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Numerical solution of integro-differential equations of high order by wavelet basis, its algorithm and convergence analysis

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Abstract

This paper presents, for the first time, numerical solutions for this particular type of integro-differential equations. According to equations which will be introduced, suitable wavelet Galerkin method is provided using wavelet basis in the space $C^{\alpha}(R)\bigcap L^2(R)$, $\alpha > 0$, that $C^{\alpha}(R)$ is the Hölder space of exponent α . This approach has two advantages. First, the wavelets basis are arbitrary. It means that any differentiable wavelets basis can be used. Second, the desired orders for this equation are the reasons for involving a wide variety of this types of equations. The Algorithm and convergence analysis of this scheme are described. Numerical examples, plots and tablets of errors confirm the applicability and the validity of the proposed method.

Keywords: `Fredholm integro-differential equation; wavelet Galerkin method; wavelets basis; convergence analysis.`

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1. Introduction

. Integro-differential equations whose numerical solution would be achieved by an arbitrary wavelet scaling function basis $\phi_{-J,k}(x)$ are as follows.

$$\begin{cases} f^{(n)}(x) = g(x) + \int_0^{2\pi} K(x,t) f^{(s)}(t)\, dt; & n > s;\ s = 0, 1, ..., m;\ x \in [0, 2\pi]. \\ \alpha_i = f^{(n-i)}(0), & \mathrm{i} = 1, 2, ..., \mathrm{n}. \end{cases} \tag{1}$$

where the functions $f^{(n)}(x)$, $f^{(s)}(x)$ indicate the n^{th}-order and s^{th}-order derivative of $f(x)$, and n, s, $g(x)$, $K(x,t)$ are known, $f(x)$ is the unknown function. The integro-differential equations are applied in many branches of science, such as physics, engineering, biochemistry and etc. In this study, the numerical solution of integro-differential equations (1) is discussed using the wavelet Galerkin method which is solved numerically by any differentiable wavelets function. A wide range of these types of equations are included because n and s are arbitrary. A quick way to solve the integral equations is offered in [1-4].
Methods for solving nonlinear integro-differential equations and integral equations using Haar wavelet is provided in [5-7,11,15]. Comparisons between homotopy method and Galerkin method are done. In [16] Haar wavelets for the solution of fractional integral equations are applied. Another type of integro-differential equations of order n were solved and calculations and formulas were presented by wavelet Galerkin method in [8,9]. In [12,13], based on Gaussian wavelet basis, Volterra integro-differential equations and integro-differential equations with weakly singular kernels were solved. Wavelet Galerkin method is presented for integral equations, specifically B-spline wavelets or block-pulse functions in [14,15,17]. In [10] trigonometric wavelets have been introduced.

2. Preliminaries of wavelet

Definition 1. (Multi-Resolution Analysis (MRA)). let $V_j,\ j = ..., -1, 0, 1, ...$ be a sequence of subspaces of functions in $L^2(R)$, we say that $\{V_j,\ j \in\ Z\}$ is a MRA if the following conditions hold:

1) $V_j \subset V_{j+1}$

2) $\overline{\bigcup V_j} = L^2(R)$

3) $\bigcap V_j = \{0\}$

4) $f(x) \in V_j \Leftrightarrow f\left(2^{-j}x\right) \in V_0$

5) Function ϕ belongs to V_0 and the set $\{\phi(x-k),\ k \in\ Z\}$ with inner product is an orthonormal basis for V_0 .

In fact, the function which is generate a MRA is a scaling function.

Theorem 1. Suppose $\{V_j, j \in\ Z\}$ is a MRA with scaling function ϕ that ϕ belongs to $L^2(R)$. The following scaling relation is established.
$\phi(x) = \sum_k a_k \phi(2x-k)$, $a_k = \int_{-\infty}^{+\infty} \phi(x)\overline{\phi(2x-k)\,dx}$.

Proof. [18].

Theorem 2. Suppose $\{V_j, j \in\ Z\}$ is a MRA with scaling function ϕ that ϕ belongs to $L^2(R)$, then $\left\{\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi\left(2^j x-k\right); k \in\ Z\right\}$ is a orthonormal basis for V_j.

Proof. [18].

Suppose V_{j+1} is decomposed to an orthogonal set of V_j and an orthogonal complement set W_j. Basis function related to W_j space is shown by ψ_j, ψ_j will be wavelet basis function. Thus

$V_{j+1} = V_j \oplus W_j = W_j \oplus V_{j-1} \oplus W_{j-1} = ... = W_j \oplus W_{j-1} \oplus ... \oplus W_0 \oplus V_0$

$L^2_{2\pi} = V_0 \oplus \left(\oplus_{j=0}^{\infty} W_j\right)$

Theorem 3. Suppose $\{V_j, j \in\ Z\}$ is a MRA with wavelet function ψ that ψ belongs to $L^2(R)$, then $\left\{\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi\left(2^j x-k\right); k \in\ Z\right\}$is a orthonormal basis for V_j.

Proof. [18].

Definition 2. (Hölder space). Hölder space of order $0 < \alpha < 1$ is defined as
$C^\alpha(R) = \left\{f \in L^\infty(R); \sup \frac{|f(x+h)-f(x)|}{|h|^\alpha} < \infty\right\}$, and if $\alpha = n + s'$: $0 < s' < 1$.

Or

$C^{\alpha}(R) = \left\{ f \in L^{\infty}(R) \bigcap C^{n}(R); \frac{d^n}{dx^n} f \in C^{s'}(R) \right\}.$

Now we define orthogonal projection operator (P_J) by the following Theorem.

Theorem 5. Let f be a function belonging to $(C^{\alpha} \bigcap L^2)(R)$, $\alpha > 0$, and ϕ, ψ be a scaling and a wavelet basis function. Assume that $\phi, \psi \in C^r$ for some $0 < \alpha < r$. For $J > 0$, define the projection operatorP_J by

$$P_J\left(f\right)\left(x\right) = \sum_{k \in z} \left\langle f, \phi_{0,k} \right\rangle \phi_{0,k}\left(x\right) + \sum_{j=-J+1}^{-1} \sum_{k \in z} \left\langle f, \psi_{j,k} \right\rangle \psi_{j,k}\left(x\right)$$

Then $P_J(f)$ converges to f in the $\|.\|_{\infty}$ norm. Moreover, $\|P_J\left(f\right) - f\|_{\infty} \leq c2^{-J\alpha}$, for some constant c depending only of f. (**Proof.** [19].)

Hint 1. [19]. Using the multi-resolution analysis definition 1, we can write above equality in the following form.

$$f(x) \cong P_J f\left(x\right) = \sum_{k=1}^{2^{(J+2)}} c_{-J,k} \phi_{-J,k}\left(x\right).$$

In this paper because of we want coefficients matrix dimension in wavelet spaces be finite, so we assume our wavelets have compact support, dimension of V_j space be finite and $j = J \in N$. Generally, dimension of V_J space and coefficients matrices dimension depend on basis criteria of $\{\phi_{-J,k}\}$ (or $\{\psi_{-J,k}\}$) and their number. Here, we consider $dimV_J = 2^J$. Later, we will see that $dimV_J$ equals to 2^{J+2} in trigonometric wavelet space.

3. Method of solution:

Here, we offer a method to solve (1). Inner product of two integral function of f and g on $[0, 2\pi]$ is defined as $\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f\left(x\right) g\left(x\right) dx$.

Any square integrable function $f^{(n)}(x)$ can be written as a finite linear combination of scaling function basis as follows

$$f^{(n)}(x) = \sum_{k=1}^{2^J} c_{-J,k}\phi_{-J,k}(x) \tag{2}$$

With n times integrations of the both sides of the equation(2), we have :

$$\underbrace{\int_0^x \int_0^x ... \int_0^x}_{n} f^{(n)}(t)\, dtdt...dt = \sum_{k=1}^{2^J} c_{-J,k} \underbrace{\int_0^x \int_0^x ... \int_0^x}_{n} \phi_{-J,k}(t)\, dtdt...dt. \tag{3}$$

Now, we simply the right side of equation (3) using following formula

$$\underbrace{\int_0^x \int_0^x ... \int_0^x}_{\gamma} A(x)\, dtdt...dt = \frac{1}{(\gamma-1)!}\int_0^x (x-t)^{\gamma-1} A(t)\, dt,$$

and the left side using the fact that the n^{th}-order derivative of the function $f(x)$ is integrated n times. So we obtain the numerical solution as following .

$$f_J^{num}(x) = f_J^*(x) = \sum_{k=1}^{2^J} c_{-J,k} \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\phi_{-J,k}(t)\, dt + \sum_{i=0}^{n-1} \frac{x^i}{i!} f^{(i)}(0). \; (J \in N) \tag{4}$$

Our goal is calculating coefficients of $c_{-J,k}$, then by substituting $c_{-J,k}$ in (4), we obtain numerical solution $f_J^*(x)$.

Now we obtain $f^{(m)}(x)$, so we integrates n-m times of the both sides of the equation (2) and similar to the before process, we find $f^{(m)}(x)$ that would be as follows.

$$f^{(m)}(x) = \sum_{k=1}^{2^J} c_{-J,k} \int_0^x \frac{(x-t)^{n-m-1}}{(n-m-1)!}\phi_{-J,k}(t)\, dt + \sum_{i=0}^{n-m-1} \frac{x^i}{i!} f^{(i)}(0). \tag{5}$$

With substituting (2) and (5) in (1) we have

$$\sum_{k=1}^{2^J} c_{-J,k}\left(\phi_{-J,k}(x) - \int_0^{2\pi} K(x,t)\left(\int_0^t \frac{(t-r)^{n-m-1}}{(n-m-1)!}\phi_{-J,k}(r)\, dr\right) dt\right) =$$

$$g(x) + \sum_{i=0}^{n-m-1} \frac{f^{(i)}(0)}{i!} \int_0^{2\pi} K(x,t)\, t^i dt.$$

As a result, the residual function will be as follows:

$$R_n(x) = \sum_{k=1}^{2^J} c_{-J,k} \left(\phi_{-J,k}(x) - \int_0^{2\pi} K(x,t) \left(\int_0^t \frac{(t-r)^{n-m-1}}{(n-m-1)!} \phi_{-J,k}(r)\, dr \right) dt \right)$$
$$-g(x) - \sum_{i=0}^{n-m-1} \frac{f^{(i)}(0)}{i!} \int_0^{2\pi} K(x,t)\, t^i dt.$$

Now, if we apply the Galerkin method, we get

$$\langle R_n(x), \phi_p(x) \rangle = 0, \qquad p = 1, 2, ..., 2^J.\ And\ J \in N.$$

The following system of linear equations is obtained from the above relation:

$$\sum_{k=1}^{2^J} c_{-J,k} \left\langle \phi_{-J,k}(x) - \int_0^{2\pi} K(x,t) \left(\int_0^t \frac{(t-r)^{n-m-1}}{(n-m-1)!} \phi_{-J,k}(r)\, dr \right) dt, \phi_{-J,p}(x) \right\rangle =$$
$$\left\langle g(x) + \sum_{i=0}^{n-m-1} \frac{f^{(i)}(0)}{i!} \int_0^{2\pi} K(x,t)\, t^i dt, \phi_{-J,p}(x) \right\rangle, \qquad p = 1, 2, ..., 2^J. \quad (6)$$

Suppose $[a_{p,k}][c_{-J,k}] = [b_p]$ be matrix form of the linear system of equation (6), $a_{p,k}$, c_k, and b_p are the elements of coefficient matrix, elements of unknowns, and the elements of known values. Obviously, $a_{p,k}$ and b_p are obtained from the linear system (6) as (7) and (8), and the wavelet basis $\phi_{-J,k}(x)$ and $\phi_{-J,p}(x)$ are known functions considering the wavelet basis type selected.

$$a_{p,k} = \int_0^{2\pi} \phi_{-J,k}(x)\, \phi_{-J,p}(x)\, dx -$$
$$\int_0^{2\pi} \int_0^{2\pi} K(x,t)\, \phi_{-J,p}(x) \left(\int_0^t \frac{(t-r)^{n-m-1}}{(n-m-1)!} \phi_{-J,k}(r)\, dr \right) dxdt. \quad (7)$$

That $p, k = 1, 2, ..., 2^J$. And also,

$$b_p = \int_0^{2\pi} g(x)\, \phi_{-J,p}(x)\, dx + \sum_{i=0}^{n-m-1} \frac{f^{(i)}(0)}{i!} \int_0^{2\pi} \int_0^{2\pi} K(x,t)\, t^i \phi_{-J,p}(x)\, dtdx. \quad (8)$$

That $p = 1, 2, ..., 2^J$.

Eventually, the values of the unknown elements $c_{J,k}$ are obtained using the matrix equation $[c_{-J,k}] = [a_{p,k}]^{-1}[b_p]$, then by substituting $c_{-J,k}$ in (4), we obtain numerical solution $f_J^*(x)$. Dimension of matrices $[a_{p,k}]$, $[c_{-J,k}]$, $[b_p]$ are in order $2^J \times 2^J$, $2^J \times 1$, $2^J \times 1$. $(J \in N)$

4. Algorithm of the method

Here, an appropriate algorithm for solving equation (1) is offered.

1) Let $\{\phi_{-J,k}(x)\}_{k=1}^{2^J}$ is the scaling functions basis and $\phi_{-J,k}(x) \in C^n[0,2\pi]$.

A $= [a_{p,k}]$ that $a_{p,k} = \int_0^{2\pi} \phi_{-J,k}(x)\,\phi_{-J,p}(x)\,dx -$ $\int_0^{2\pi}\int_0^{2\pi} K(x,t)\,\phi_{-J,p}(x)\left(\int_0^t \frac{(t-r)^{n-m-1}}{(n-m-1)!}\phi_{-J,k}(r)\,dr\right)dxdt.$

B $= [b_p]$ that $b_p = \int_0^{2\pi} g(x)\,\phi_{-J,p}(x)\,dx + \sum_{i=0}^{n-m-1}\frac{f^{(i)}(0)}{i!}\int_0^{2\pi}\int_0^{2\pi} K(x,t)\,t^i\phi_{-J,p}(x)\,dtdx.$

$f_J^*(x) = \sum_{k=1}^{2^J} c_{-J,k}\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\phi_{-J,k}(t)\,dt + \sum_{i=0}^{n-1}\frac{x^i}{i!}f^{(i)}(0)$. $(J \in N)$ be the numerical solution.

$C = [c_{-J,k}]$ that $c_{-J,k}$ are unknown.

2) If $s \geq n$, $(s = 0,1,2,...,m.)$ the solution is not discussed.

3) If s < n By solving the system of C = A^{-1}B, the unknown coefficients $c_{-J,k}$ are obtained and go to 4 phase.

4) Substitute $\{c_{-J,k}\}_{k=1}^{2^J}$ in $f_J^*(x) = \sum_{k=1}^{2^J} c_{-J,k}\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\phi_{-J,k}(t)\,dt + \sum_{i=0}^{n-1}\frac{x^i}{i!}f^{(i)}(0)$. $(J \in N)$.

5. Convergence Analysis

We use orthogonal projection operator $P_J(f)$ to provide the convergence analysis that $J \in N$ and known. Assume that matrix $[a]_{p\times k}$ is invertible, $A_J^{-1} = [a_{p,k}]^{-1}$, $c_{-J,k}$ is exact coefficient, $c_{-J,k}^*$ denotes the numerical approximation to the wavelet coefficients $c_{-J,k}$, $P_J(f)(x) = \sum_{k=1}^{2^J} c_{-J,k}\phi_{-J,k}(x)$ is exact solution projection operator, let $P_J^*(f)(x) = \sum_{k=1}^{2^J} c_{-J,k}^*\phi_{-J,k}(x)$ be a numerical approximation to $P_J(f)$. $P_J^{(n)}(f)$ and $P_J^{(s)}(f)$ are functions projection operators $f^{(n)}(x)$ and $f^{(s)}(x)$.

We proof if $J, J_1, J_2 \to \infty$ then $f(x) \to P_J^*(f)$ or $\|f(x) - P_J^*(f)\|_\infty = 0$.

$\|f(x) - P_J^*(f)\|_\infty \leq \|f(x) - P_J(f)\|_\infty + \|P_J(f) - P_J^*(f)\|_\infty$. According to the Theorem 5 we have $\|f(x) - P_J(f)\|_\infty \leq c2^{-J\alpha}$, for some constant c. Now, we find an upper bound for $\|P_J(f) - P_J^*(f)\|_\infty$. According to Hint 1, we have:

$$\|P_J(f) - P_J^*(f)\|_\infty \leq \left|\sum_{k=1}^{2^J}\left(c_{-J,k} - c_{-J,k}^*\right)\phi_{-J,k}(x)\right| \leq$$

$$\left\|c_{-J,k}-c^{*}_{-J,k}\right\|_{\infty}\sum_{k=1}^{2^{J}}\left|\phi_{-J,k}\left(x\right)\right|\leq$$

$$\left\|c_{-J,k}-c^{*}_{-J,k}\right\|_{\infty}2^{\frac{-J}{2}}\sum_{k=1}^{2^{J}}\left|\phi\left(2^{-J}x-k\right)\right|\leq\left\|c_{-J,k}-c^{*}_{-J,k}\right\|_{\infty}2^{\frac{-J}{2}}M_{1}. \qquad (9)$$

Now, we find a upper bound for $\left\|c_{-J,k}-c^{*}_{-J,k}\right\|_{\infty}$. According to (1) we had,

$$f^{(n)}\left(x\right)=g\left(x\right)+\int_{0}^{2\pi}K\left(x,t\right)f^{(s)}\left(t\right)dt.$$

So

$$P_{J}^{(n)}\left(f\right)-P_{J}^{(n)}\left(f\right)+f^{(n)}\left(x\right)=\int_{0}^{2\pi}K\left(x,t\right)\left[f^{(s)}\left(t\right)-P_{J}^{(s)}\left(f\right)+P_{J}^{(s)}\left(f\right)\right]dt+g\left(x\right).$$

Or

$$P_{J}^{(n)}\left(f\right)\left(x\right)=\int_{0}^{2\pi}K\left(x,t\right)P_{J}^{(s)}\left(f\right)\left(t\right)dt+$$

$$\underbrace{g\left(x\right)+P_{J}^{(n)}\left(f\right)\left(x\right)-f^{(n)}\left(x\right)+\int_{0}^{2\pi}K\left(x,t\right)\left(f^{(s)}\left(t\right)-P_{J}^{(s)}\left(f\right)\left(t\right)\right)dt}_{\hat{g}(x)} \qquad (10)$$

If $\hat{g}\left(x\right)=g\left(x\right)+P_{J}^{(n)}\left(f\right)\left(x\right)-f^{(n)}\left(x\right)+\int_{0}^{2\pi}K\left(x,t\right)\left(f^{(s)}\left(t\right)-P_{J}^{(s)}\left(f\right)\left(t\right)\right)dt.$ (11)
From (10) we can write,

$$P_{J}^{(n)}\left(f\right)\left(x\right)=\int_{0}^{2\pi}K\left(x,t\right)P_{J}^{(s)}\left(f\right)\left(t\right)dt+\hat{g}\left(x\right). \qquad x\in[0,2\pi] \quad (12)$$

And similar (4.8) from [19] we can write

$$P_{J}^{(n*)}\left(f\right)\left(x\right)=\int_{0}^{2\pi}K\left(x,t\right)P_{J}^{(m*)}\left(f\right)\left(t\right)dt+\mathrm{g}\left(x\right). \qquad x\in[0,2\pi] \quad (13)$$

Although (4.8) is written for a limited number of points, but (4.8) is true for every $x\in[0,2\pi]$.

By solving the integro-differential equations (12) and (13) according to the method in Section 3 ($[c_k]=[a_{p,k}]^{-1}[b_p]$) and using (8), we have,

$$c_{-J,k}=A_{J}^{-1}\left[\int_{0}^{2\pi}\hat{g}\left(x\right)\phi_{-J,p}\left(x\right)dx+\sum_{i=0}^{n-m-1}\frac{f^{(i)}(0)}{i!}\int_{0}^{2\pi}\int_{0}^{2\pi}K\left(x,t\right)t^{i}\phi_{-J,p}\left(x\right)dtdx.\right]_{p\times1}$$

$c^*_{-J,k} = A_J^{-1}\left[\int_0^{2\pi} g\left(x\right)\phi_{-J,p}\left(x\right)dx + \sum_{i=0}^{n-m-1}\frac{f^{(i)}(0)}{i!}\int_0^{2\pi}\int_0^{2\pi} K\left(x,t\right)t^i\phi_{-J,p}\left(x\right)dtdx\right]_{p\times 1}$.

Thus

$$\sup\left|c_{-J,k} - c^*_{-J,k}\right| \leq \left\|A_J^{-1}\right\|_\infty \sup\left|\int_0^{2\pi}\left(g\left(x\right) - \hat{g}\left(x\right)\right)\phi_{-J,p}\left(x\right)dx\right| \leq$$

$$2\pi\left\|A_J^{-1}\right\|_\infty \sup\left|g\left(x\right) - \hat{g}\left(x\right)\right| \sup\left|\phi_{-J,p}\left(x\right)\right| \leq 2\pi M_2\left\|A_J^{-1}\right\|_\infty \sup\left|g\left(x\right) - \hat{g}\left(x\right)\right| \quad (14)$$

Now we find a bound for $\sup\left|g\left(x\right) - \hat{g}\left(x\right)\right|$. Equation (11) was the following form.

$$\hat{g}\left(x\right) = g\left(x\right) + P_J^{(n)}\left(f\right)\left(x\right) - f^{(n)}\left(x\right) + \int_0^{2\pi} K\left(x,t\right)\left(f^{(s)}\left(t\right) - P_J^{(s)}\left(f\right)\left(t\right)\right)dt.$$

So

$$g\left(x\right) - \hat{g}\left(x\right) = f^{(n)}\left(x\right) - P_J^{(n)}\left(f\right)\left(x\right) + \int_0^{2\pi} K\left(x,t\right)\left(P_J^{(s)}\left(f\right)\left(t\right) - f^{(s)}\left(t\right)\right)dt.$$

Thus,

$$\sup\left|g\left(x\right) - \hat{g}\left(x\right)\right| \leq \sup\left|f^{(n)}\left(x\right) - P_J^{(n)}\left(f\right)\left(x\right)\right| +$$

$$\sup\int_0^{2\pi}\left|K\left(x,t\right)\right|\left|f^{(s)}\left(t\right) - P_J^{(s)}\left(f\right)\left(t\right)\right|dt \leq$$

$$\sup|f^{(n)}\left(x\right) - P_J^{(n)}\left(f\right)\left(x\right)| + 2\pi\sup|K(s,t)||f^{(s)}\left(t\right) - P_J^{(s)}\left(f\right)\left(t\right)|. \quad (15)$$

According Theorem 5 when $J \longrightarrow \infty$, $\left\|f\left(x\right) - P_J\left(f\right)\left(x\right)\right\|_\infty = 0$, or $P_J\left(f\right)\left(x\right) = f\left(x\right)$, thus $\left\|f^{(n)}\left(x\right) - P_J^{(n)}\left(f\right)\left(x\right)\right\|_\infty = \left\|f^{(n)}\left(x\right) - f^{(n)}\left(x\right)\right\|_\infty = 0$, and also $\left\|f^{(s)}\left(x\right) - P_J^{(s)}\left(f\right)\left(x\right)\right\|_\infty = \left\|f^{(s)}\left(x\right) - f^{(s)}\left(x\right)\right\|_\infty = 0$. Thus there are J_1 and J_2 that

$\sup|f^{(n)}\left(x\right) - P_J^{(n)}\left(f\right)\left(x\right)| + 2\pi\sup|K(s,t)||f^{(s)}\left(t\right) - P_J^{(s)}\left(f\right)\left(t\right)| \leq$

$c_1 2^{-J_1} + 2\pi M c_2 2^{-J_2}$, for some constant c_1 and c_2. (16)

we can obtain a bound using relations (16), (14) and (9) for $\left\|P_J\left(f\right) - P_J^*\left(f\right)\right\|_\infty$ as following

$\left\|P_J\left(f\right) - P_J^*\left(f\right)\right\|_\infty \leq 2^{(\frac{-J}{2}+1)}\pi M_1 M_2\|A^{-1}\|_\infty(c_1 2^{-J_1} + 2\pi M c_2 2^{-J_2})$, for some constant c_1 and c_2. Thus

$\|f(x) - P_J^*(f)\|_\infty \le \|f(x) - P_J(f)\|_\infty + \|P_J(f) - P_J^*(f)\|_\infty \le$
$c2^{-J\alpha} + 2^{(\frac{-J}{2}+1)}\pi M_1 M_2 \|A_J^{-1}\|_\infty (c_1 2^{-J_1} + 2\pi M c_2 2^{-J_2})$. For some constant c, c_1, c_2, and the convergence analysis is complete.

(The second argument for (16) inequality: $f^{(n)}(x)$ and $P_J^{(n)}(f)(x)$ satisfy in condition of Theorem 4, thus there is J' that $\left\|f^{(n)}(x) - P_J^{(n)}(f)(x)\right\|_\infty \le c2^{-J'}$).

6. Numerical examples

In this section we present some numerical examples to illustrate the efficiency of method. In all of the examples, trigonometric scaling functions have been used as wavelet basis. Hence, prior to presenting the examples, explanations are provided on trigonometric wavelets from [10]. In order to analyze the error of the method, we use the following relations in Examples 1 and 2.

$\|E_J\|_\infty = \|f(x_j) - f_J^*(x_j)\|_\infty = sup\{|f(x_j) - f_J^*(x_j)| : \ x_j = \frac{j\pi}{2000},\ j = 0, 1, 2, ..., 1000\}$.

Absolute error$=|f(x_j) - f_J^*(x_j)| : \ x_j = \pi/20, 3\pi/20, \pi/4, 7\pi/20, 9\pi/20$.

To start talking about trigonometric wavelets, we introduce Dirichlet kernel and conjugate Dirichlet kernel

$$D_\ell(x) = \frac{1}{2} + \sum_{k=1}^{\ell} \cos(kx) = \begin{cases} \frac{\sin(\ell+1/2)x}{2\sin(x/2)}, & x \notin 2\pi Z \\ \ell + \frac{1}{2}, & x \in 2\pi Z. \end{cases}$$

$$\widetilde{D}(x) = \sum_{k=1}^{\ell} \sin(kx) = \begin{cases} \frac{\cos(x/2) - \cos(\ell+1/2)x}{2\sin(x/2)}, & x \notin 2\pi Z \\ 0, & x \in 2\pi Z. \end{cases}$$

Trigonometric wavelet scaling functions $\phi^0_{-J,n}(x)$, $\phi^1_{-J,n}(x)$ and their details are defined as follows:

$$\phi^0_{-J,0}(x) = \frac{1}{2^{2J+1}} \sum_{\ell=0}^{2^{J+1}-1} D_\ell(x).$$

$$\phi^1_{-J,0}(x) = \frac{1}{2^{2J+1}} (\tilde{D}_{2^{J+1}-1}(x) + \frac{1}{2}\sin(2^{J+1}x)).$$

Lemma 5.1 For any $J \in N$, we have

$$\phi_{J,0}^{0}(x) = \begin{cases} \frac{1}{2^{2J+2}} \frac{\sin^2(2^J x)}{\sin^2(\frac{x}{2})} & ,\mathrm{x} \notin 2\pi\mathrm{Z} \\ 1 & ,\mathrm{x} \in 2\pi\mathrm{Z} \end{cases}$$

$$\phi_{J,0}^{1}(x) = \begin{cases} \frac{1}{2^{2J+2}} \left(1 - \cos\left(2^{J+1}x\right)\right) \cot\frac{x}{2} & ,\mathrm{x} \notin 2\pi\mathrm{Z} \\ 0 & ,\mathrm{x} \in 2\pi\mathrm{Z} \end{cases}$$

(**Proof.** [10].)

And also

$$\phi^{0}_{-J,n}(x) = \phi^{0}_{-J,0}(x - x_{-J,n}),\ \phi^{1}_{-J,n}(x) = \phi^{1}_{-J,0}(x - x_{-J,n}).$$

Where $x_{-J,n} = \frac{n\pi}{2^J}$, $n = 1, 2, \ldots, 2^{J-1}$. And $J \in N$.

For $J \in N$, the spaces V_J are defined by

$$V_J = span\left\{1, cosx, ..., cos(2^{J+1} - 1)x, sinx, ..., sin2^{J+1}x\right\}.$$

Thus $dimV_J = 2^{(J+2)}$, and coefficients matrix dimension of linear system (6) in trigonometric wavelet space is $2^{J+2} \times 2^{J+2}$. [10].

Consequently, $[c_{-J,k}]_{2^{J+2}\times 1} = [a_{p,k}]^{-1}_{2^{J+2}\times 2^{J+2}}[b_p]_{2^{J+2}\times 1}$.

Example 1. We consider integro-differential equation as follow

$$\begin{cases} f^{(10)}(x) = cosx - (1 - \frac{\pi^2}{4})sinx + 2x^3 + (\pi - 4) + \int_0^{\pi/2}(x^3 + t^2 sinx) f^{(7)}(t)\, dt. \\ f^{(i)}(0) = \begin{cases} 1, & i = 1, 2, 5, 6, 9. \\ -1, & i = 0, 3, 4, 7, 8. \end{cases} \end{cases}$$

The exact solution is $f(x) = sinx - cosx$. The results are shown in Fig 1 and Table 1.

Example 2. We consider integro-differential equation as follow

$$\begin{cases} f^{(7)}(x) = g(x) + \int_0^{\pi/2} (x^2 t^3 - e^{2t}) f^{(4)}(t)\, dt. \\ f^{(i)}(0) = \begin{cases} 0, & i = 1, 5. \\ -1, & i = 0, 2, 4, 6. \\ -2, & i = 3. \end{cases} \end{cases}$$

That

$g(x) = \frac{8}{15} + \frac{2}{5}e^{\pi} - \frac{1}{3}e^{\frac{3\pi}{2}} - e^x + \frac{1}{8}(96 - 6\pi^2 + e^{\frac{\pi}{2}}(-48 + 24\pi - 6\pi^2 + \pi^3))x^2 - cosx.$
The exact solution is $f(x) = sinx - e^x$. The results are shown in Fig 2 and Table 2.

Table 1: Absolute errors and $\|E_J\|_\infty$ for numerical solution of Example 1.

x	$J=1$	$J=2$	$J=3$	$J=10$	$J=15$
$\pi/20$	7.32E-1	7.43E-2	7.72E-3	8.01E-10	3.12E-15
$3\pi/20$	2.45E-1	3.92E-2	2.61E-3	2.70E-10	2.31E-15
$\pi/4$	3.12E-2	1.21E-3	2.21E-4	2.93E-11	2.51E-16
$7\pi/20$	3.79E-1	2.94E-2	2.95E-3	7.22E-10	8.93E-15
$9\pi/20$	8.21E-1	7.92E-2	9.12E-3	1.71E-9	3.23E-14
$\|f(x_j) - f_J^*(x_j)\|_\infty$	9.68E-1	9.76E-2	2.28E-2	7.17E-9	1.57E-14

Table 2: Absolute errors and $\|E_J\|_\infty$ for numerical solution of Example 2.

x	$J=1$	$J=2$	$J=3$	$J=10$	$J=15$
$\pi/20$	2.40E-1	9.90E-2	2.27E-3	4.96E-10	7.96E-16
$3\pi/20$	9.17E-2	6.07E-2	8.72E-4	2.95E-10	2.08E-16
$\pi/4$	1.03E-1	7.18E-3	1.24E-3	1.94E-11	8.11E-17
$7\pi/20$	3.42E-1	6.14E-2	4.25E-3	3.24E-10	4.68E-16
$9\pi/20$	6.28E-1	1.46E-1	8.45E-3	7.27E-10	8.73E-16
$\|f(x_j) - f_J^*(x_j)\|_\infty$	7.14E-1	2.59E-1	1.32E-2	9.34E-10	1.04E-15

7. Conclusion

In this paper, we reduce integro-differential equation of an arbitrary order to a linear system of equations using the Galerkin method by an arbitrary wavelet basis. For the convergence analysis, we proof that the numerical solution of integro-differential equation converge to the exact solution. Figs. 1, 2, shown the numerical solution convergence to exact solution.

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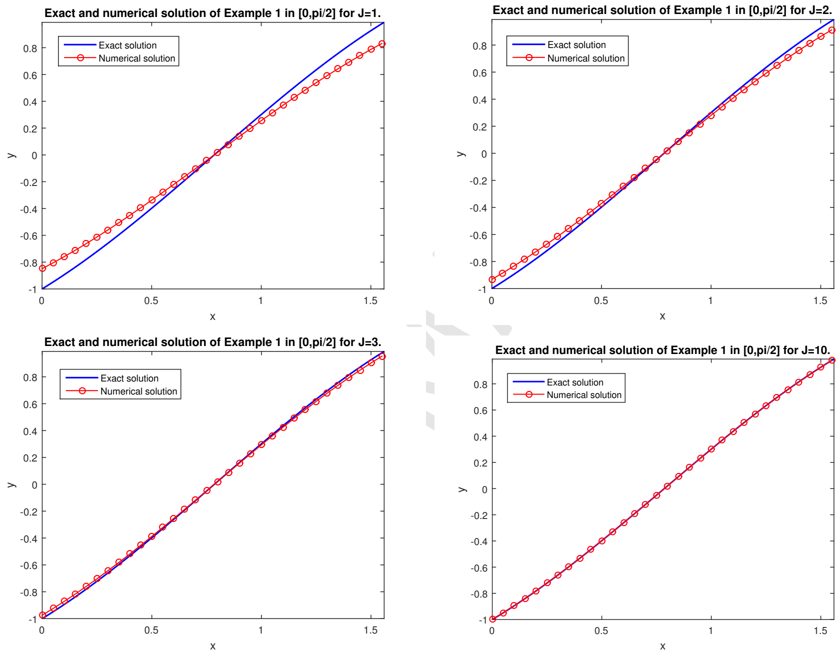

Figure 1: Plots of the exact and numerical solutions for Example 1.

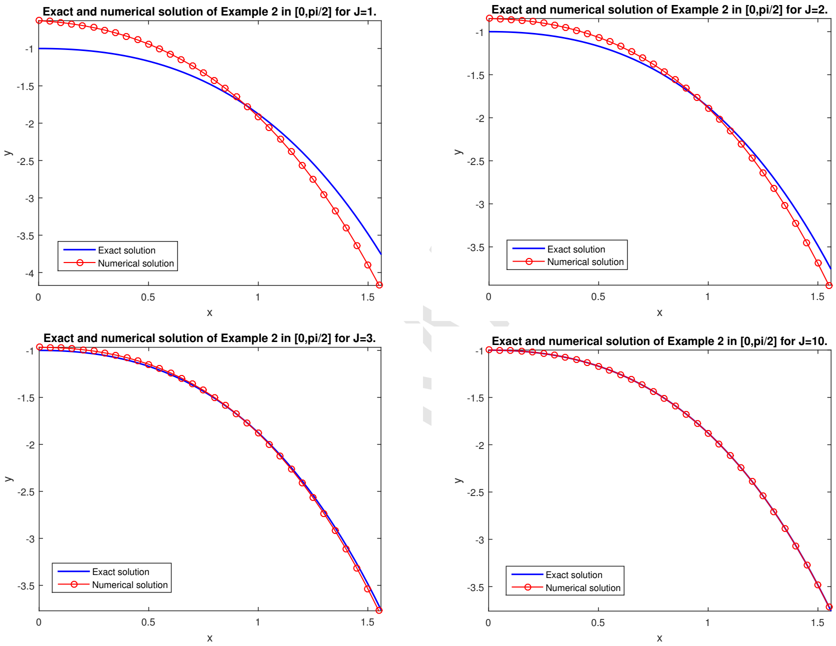

Figure 2: Plots of the exact and numerical solutions for Example 2.