}\sigma_2^2 B_2^2 + (iu\rho_2\sigma_2 - k_2)B_2 - \frac{1}{2}u(i+u) = 0, \\
C_t + iu + \frac{1}{2}\sigma_r^2 C^2 - Ck_r - 1 = 0, \\
A_t + B_1 k_1 \theta_1 + B_2 k_2 \theta_2 + C k_r \theta_r + \lambda \Lambda(u, \zeta) = 0,
\end{cases} \quad (10)$$

where $\Lambda(u, \zeta) = \frac{p}{\eta_1 - iu} + \frac{q}{\eta_2 + iu} - 1 - iu \left(\frac{p}{\eta_1 - 1} + \frac{q}{\eta_2 + 1} - 1 \right)$. By solving the above ODEs we can obtain the characteristic function (7).

Lemma 3.2. Suppose that the stochastic process $V(t)$ satisfies the following CIR model:

$$dV(t) = k(\theta - V(t))dt + \sigma\sqrt{V(t)}dW(t).$$

Under the T-forward measure Q , the characteristic function of $V(t)$ is given by

$$E^Q[\exp(iuV(t))] = \left(1 - iu\sigma^2 \frac{1 - e^{-kt}}{2k} \right)^{-\frac{2k\theta}{\sigma^2}} \exp \left(\frac{iu e^{-kt} V(0)}{1 - iu\sigma^2 \frac{1 - e^{-kt}}{2k}} \right). \quad (11)$$

Proof See Ioffe [18].

Theorem 3.1. Under the T-forward measure Q , the forward characteristic function of $\ln \frac{S(T)}{S(t_0)}$ with determination time t_0 and maturity T is given by

$$\varphi_F(u) = \exp \left(A(u, \tau) + \sum_{j=1}^2 B_{Fj}(u, \tau) + C_F(u, \tau) \right), \quad (12)$$

where

$$B_{Fj}(u, \tau) = -\frac{2k_j\theta_j}{\sigma_j^2} \ln \left(1 - \frac{B_j(u, \tau)}{h_{tj}} \right) + \frac{B_j(u, \tau)e^{-k_j t_0}}{1 - \frac{B_j(u, \tau)}{h_{tj}}} V_j,$$

$$\begin{aligned}
 C_F(u, \tau) &= -\frac{2k_r\theta_r}{\sigma_r^2} \ln \left(1 - \frac{C(u, \tau)}{h_{tr}} \right) + \frac{C(u, \tau)e^{-k_r t_0}}{1 - \frac{C(u, \tau)}{h_{tr}}} r, \\
 h_{tj} &= \frac{2k_j}{\sigma_j^2(1 - e^{-k_j t_0})}, \\
 h_{tr} &= \frac{2k_r}{\sigma_r^2(1 - e^{-k_r t_0})}, \\
 \tau &= T - t_0.
 \end{aligned}$$

Proof Using the tower law of conditional expectations, one can obtain

$$\begin{aligned}
 \varphi_F(u) &= E^Q \left(e^{iu \ln \frac{S(T)}{S(t_0)}} \right) \\
 &= E^Q \left(E^Q \left(e^{iu \ln S(T) - iu \ln S(t_0)} | \mathcal{F}_{t_0} \right) | \mathcal{F}_0 \right).
 \end{aligned}$$

Note that the inner expectation is the characteristic function of $\ln S(T)$ evaluated in the point t_0 given by lemma 3.1. Substituting (7) into the above expression yields

$$\begin{aligned}
 \varphi_F(u) &= E^Q \left(e^{A(u, \tau) + \sum_{j=1}^2 B_j(u, \tau) V_j(t_0) + C(u, \tau) r(t_0)} | \mathcal{F}_0 \right) \\
 &= e^{A(u, \tau)} E^Q \left(e^{\sum_{j=1}^2 B_j(u, \tau) V_j(t_0) + C(u, \tau) r(t_0)} | \mathcal{F}_0 \right). \quad (13)
 \end{aligned}$$

Using lemma 3.2 and setting $h_{tj} = \frac{2k_j}{\sigma_j^2(1 - e^{-k_j t_0})}$, $h_{tr} = \frac{2k_r}{\sigma_r^2(1 - e^{-k_r t_0})}$. the following result can be obtained:

$$\begin{aligned}
 E^Q \left(e^{\sum_{j=1}^2 B_j(u, t) V_j(t_0)} | \mathcal{F}_0 \right) &= \sum_{j=1}^2 -\frac{2k_j\theta_j}{\sigma_j^2} \ln \left(1 - \frac{B_j(u, \tau)}{h_{tj}} \right) + \frac{B_j(u, \tau)e^{-k_j t_0}}{1 - \frac{B_j(u, \tau)}{h_{tj}}} V_j, \\
 E^Q \left(e^{C(u, \tau) r(t_0)} | \mathcal{F}_0 \right) &= -\frac{2k_r\theta_r}{\sigma_r^2} \ln \left(1 - \frac{C(u, \tau)}{h_{tr}} \right) + \frac{C(u, \tau)e^{-k_r t_0}}{1 - \frac{C(u, \tau)}{h_{tr}}} r.
 \end{aligned}$$

Combining (13) and the above two expression theorem 3.1 follows.

4. Pricing method

Under the T-forward measure Q , one can express the price of a forward starting option on the underlying asset in the following expectation

$$C_F(t_0, T) = P(0, T) E^Q \left[\alpha \left(\frac{S(T)}{S(t_0)} - K \right)^+ \right]$$

with strike price K , determination time t_0 and maturity T . $\alpha = 1$ for calls and $\alpha = -1$ for puts. Let $\xi = \ln \frac{S(T)}{S(t_0)} - \ln K$, one can express the above formula as

$$C_F(t_0, T) = P(0, T)K \int_{-\infty}^{\infty} \alpha(e^\xi - 1)^+ f(\xi|z) d\xi, \quad (14)$$

where $f(\xi|z)$ is the risk-neutral density of ξ conditional on $z = \ln S - \ln K$.

Since $f(\xi|z)$ decays to zero as $\xi \rightarrow \pm\infty$, one can approximate $f(\xi|z)$ by a truncated Fourier cosine series expansion on domain $[m, n]$, that is

$$f(\xi|z) \approx \frac{2}{n-m} \sum_{j=0}^{N-1} \Re \left[\varphi_F \left(\frac{j\pi}{n-m} \right) e^{-i \frac{j\pi(\ln K+m)}{n-m}} \right] \cos \left[\frac{j\pi(\xi-m)}{n-m} \right], \quad (15)$$

where $\varphi_F(\cdot)$ is the characteristic function of $f(\xi|z)$. $\Re[\cdot]$ denotes taking the real part of the argument. Inserting (15) in (14) and interchanging integration and summation gives the following formula

$$C_F(t_0, T) \approx \frac{2}{n-m} KP(0, T) \sum_{j=0}^{N-1} \Re \left[\varphi_F \left(\frac{j\pi}{n-m} \right) e^{-i \frac{j\pi m}{n-m}} \right] U_j, \quad (16)$$

where $U_j = \int_m^n \alpha(e^\xi - 1)^+ \cos \left(\frac{\xi-m}{n-m} j\pi \right) d\xi$. For call options, basic calculus gives the following analytical solutions:

$$U_j = \begin{cases} \frac{e^n \cos j\pi - \cos(\frac{mj\pi}{n-m}) + \frac{j\pi}{n-m} \sin(\frac{j\pi}{n-m})}{1 + (\frac{j\pi}{n-m})^2} - \frac{n-m}{j\pi} \sin(\frac{j\pi}{n-m}), & j \neq 0 \\ e^n - 1 - n. & j = 0 \end{cases}$$

According to Fang and Oosterlee [16], the integration interval $[m, n]$ is chosen as follows:

$$[m, n] = [c_1 - L\sqrt{|c_2|}, c_1 + L\sqrt{|c_2|}], \quad (17)$$

where L is an appropriate constant, c_n can be obtained by

$$c_n = \frac{1}{i^n} \frac{\partial^n (\ln \varphi_F(u))}{\partial u^n} \Big|_{u=0}. \quad (18)$$

For the DSVSIDJ model,

$$c_1 = \sum_{j=1}^2 \left[\frac{\theta_j(1 - e^{-k_j\tau} - k_j\tau)}{2k_j} + \frac{e^{-k_j\tau} - 1}{2k_j} \left(\frac{2k_j\theta_j}{\sigma_j^2 h_{tj}} + e^{-k_j t} V_j \right) \right]$$

$$\begin{aligned}
 & + \frac{k_r \theta_r}{\gamma} \left\{ \tau + \frac{2 \left[\frac{k_r}{\gamma} (e^{-\gamma \tau} - 1) + \tau (k_r - \gamma) e^{-\gamma \tau} \right]}{2\gamma + (k_r - \gamma)(1 - e^{-\gamma \tau})} \right\} - i \left(\frac{2k_r \theta_r}{\sigma_r^2 h_{tr}} + e^{-k_r t_r} \right) C' \\
 & + \lambda \tau \left(\frac{p}{\eta_1} - \frac{q}{\eta_2} - \frac{p\eta_1}{\eta_1 - 1} - \frac{q\eta_2}{\eta_2 + 1} + 1 \right).
 \end{aligned}$$

$$\begin{aligned}
 c_2 = & \sum_{j=1}^2 \left[\frac{k_j \theta_j \tau \gamma_j''}{\sigma_j^2} - \frac{5\sigma_j^2 \theta_j}{8k_j^3} + \frac{2\sigma_j \theta_j \rho_j}{k_j^2} - \frac{\theta_j}{k_j} + e^{-k_j \tau} \left(\frac{\theta_j \sigma_j^2 + \theta_j \sigma_j^2 \tau k_j}{2k_j^3} \right. \right. \\
 & \left. \left. + \frac{\theta_j - \theta_j \sigma_j \rho_j \tau}{k_j} - \frac{2\theta_j \sigma_j \rho_j}{k_j^2} \right) + \frac{\sigma_j^2 \theta_j}{8k_j^3} e^{-2k_j \tau} - \frac{2k_j \theta_j}{\sigma_j^2} \frac{B_j'' h_{tj} + B_j'^2}{h_{tj}^2} \right. \\
 & \left. - e^{-k_j t} V_j \frac{B_j'' h_{tj} + 2B_j'^2}{h_{tj}} \right] + \frac{k_r \theta_r}{\sigma_r^2} (\gamma_u'' \tau - \Gamma) - \frac{2k_r \theta_r}{\sigma_r^2} \frac{C'' h_{tr} + C'^2}{h_{tr}^2} \\
 & - e^{-k_r t_r} \frac{C'' h_{tr} + 2C'^2}{h_{tr}} + 2\lambda \tau \left(\frac{p}{\eta_1^2} + \frac{q}{\eta_2^2} \right),
 \end{aligned}$$

where

$$\begin{aligned}
 \gamma_j'' &= \frac{\sigma_j^2}{k_j} - \frac{\rho_j \sigma_j^3}{k_j^2} + \frac{\sigma_j^4}{4k_j^3}, \quad \gamma_u' = -\frac{i\sigma_r^2}{\gamma}, \quad \gamma_u'' = \frac{\sigma_r^4}{\gamma^3}, \quad B_j' = -\frac{i(1 - e^{-k_j \tau})}{2k_j}, \\
 B_j'' &= \frac{\sigma_j^2}{4k_j^3} e^{-2k_j \tau} + e^{-k_j \tau} \left(\frac{\sigma_j^2 \tau}{2k_j^2} + \frac{1 - \sigma_j \rho_j \tau}{k_j} - \frac{\sigma_j \rho_j}{k_j^2} \right) - \frac{1}{k_j} - \frac{\sigma_j^2}{4k_j^3} + \frac{\sigma_j \rho_j}{k_j^2}, \\
 C' &= \frac{2i \left[\gamma + k_r - \frac{\sigma_r^2}{\gamma} + (2\sigma_r^2 \tau - 2k_r) e^{-\gamma \tau} + (k_r - \gamma + \frac{\sigma_r^2}{\gamma}) e^{-2\gamma \tau} \right]}{[2\gamma + (k_r - \gamma)(1 - e^{-\gamma \tau})]^2}, \\
 C'' &= \frac{-2[-(k_r + \gamma)\gamma_u'' + 2\gamma_u'^2 + 2i(k_r + \gamma)\gamma_u' + e^{-\gamma \tau} \Gamma_1 - e^{-2\gamma \tau} \Gamma_2 + e^{-3\gamma \tau} \Gamma_3]}{[2\gamma + (k_r - \gamma)(1 - e^{-\gamma \tau})]^3}, \\
 \Gamma_1 &= (k_r - \gamma)\gamma_u'' + (k_r + \gamma)[2\gamma_u' \tau - 2\gamma_u' \tau i(k_r + \gamma) + 2\gamma(\gamma_u'' \tau - \gamma_u'^2 \tau^2)] \\
 &\quad - 2\gamma_u'(2ik_r + 2\gamma\gamma_u' \tau) + 2[i(k_r + \gamma) + \gamma_u'][\gamma_u' + (k_r - \gamma)\tau\gamma_u'], \\
 \Gamma_2 &= (k_r + \gamma)[- \gamma_u'' + 2\gamma_u'^2 \tau - 2(k_r - \gamma)\gamma_u' \tau i] + 2\gamma_u'[i(\gamma - k_r) + \gamma_u'] \\
 &\quad + (k_r - \gamma)[2\gamma_u'^2 \tau - 2\gamma_u' \tau i(k_r + \gamma) + 2\gamma(\gamma_u'' \tau - \gamma_u'^2 \tau^2)] \\
 &\quad + 2(2ik_r + 2\gamma\gamma_u' \tau)[\gamma_u' + (k_r - \gamma)\tau\gamma_u'], \\
 \Gamma_3 &= (k_r - \gamma)[- \gamma_u'' + 2\gamma_u'^2 \tau - 2(k_r - \gamma)\gamma_u' \tau i] - 2[i(\gamma - k_r) + \gamma_u'][\gamma_u' + (k_r - \gamma)\tau\gamma_u'], \\
 \Gamma &= \frac{1}{\gamma^2 [2\gamma + (k_r - \gamma)(1 - e^{-\gamma \tau})]^2} \left\{ [2k_r \gamma_u'' + e^{-\gamma \tau} ((2\tau \gamma^2 - 2k_r - 2\tau k_r \gamma) \gamma_u''] \right.
 \end{aligned}$$

$$\begin{aligned} & + (4\tau\gamma + 2k_r\gamma\tau^2 - 2\tau^2\gamma^2)\gamma_u'^2][2\gamma^2 + \gamma(k_r - \gamma)(1 - e^{-\gamma\tau})] \\ & - [2k_r\gamma_u'(1 - e^{-\gamma\tau}) - 2\gamma\gamma_u'\tau(k_r - \gamma)e^{-\gamma\tau}][2\gamma\gamma_u'(1 + e^{-\gamma\tau}) \\ & + k_r\gamma_u'(1 - e^{-\gamma\tau}) + \gamma\gamma_u'\tau e^{-\gamma\tau}(k_r - \gamma)] \}. \end{aligned}$$

5. Simulation experiment and numerical results

In this section, we use the DSVSIDJ model to analyze the valuation of forward starting call options. We first evaluate the options by the COS method. We use $N = 64$ points in the (15) and $L = 10$ in the (17). Parameters are set as follows: $\eta_1 = \eta_2 = 20$, $\lambda = 0.5$, $p = 0.6$, $k_r = 10$, $\theta_r = 0.06$, $\sigma_r = 0.5$, $k_1 = 12$, $\theta_1 = 0.05$, $\sigma_1 = 0.9$, $\rho_1 = -0.5$, $V_1 = 0.05$, $k_2 = 16$, $\theta_2 = 0.08$, $\sigma_2 = 0.9$, $\rho_2 = -0.5$, $V_2 = 0.02$, $t = 0$, $t_0 = 2$, $T = 10$, $S = 100$, $r = 0.012$. The results are shown in Table 1.

As the true values of forward starting options are not obtainable the paper takes Monte Carlo solutions as benchmark. We also evaluate forward starting call options based on the QE scheme proposed by Andersen [19]. and adapted to the double Heston model by Gauthier and Possamai [20]. Suppose that $Z_j(j = 1, 2)$ are independent standard Gaussian random variables, $U_j(j = 1, 2)$ are independent uniform random numbers, $Y_j(j = 1, 2, \dots)$ are independent double exponential random variables. Given a positive threshold Ψ_c , the QE scheme for $S(t)$ under the DSVSIDJ model can be written as

$$\begin{aligned} S(t + \Delta) &= \frac{S(t)P(0, t)}{P(0, t + \Delta)} \exp \left[\lambda \delta \Delta + \sum_{j=1}^2 (K_j^0 + K_j^1 V_j(t) + K_j^2 V_j(t + \Delta)) \right. \\ &\quad \left. + \sqrt{K_j^3 V_j(t) + K_j^4 V_j(t + \Delta)} Z_j + \sum_{j=N(t)+1}^{N(t+\Delta)} \zeta_j - 1 \right], \\ V_j(t + \Delta) &= \mathbf{1}_{\Psi_j \geq \Psi_c} [a_j(b_j + Z_j)^2] + \mathbf{1}_{\Psi_j < \Psi_c} \left(\mathbf{1}_{p_j < U_j \leq 1} \epsilon_j^{-1} \ln \frac{1 - p_j}{1 - U_j} \right), \end{aligned}$$

where $K_j^0 = -\rho_j k_j \theta_j \Delta / \sigma_j$, $K_j^1 = \beta \Delta (k_j \rho_j / \sigma_j - 0.5) - \rho_j / \sigma_j$, $K_j^2 = (1 - \beta) \Delta (k_j \rho_j / \sigma_j - 0.5) + \rho_j / \sigma_j$, $K_j^3 = \beta \Delta (1 - \rho_j^2)$, $K_j^4 = (1 - \beta) \Delta (1 - \rho_j^2)$, $\beta \in [0, 1]$, $a_j = m_j / (1 + b_j^2)$, $b_j = 2\Psi_j^{-1} + 2\sqrt{2\Psi_j^{-1}(2\Psi_j^{-1} - 1)}$, $\Psi_j = s_j^2 / m_j^2$, $m_j = \theta_j + (V_j(t) - \theta_j)e^{-k_j \Delta}$, $s_j^2 = V_j(t)\sigma_j^2 e^{-k_j \Delta} (1 - e^{-k_j \Delta}) / k_j + \theta_j \sigma_j^2 (1 - e^{-k_j \Delta})^2 / (2k_j)$, $p_j = (\Psi_j - 1) / (\Psi_j + 1)$, $\epsilon_j = (1 - p_j) / m_j$.

Table 1. Comparisons of the accuracy and speed of the COS, FFT and Monte Carlo metod.

K	COS (N=64)	FFT (N = 4096)	MC± standard deviation (M=1000000, N=1000)
80	37.9226	37.5368	37.9220± 0.0679
90	31.5440	31.7347	31.5443±0.0666
100	25.3989	25.4643	25.3987± 0.0653
115	19.4648	19.5032	19.4645±0.0640
120	13.7217	13.9912	13.7218±0.0627
Computing time (sec)	0.0151	0.0110	493.1261

For Monte Carlo method, we use $\Psi_C = 1.5, \beta = 0.5$, the number of simulation $M = 1000000$ and the number of time steps $N = 1000$. For comparison, FFT is also used in numerical experiments. For FFT, we use $N = 4096$ points in the quadrature, implying a log strike spacing of $h = \frac{\pi}{300} = 0.01047$ which is adequate for practice. The results are also shown in Table 1.

Table 1 shows that the COS method and FFT is considerably faster than the MC method. For a given set of parameters, both the COS method and the FFT method calculate prices for 200 different strikes in approximately 0.01 seconds. However, the solutions produced by FFT are depend on the choice of the dampening factor which is depend on the set of model parameters, therefore, the accuracy of FFT is obviously lower than the other two methods. Our numerical results show that the COS method is fast and accurate for pricing forward starting options.

6. Conclusion

The double Heston stochastic volatility model with stochastic interest rates and double exponential jumps incorporates several important features of stock return. We derieve the characteristic function and forward characteristic function of the log-asset price and obtain numerical solutions to forward starting options pricing based on the COS method. We also provide Monte Carlo solutions to forward starting options based on the QE scheme. Numerical results show that the COS method is fast and accurate.

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