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High dimensional finite element method for multiscale nonlinear monotone parabolic equations

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Abstract

We develop in this paper a finite element (FE) method for solving nonlinear monotone parabolic equations in a domain $D \subset \mathbb{R}^d$ that depends on n separable microscopic scales. The method employs an essentially optimal number of degrees of freedom. For nonlinear multiscale equations, it is not possible to form the homogenized equation explicitly numerically. The method solves the multiscale homogenized equation which is obtained from multiscale convergence. This equation contains all the necessary information: the solution to the homogenized equation which approximates the solution to the multiscale equation macroscopically, and the scale interacting terms which provide the microscopic information. However, it is posed in a high dimensional tensorized domain. We develop the sparse tensor product FE method for this equation that uses an essentially optimal number of degrees of freedom to obtain an approximation for the solution of this equation within a prescribed accuracy. We then construct numerical correctors from the FE solution. In the two scale case, we derive a new homogenization error from which an explicit error for the numerical corrector is established: it is the sum of the FE error and the homogenization error. Numerical examples illustrate the theoretical results.

1 Introduction

We develop in this paper an essentially optimal numerical method for solving locally periodic multiscale monotone parabolic equations in a domain $D \subset \mathbb{R}^d$. As for other multiscale problems, a direct numerical method that uses fine mesh to capture the microscopic scales is prohibitively expensive. The homogenization approach ([8, 7, 29]) establishes an equivalent problem that approximates the solution of the multiscale equation in the average sense. This homogenized equation is obtained in the limit when all the microscopic scales tend to zero. For linear problems, the homogenized equation can be obtained from the solution of a number of cell problems. When the equation is only locally periodic, the cost of forming the homogenized coefficient can be expensive as cell problems need to be solved for each macroscopic point. For nonlinear problems, it is virtually impossible to form the homogenized equation numerically as for each vector in $\mathbb{R}^d$ a nonlinear cell problem has to be solved ([17, 25]). There have been several attempts to solve multiscale nonlinear problems without forming the homogenized equation. Efendiev and Pankov ([21, 22]) employ the framework of the Multiscale Finite Element method (MsFEM) (see [28, 20]) to solve time independent and time dependent multiscale monotone equations. Generalized Multiscale Finite Element method is employed in [13]. Abdulle et al. [4], Abdule and Huber [2, 3] employ the Heterogeneous Multiscale method (HMM) (see [19], [1]) to solve time independent and time dependent monotone problems. An aposterior error estimate for the HMM method for multiscale monotone problems is studied in Henning and Ohlberger [24]. For locally periodic multiscale monotone problems, Hoang [25] develops the sparse tensor product FE method for solving the multiscale homogenized equation. The major advantage of this method, in comparison to the above mentioned papers, is that for locally periodic problems, it requires an essentially optimal number of degrees of freedom to achieve a prescribed level of accuracy. This method is originally developed for linear multiscale elliptic equations in Hoang and Schwab [26] (see also Harbretch and Schwab [23]). It solves the multiscale homogenized equation which contains the solution of the homogenized equation which approximates the solution of the multiscale problem macroscopically, and the scale interacting terms that contain the microscopic information. This equation is posed in a high dimensional tensorized space. A full tensor product FE approach is highly expensive. The sparse tensor product FE solves this equation with an equivalent level of accuracy, but requires only an essentially optimal number of degrees of freedom. The sparse tensor product FE approach has been used successfully for other equations, e.g. multiscale wave equation [33], multiscale elasticity equation [34], multiscale elastic wave equation [35] and Maxwell equation [12].

In this paper, we develop the sparse tensor product FE method for monotone parabolic problems. In Section 2, we set up the multiscale monotone parabolic problem that depends on n separated microscopic scales. We then introduce the multiscale homogenized equation which contains all the macroscopic and microscopic information. The equation is derived from multiscale convergence (see Nguetseng [31], Allaire [5] and Allaire and Briane [6]). We consider FE approximation for this high dimensional multiscale homogenized equation in Section 3. First we develop the backward Euler method for general FE spaces. We then apply the full tensor product FEs and the sparse tensor product FEs for the backward Euler method. We show that the sparse tensor product FE method achieves an essentially equivalent level of accuracy as the full tensor product FE, but needs only an essentially optimal number of degrees of freedom, which is far less than that required by the full tensor product FEs. We then develop the Crank-Nicholson method for the multiscale homogenized equation for general FE spaces, and then use the full tensor and sparse tensor product FE spaces. Section 4 develops numerical correctors for the problem. For two scale problems, when the solution is sufficiently regular, we prove a new homogenized error estimate. From this we derive an error estimate for the numerical corrector in terms of the homogenization error and the FE error. For problems with more than two scales, such a homogenization error is not available. We construct a numerical corrector without an error estimate. Section 5 presents some numerical examples that illustrate the theoretical results.

Throughout the paper, by ∇ without explicitly indicating the variable, we denote the gradient with respect to x of a function that depends only on the spatial variable x and the temporal variable t, and by ∇_x we mean the partial gradient with respect to x of a function that depends on x and t and also other variables. Repeated indices indicate summation; # denotes spaces of periodic functions.

2 Multiscale monotone parabolic problems

2.1 Problem setting

Let $D \subset \mathbb{R}^d$ be a bounded domain; let $Y = (0,1)^d$ be the unit cube in $\mathbb{R}^d$. Let n be a positive integer. Let $Y_1, \dots, Y_n$ be n copies of the unit cube Y. For conciseness, we denote by $\boldsymbol{y}_i = (y_1, \dots, y_i)$ a vector in $\mathbf{Y}_i = Y_1 \times \dots \times Y_i$. We denote by $\boldsymbol{y} = \boldsymbol{y}_n$ and $\mathbf{Y} = \mathbf{Y}_n$. Let $A(t, x, y_1, \dots, y_n, \xi) : (0,T) \times D \times Y_1 \times \dots \times Y_n \times \mathbb{R}^d \to \mathbb{R}^d$ be a smooth function and Y_i periodic with respect to y_i. We assume that A is monotone and locally Lipschitz. In particular, we assume that there are constants $p \geq 2$, $\alpha > 0$ and $\beta > 0$ so that for all $t \in (0,T)$, $x \in D$, $y_i \in Y_i$ $(i = 1, \dots, n)$, and $\xi_1, \xi_2 \in \mathbb{R}^d$, we have

$$(A(t, x, y_1, \dots, y_n, \xi_1) - A(t, x, y_1, \dots, y_n, \xi_2), \xi_1 - \xi_2) \geq \alpha|\xi_1 - \xi_2|^p, \tag{2.1}$$

and

$$|A(t, x, y_1, \dots, y_n, \xi_1) - A(t, x, y_1, \dots, y_n, \xi_2)| \leq \beta(|\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2| \tag{2.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^d$. Let $\varepsilon_1, \dots, \varepsilon_n$ be n functions of a small and positive quantity ε that represent n microscopic scales on which the problem depends. We assume scale separation (see Bensoussan et al. [8]), i.e. for $i = 1, \dots, n-1$

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{i+1}(\varepsilon)}{\varepsilon_i(\varepsilon)} = 0.$$

Without loss of generality, we assume that $\varepsilon_1 = \varepsilon$. The multiscale monotone function is defined as

$$A^\varepsilon(t, x, \xi) = A(t, x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}, \xi).$$

Let $V = W_0^{1,p}(D)$. Let $H = L^2(D)$. We have that $V \subset H \subset V'$. By $\langle \cdot, \cdot \rangle_H$, we denote the inner product in H extended to the duality pairing between V' and V. Let $f \in L^q((0,T), V')$ where $1/p + 1/q = 1$. Let $g \in H$. Let $T > 0$. We consider the following multiscale monotone parabolic problem:

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} - \nabla \cdot (A^\varepsilon(t, x, \nabla u^\varepsilon)) &= f, \quad \text{in } D \times (0,T) \\ u^\varepsilon(0, x) &= g \end{aligned} \tag{2.3}$$

with the Dirichlet boundary condition on ∂D. Problem (2.3) has a unique solution that satisfies

$$\|u^\varepsilon\|_{L^p((0,T),V)} + \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^q((0,T),V')} \le c(\|f\|_{L^q((0,T),V')} + \|g\|_H)$$

where the constant c only depends on T, α and β in (2.1) and (2.2), and on $\sup_{t\in[0,T], x\in D, \boldsymbol{y}\in\mathbf{Y})} |A(t, x, \boldsymbol{y}, 0)|$ (see [30] and [36]). We will study homogenization of (2.3) by multiscale convergence. We thus recall the concept of multiscale convergence in the L^p setting.

2.2 Multiscale homogenization

Multiscale convergence was initiated by Nguetseng in [31] and developed further by Allaire [5] and Allaire and Briane [6]. The definition is extend to functions that depend on time as follows.

Definition 2.1 *A sequence $\{w^\varepsilon\}_\varepsilon \in L^p((0,T)\times D)$ $(n+1)$-scale converges to a function $w_0(t, x, y_1, \ldots, y_n) \in L^p((0,T) \times D \times Y_1 \times \ldots \times Y_n)$ if*

$$\lim_{\varepsilon\to 0} \int_0^T \int_D w^\varepsilon(t, x)\phi(t, x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n})dx = \int_0^T \int_D \int_{Y_1} \ldots \int_{Y_n} w_0(t, x, y_1, \ldots, y_n)\phi(t, x, y_1, \ldots, y_n)dy_n \ldots dy_1 dx dt,$$

for all functions $\phi \in C((0,T) \times D \times Y_1 \times \ldots \times Y_n)$ which are Y_i-periodic with respect to y_i.

Definition 2.1 makes sense due to the following proposition.

Proposition 2.2 *From each bounded sequence in $L^p((0,T) \times D)$, we can extract a subsequence that $(n+1)$-scale converges.*

The proposition for the case of time independent functions is proved in [5] and [6]. For time dependent functions, the proof is similar, see, e.g. [27]. We denote by

$$V_i = L^p(D \times Y_1 \times \ldots \times Y_{i-1}, W_\#^{1,p}(Y_i)/\mathbb{R}), \quad (i = 1, \ldots, n).$$

For a bounded sequence $\{w^\varepsilon\}_\varepsilon \subset L^p((0,T), V)$ such that $\frac{\partial w^\varepsilon}{\partial t}$ is bounded in $L^q((0,T), V')$, we have the following results.

Proposition 2.3 *From a bounded sequence $\{w^\varepsilon\} \subset L^p((0,T), V)$ with $\frac{\partial w^\varepsilon}{\partial t}$ being bounded in $L^q((0,T), V')$, we can extract a subsequence (not renumbered) such that ∇w^ε $(n+1)$-scale converges to*

$$\nabla w_0 + \sum_{i=1}^n \nabla_{y_i} w_i$$

where $w_0 \in L^p((0,T), V)$ and $w_i \in L^p((0,T), V_i)$ $(i = 1, \ldots, n)$.

The proof of this result is standard. It follows the standard result of [5] and [6]; see for example [27] or [32] We define the following space

$$\mathbf{V} = \{(\phi_0, \phi_1, \ldots, \phi_n) : \phi_0 \in V, \ \phi_i \in V_i\}$$

which is equipped with the norm

$$|||(\phi_0, \{\phi_i\})||| = \|\nabla\phi_0\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i}\phi_i\|_{L^p(D\times Y_1\times\ldots\times Y_i)}. \tag{2.4}$$

We have the norm equivalence:

Lemma 2.4 *There are positive constants c_1 and c_2 such that for all $(\phi_0, \{\phi_i\}) \in \mathbf{V}$,*

$$\begin{aligned} & c_1 |||(\phi_0, \{\phi_i\})||| \\ & \leq \left(\int_D \int_{Y_1} \dots \int_{Y_n} |\nabla_x \phi_0 + \nabla_{y_1}\phi_1 + \ldots + \nabla_{y_n}\phi_n|^p dx dy_1 \ldots dy_n \right)^{1/p} \\ & \leq c_2 |||(\phi_0, \{\phi_i\}|||. \end{aligned}$$

A concise proof can be found in [25]. We have the following result:

Theorem 2.5 *The solution u^ε of problem* (2.3) *converges weakly in $L^p((0,T),V)$ to a function u_0, and ∇u^ε $(n+1)$-scale converges to $\nabla u_0 + \nabla_{y_1} u_1 + \ldots + \nabla_{y_n} u_n$ where $(u_0, u_1, \ldots, u_n) \in \mathbf{V}$ satisfies the problem*

$$\begin{aligned} & \left\langle \frac{\partial u_0}{\partial t}(t), \phi_0 \right\rangle_H + \\ & \int_D \int_{\mathbf{Y}} A(t, x, \boldsymbol{y}, \nabla u_0(t) + \nabla_{y_1} u_1(t) + \ldots + \nabla_{y_n} u_n(t)) \cdot (\nabla \phi_0 + \nabla_{y_1}\phi_1 + \ldots + \nabla_{y_n}\phi_n) d\boldsymbol{y} dx \\ & = \int_D f(t)\phi_0 dx \end{aligned} \tag{2.5}$$

for all $(\phi_0, \phi_1, \ldots, \phi_n) \in \mathbf{V}$.

The proof of this result is quite standard. We refer to, e.g., Allaire [5] or Woukeng [32].

3 Finite element discretization

We approximate problem (2.5) by FEs in this section. We consider both the backward Euler method and the Crank-Nicholson method.

3.1 Backward Euler method

We first consider the backward Euler method for general FE spaces. We then restrict our consideration to the cases of full and sparse tensor product FEs.

3.1.1 Backward Euler method for general FE spaces

Let $V^L \subset V$ and $V_i^L \subset V_i$ $(i = 1, \ldots, n)$ be finite dimensional spaces where the superscript L indicates the level of resolution. Let M be an integer. Let $\Delta t = T/M$. We consider the time sequence $0 = t_0 < t_1 < \ldots < t_M$ where $t_m = m\Delta t$ for $m = 0, 1, \ldots, M$. Let $g^L \in V^L$ be an approximation of g. We consider the problem: Find $u_{0,m}^L \in V^L$ and $u_{i,m}^L \in V_i^L$ for $i = 1, \ldots, n$ such that

$$\begin{aligned} & \left\langle \frac{u_{0,m+1}^L - u_{0,m}^L}{\Delta t}, \phi_0 \right\rangle_H \\ & + \int_D \int_{\mathbf{Y}} A(t_{m+1}, x, \boldsymbol{y}, \nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L) \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i}\phi_i) d\boldsymbol{y} dx \\ & = \int_D f(t_{m+1}, x)\phi_0(x) dx \end{aligned} \tag{3.1}$$

for all $\phi_0 \in V^L$ and $\phi_i \in V_i^L$ $(i = 1, \ldots, n)$. We first show that (3.1) has a unique solution.

Proposition 3.1 *Problem* (3.1) *has a unique solution.*

Proof Let $\mathbf{c}_m = (\mathbf{c}_{0,m}, \{\mathbf{c}_{i,m}\})$ and $\mathbf{d} = (\mathbf{d}_0, \{\mathbf{d}_i\})$ $(i = 1, \ldots, n)$ in $\mathbb{R}^{\dim V^L} \times \mathbb{R}^{\dim V_1^L} \times \ldots \times \mathbb{R}^{\dim V_n^L}$ be the coordinate vectors of $(u_{0,m}^L, \{u_{i,m}^L\})$ and $(\phi_0, \{\phi_i\})$ respectively in the expansion with respect to the basis functions of $V^L \times V_1^L \times \ldots \times V_n^L$. Let $\mathcal{A}(\mathbf{c}_{m+1})$ be the vector describing the interaction of $A(t_{m+1}, x, \boldsymbol{y}, \nabla u_{0,m+1}^L + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}^L)$ with the basis functions of $V^L \times V_1^L \times \ldots \times V_n^L$ in the second term on the left hand side of (3.1). Let $\mathcal{B}$ be the Gramm matrix describing the interaction of basis functions of V^L with themselves in the inner product of H. Let F_{m+1} be the interaction of $f(t_{m+1})$ with the basis functions of V^L with respect to the inner product of H. We can write (3.1) as

$$\frac{1}{\Delta t}\mathcal{B}\mathbf{c}_{0,m+1}\cdot\mathbf{d}_0+\mathcal{A}(\mathbf{c}_{m+1})\cdot\mathbf{d}=F_{m+1}\cdot\mathbf{d}_0+\frac{1}{\Delta t}\mathcal{B}\mathbf{c}_{0,m}\cdot\mathbf{d}_0. \tag{3.2}$$

The left hand side of (3.2) represents a monotone function. Indeed, for any $(v_0,\{v_i\})$ and $(w_0,\{w_i\})$ in $V^L\times V_1^L\times\ldots\times V_n^L$,

$$\begin{aligned}
&\frac{1}{\Delta t}\langle v_0-w_0,v_0-w_0\rangle_H\\
&+\int_D\int_{\mathbf{Y}}(A(t_{m+1},x,\boldsymbol{y},\nabla v_0+\sum_{i=1}^n\nabla_{y_i}v_i)-A(t_{m+1},x,\boldsymbol{y},\nabla w_0+\sum_{i=1}^n\nabla_{y_i}w_i))\\
&\qquad\cdot(\nabla(v_0-w_0)+\sum_{i=1}^n\nabla_{y_i}(v_i-w_i))d\boldsymbol{y}dx\\
&\geq\frac{1}{\Delta t}\|v_0-w_0\|_H^2+\alpha\int_D\int_{\mathbf{Y}}|(\nabla v_0+\sum_{i=1}^n\nabla_{y_i}v_i)-(\nabla w_0+\sum_{i=1}^n\nabla_{y_i}w_i)|^p d\boldsymbol{y}dx
\end{aligned}$$

i.e. for any two vectors $\mathbf{p}=(\mathbf{p}_0,\{\mathbf{p}_i\})$ and $\mathbf{q}=(\mathbf{q}_0,\{\mathbf{q}_i\})$ in $\mathbb{R}^{\dim V^L}\times\mathbb{R}^{\dim V_1^L}\times\ldots\times\mathbb{R}^{\dim V_n^L}$,

$$\frac{1}{\Delta t}\mathcal{B}(\mathbf{p}_0-\mathbf{q}_0)\cdot(\mathbf{p}_0-\mathbf{q}_0)+\mathcal{A}(\mathbf{p}-\mathbf{q})\cdot(\mathbf{p}-\mathbf{q})\geq c(\Delta t)(|\mathbf{p}-\mathbf{q}|^2+|\mathbf{p}-\mathbf{q}|^p)$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{\dim V^L}\times\mathbb{R}^{\dim V_1^L}\times\ldots\times\mathbb{R}^{\dim V_n^L}$. Thus problem (3.2) has a unique solution $\mathbf{c}_{m+1}$. $\square$

We denote by $u_0(t_m)=u_{0,m}$, $u_i(t_m)=u_{i,m}$, $z_{0,m}^L=u_{0,m}-u_{0,m}^L$, $z_{i,m}^L=u_{i,m}-u_{i,m}^L$ $(i=1,\ldots,n)$. We then have the following result.

Theorem 3.2 *Assume that $u_0\in C^2([0,T],H)\cap C([0,T],W^{1,p}(D))$ and $u_i\in C([0,T],V_i)$, then*

$$\begin{aligned}
&\|z_{0,M}^L\|_H^2+\Delta t\sum_{m=1}^M(\|z_{0,m}^L\|_V^p+\sum_{i=1}^n\|z_{i,m}^L\|_{V_i}^p)\\
&\leq c\Delta t\left(\sum_{m=1}^M\left(\|u_{0,m}-\tilde u_{0,m}\|_V^{p/(p-1)}+\sum_{i=1}^n\|u_{i,m}-\tilde u_{i,m}\|_{V_i}^{p/(p-1)}\right.\right.\\
&\qquad\left.+\|u_{0,m}-\tilde u_{0,m}\|_V^p+\sum_{i=1}^n\|u_{i,m}-\tilde u_{i,m}\|_{V_i}^p\right)\\
&\quad\left.+\sum_{m=1}^{M-1}\left\|\frac{(u_{0,m+1}-\tilde u_{0,m+1})-(u_{0,m}-\tilde u_{0,m})}{\Delta t}\right\|_H^2\right)\\
&\quad+\max_{m=1,\ldots,M}\|u_{0,m}-\tilde u_{0,m}\|_H^2+\|g-g^L\|_H^2+c(\Delta t)^{p/p-1}.
\end{aligned} \tag{3.3}$$

for all sequences $\{\tilde u_{0,m},\ m=1,\ldots,M\}\subset V^L$ and $\{\tilde u_{i,m},\ m=1,\ldots,M\}\subset V_i^L$ for $i=1,\ldots,n$.

Proof We denote by $\rho_m=\frac{\partial u_0}{\partial t}(t_{m+1})-(u_0(t_{m+1})-u_0(t_m))/\Delta t$. As $u_0\in C^2([0,T],H)$ we have that $\|\rho_m\|_H\leq c\Delta t$ for all $m=1,\ldots,M$, where c is independent of m. We then have from (2.5) and (3.1) that

$$\begin{aligned}
&\left\langle\frac{z_{0,m+1}^L-z_{0,m}^L}{\Delta t},\phi_0\right\rangle_H+\langle\rho_m,\phi_0\rangle_H\\
&+\int_D\int_{\mathbf{Y}}(A(t_{m+1},x,\boldsymbol{y},\nabla u_{0,m+1}+\sum_{i=1}^n\nabla_{y_i}u_{i,m+1})-A(t_{m+1},x,\boldsymbol{y},\nabla u_{0,m+1}^L+\sum_{i=1}^n\nabla_{y_i}u_{i,m+1}^L))\\
&\qquad\cdot(\nabla\phi_0+\sum_{i=1}^n\nabla_{y_i}\phi_i)d\boldsymbol{y}dx=0
\end{aligned} \tag{3.4}$$

for all $\phi_0 \in V^L$ and $\phi_i \in V_i^L$, $i = 1, \dots, n$. Therefore, for all $\{\tilde{u}_{0,m}\} \subset V^L$ and $\{\tilde{u}_{i,m}\} \subset V_i^L$, we have

$$
\begin{aligned}
& \left\langle \frac{z^L_{0,m+1} - z^L_{0,m}}{\Delta t}, z^L_{0,m+1} \right\rangle_H \\
& + \int_D \int_{\mathbf{Y}} (A(t_{m+1}, x, \boldsymbol{y}, \nabla u_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}) - A(t_{m+1}, x, \boldsymbol{y}, \nabla u^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u^L_{i,m+1})) \\
& \qquad \cdot (\nabla z^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} z^L_{i,m+1}) d\boldsymbol{y} dx \\
= & \left\langle \frac{z^L_{0,m+1} - z^L_{0,m}}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\
& + \int_D \int_{\mathbf{Y}} (A(t_{m+1}, x, \boldsymbol{y}, \nabla u_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}) - A(t_{m+1}, x, \boldsymbol{y}, \nabla u^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u^L_{i,m+1})) \\
& \qquad \cdot ((\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}) + \sum_{i=1}^n (\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1})) d\boldsymbol{y} dx \\
& + \langle \rho_m, (u^L_{0,m+1} - \tilde{u}_{0,m+1}) \rangle_H.
\end{aligned}
$$

Using $u^L_{0,m+1} - \tilde{u}_{0,m+1} = u^L_{0,m+1} - u_{0,m+1} + u_{0,m+1} - \tilde{u}_{0,m+1}$, from (2.2) we have

$$
\begin{aligned}
& \frac{1}{2\Delta t} (\|z^L_{0,m+1}\|^2_H - \|z^L_{0,m}\|^2_H) + c \|\nabla z^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} z^L_{i,m+1})\|^p_{L^p(D\times\mathbf{Y})} \\
\le & \left\langle \frac{z^L_{0,m+1} - z^L_{0,m}}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H \\
& + c (\|\nabla u_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}\|^{p-2}_{L^p(D\times\mathbf{Y})} + \|\nabla u^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u^L_{i,m+1}\|^{p-2}_{L^p(D\times\mathbf{Y})}) \\
& \qquad \cdot \|\nabla z^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} z^L_{i,m+1}\|_{L^p(D\times\mathbf{Y})} \\
& \qquad \cdot (\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D\times\mathbf{Y})}) \\
& + c\Delta t (\|u_{0,m+1} - \tilde{u}_{0,m+1}\|_H + \|z^L_{0,m+1}\|_H).
\end{aligned}
$$

From the hypothesis, $\|\nabla u_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} u_{i,m+1}\|_{L^p(D\times\mathbf{Y})}$ is uniformly bounded for all m. Thus

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|z^L_{0,m+1}\|^2_H - \|z^L_{0,m}\|^2_H) + c\|\nabla z^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} z^L_{i,m+1})\|^p_{L^p(D\times\mathbf{Y})}\\
\leq\ & \left\langle \frac{z^L_{0,m+1} - z^L_{0,m}}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H\\
&+c(\|\nabla z^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} z^L_{i,m+1}\|_{L^p(D\times\mathbf{Y})} + \|\nabla z^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} z^L_{i,m+1}\|^{p-1}_{L^p(D\times\mathbf{Y})})\\
&\quad\cdot(\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D\times\mathbf{Y})})\\
&+c\Delta t(\|u_{0,m+1} - \tilde{u}_{0,m+1}\|_H + \|z^L_{0,m+1}\|_H)\\
\leq\ & \left\langle \frac{z^L_{0,m+1} - z^L_{0,m}}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H\\
&+\delta\|\nabla z^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} z^L_{i,m+1})\|^p_{L^p(D\times\mathbf{Y})}\\
&\quad+c(\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D\times\mathbf{Y})})^q\\
&\quad+c(\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D\times\mathbf{Y})})^p\\
&+c(\Delta t)^q + \delta\|z^L_{0,m+1}\|^p_H + c\|u_{0,m+1} - \tilde{u}_{0,m+1}\|^p_H
\end{aligned}
$$

for a constant $\delta > 0$ where we have used the Young inequality; $q = p/(1-p)$. From Lemma 2.4, we have

$$
\|z^L_{0,m+1}\|^p_V + \sum_{i=1}^n \|z^L_{i,m+1}\|^p_{V_i} \leq c\|\nabla z^L_{0,m+1} + \sum_{i=1}^n \nabla_{y_i} z^L_{i,m+1}\|^p_{L^p(D\times\mathbf{Y})}.
$$

Thus for δ sufficiently small,

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|z^L_{0,m+1}\|^2_H - \|z^L_{0,m}\|^2_H) + c(\|z^L_{0,m+1}\|^p_V + \sum_{i=1}^n \|z^L_{i,m+1}\|^p_{V_i})\\
\leq\ & c\left\langle \frac{z^L_{0,m+1} - z^L_{0,m}}{\Delta t}, u_{0,m+1} - \tilde{u}_{0,m+1} \right\rangle_H\\
&+c(\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D\times\mathbf{Y})})^q\\
&+c(\|\nabla u_{0,m+1} - \nabla \tilde{u}_{0,m+1}\|_{L^p(D)} + \sum_{i=1}^n \|\nabla_{y_i} u_{i,m+1} - \nabla_{y_i} \tilde{u}_{i,m+1}\|_{L^p(D\times\mathbf{Y})})^p + c(\Delta t)^q.
\end{aligned}
$$

Therefore for any $P = 1, \dots, M$, we deduce

$$
\begin{aligned}
&\|z^L_{0,P}\|^2_H + c\Delta t \sum_{m=1}^{P} (\|z^L_{0,m}\|^p_V + \sum_{i=1}^{n} \|z^L_{i,m}\|^p_{V_i}) \\
\leq\ & c\Delta t \sum_{m=1}^{P} \Big(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|^q_{L^p(D)} + \sum_{i=1}^{n} \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|^q_{L^p(D\times\mathbf{Y})} \Big) \\
&+ c\Delta t \sum_{m=1}^{P} \Big(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|^p_{L^p(D)} + \sum_{i=1}^{n} \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|^p_{L^p(D\times\mathbf{Y})} \Big) \\
&+ c(\Delta t)^q + c\Delta t \sum_{m=1}^{P} \left\langle \frac{z^L_{0,m} - z^L_{0,m-1}}{\Delta t}, u_{0,m} - \tilde{u}_{0,m} \right\rangle_H + \|g - g^L\|^2_H.
\end{aligned}
$$

We have

$$
\begin{aligned}
& \Delta t \sum_{m=1}^{P} \left\langle \frac{z^L_{0,m} - z^L_{0,m-1}}{\Delta t}, u_{0,m} - \tilde{u}_{0,m} \right\rangle_H \\
=\ & -\langle z^L_{0,0}, u_{0,1} - \tilde{u}_{0,1}\rangle_H + \langle z^L_{0,P}, u_{0,P} - \tilde{u}_{0,P}\rangle + \sum_{m=1}^{P-1} \langle z^L_{0,m}, (u_{0,m} - \tilde{u}_{0,m}) - (u_{0,m+1} - \tilde{u}_{0,m+1})\rangle \\
\leq\ & c\|z^L_{0,0}\|^2_H + c\|u_{0,1} - \tilde{u}_{0,1}\|^2_H + \delta\|z^L_{0,P}\|^2_H + c\|u_{0,P} - \tilde{u}_{0,P}\|^2_H \\
& + \delta\Delta t \sum_{m=1}^{P-1} \|z^L_{0,m}\|^2_H + c\Delta t \sum_{m=1}^{P-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|^2_H.
\end{aligned}
$$

We have $\Delta t \sum_{m=1}^{P-1} \|z^L_{0,m}\|^2_H \leq \Delta t M \max_m \|z^L_{0,m}\|^2_H \leq T \max_m \|z^L_{0,m}\|^2_H$. Choosing δ sufficiently small, we deduce that

$$
\begin{aligned}
\|z^L_{0,P}\|^2_H \leq\ & c\Delta t \sum_{m=1}^{P} \Big(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|^q_{L^p(D)} + \sum_{i=1}^{n} \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|^q_{L^p(D\times\mathbf{Y})} \Big) \\
&+ c\Delta t \sum_{m=1}^{P} \Big(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|^p_{L^p(D)} + \sum_{i=1}^{n} \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|^p_{L^p(D\times\mathbf{Y})} \Big) \\
&+ c(\Delta t)^q + c\|g - g^L\|^2_H + c\|u_{0,1} - \tilde{u}_{0,1}\|^2_H + c\|u_{0,P} - \tilde{u}_{0,P}\|^2_H \\
&+ \delta T \max_m \|z^L_{0,m}\|^2_H + c\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|^2_H.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\max_m \|z^L_{0,m}\|^2_H \leq\ & c\Delta t \sum_{m=1}^{M} \Big(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|^q_{L^p(D)} + \sum_{i=1}^{n} \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|^q_{L^p(D\times\mathbf{Y})} \Big) \\
&+ c\Delta t \sum_{m=1}^{M} \Big(\|\nabla u_{0,m} - \nabla \tilde{u}_{0,m}\|^p_{L^p(D)} + \sum_{i=1}^{n} \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde{u}_{i,m}\|^p_{L^p(D\times\mathbf{Y})} \Big) \\
&+ c(\Delta t)^q + c\|g - g^L\|^2_H + c\|u_{0,1} - \tilde{u}_{0,1}\|^2_H + c \max_{m=1,\dots,M} \|u_{0,m} - \tilde{u}_{0,m}\|^2_H \\
&+ \delta T \max_m \|z^L_{0,m}\|^2_H + c\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_{0,m+1} - \tilde{u}_{0,m+1}) - (u_{0,m} - \tilde{u}_{0,m})}{\Delta t} \right\|^2_H
\end{aligned}
$$

so

$$
\begin{aligned}
\max_m \|z_{0,m}^L\|_H^2 \quad \le \quad & c\Delta t \sum_{m=1}^{M} \left(\|\nabla u_{0,m} - \nabla \tilde u_{0,m}\|^q_{L^p(D)} + \sum_{i=1}^{n} \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde u_{i,m}\|^q_{L^p(D\times \mathbf{Y})} \right) \\
& + c\Delta t \sum_{m=1}^{M} \left(\|\nabla u_{0,m} - \nabla \tilde u_{0,m}\|^p_{L^p(D)} + \sum_{i=1}^{n} \|\nabla_{y_i} u_{i,m} - \nabla_{y_i} \tilde u_{i,m}\|^p_{L^p(D\times \mathbf{Y})} \right) \\
& + c(\Delta t)^q + c\|g - g^L\|_H^2 + c \max_{m=1,\dots,M} \|u_{0,m} - \tilde u_{0,m}\|_H^2 \\
& + c\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_{0,m+1} - \tilde u_{0,m+1}) - (u_{0,m} - \tilde u_{0,m})}{\Delta t} \right\|_H^2
\end{aligned}
$$

The conclusion follows. □

3.1.2 Backward Euler for full tensor product FE spaces

We construct a hierarchy of finite element spaces. We assume that the domain D is a polygon. Let $\{\mathcal{T}^l\}$ for $l \ge 0$ be the hierarchy of regular triangular simplices of mesh size $h_l = O(2^{-l})$ in D. The set of simplices $\mathcal{T}^l$ is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 4 congruent triangles in the two dimension case, and is obtained by dividing each simplex in $\mathcal{T}^{l-1}$ into 8 tedrahedra in the three dimension case. In the periodic cube Y, in a similar manner, we construct the hierarchy $\{\mathcal{T}^l_\#\}$ of regular triangular simplices of mesh size $h_l = O(2^{-l})$ which are periodically distributed in Y. Let

$$
\begin{aligned}
V_0^l &= \{\phi \in W_0^{1,p}(D) : \phi|_T \in P^1(T),\ \forall\, T \in \mathcal{T}^l\}; \\
V^l &= \{\phi \in W^{1,p}(D) : \phi|_T \in P^1(T),\ \forall\, T \in \mathcal{T}^l\}; \\
V_\#^l &= \{\phi \in W_\#^{1,p}(Y) : \phi|_T \in P^1(T),\ \forall\, T \in \mathcal{T}_\#^l\}
\end{aligned}
$$

where $P^1(T)$ denotes the space of linear polynomials in the simplex T. The following approximations hold (see, e.g., [15]).

$$
\begin{aligned}
&\inf_{w^l \in V_0^l} \|w - w^l\|_{W_0^{1,p}(D)} \le c h_l \|w\|_{W^{2,p}(D)}, \quad \forall\, w \in W_0^{1,p}(D) \cap W^{2,p}(D); \\
&\inf_{w^l \in V^l} \|w - w^l\|_{L^p(D)} \le c h_l \|w\|_{W^{1,p}(D)}, \quad \forall\, w \in W^{1,p}(D); \\
&\inf_{w^l \in V_\#^l} \|w - w^l\|_{W_\#^{1,p}(Y)} \le c h_l \|w\|_{W_\#^{2,p}(Y)}, \quad \forall\, w \in W_\#^{2,p}(Y); \\
&\inf_{w^l \in V_\#^l} \|w - w^l\|_{L^p(Y)} \le c h_l \|w\|_{W_\#^{1,p}(Y)}, \quad \forall\, w \in W_\#^{1,p}(Y).
\end{aligned}
$$

As $V_i = L^p(D \times Y_1 \times \ldots \times Y_{i-1}, W_\#^{1,p}(Y_i)) \cong L^p(D) \otimes L^p(Y_1) \otimes \ldots \otimes L^p(Y_{i-1}) \otimes W_\#^{1,p}(Y_i)$, we employ the space

$$
\bar V_i^L = V^L \otimes \underbrace{V_\#^L \otimes \ldots \otimes V_\#^L}_{i \text{ times}} \tag{3.5}
$$

to approximate u_i.

To quantify the approximations of functions in V_i by functions in $\bar V_i^L$, we define the following regularity spaces. Let $\mathcal{W}_i$ be the space of functions $w \in L^p(D \times Y_1 \times \ldots \times Y_{i-1}, W^{2,p}(Y_i))$ that belong to $L^p(Y_1 \times \ldots \times Y_{i-1}, W^{1,p}(Y_i, W^{1,p}(D)))$ and $L^p(D \times \prod_{\substack{j\ne k \\ 1\le j \le i-1}} Y_j, W^{1,p}(Y_i, W^{1,p}(Y_k)))$ for all $k = 1, \ldots, i-1$. The space $\mathcal{W}_i$ is equipped with the norm

$$
\begin{aligned}
\|w\|_{\mathcal{W}_i} = \|w\|_{L^p(D\times Y_1 \times \ldots \times Y_{i-1}, W^{2,p}(Y_i))} + \|w\|_{L^p(Y_1\times \ldots \times Y_{i-1}, W^{1,p}(Y_i, W^{1,p}(D)))} + \\
\sum_{k=1}^{i-1} \|w\|_{L^p(D \times \prod_{\substack{j\ne k \\ 1\le j\le i-1}} Y_j, W^{1,p}(Y_i, W^{1,p}(Y_k)))}.
\end{aligned}
$$

We then have the following approximation.

Lemma 3.3 *For $w \in \mathcal{W}_i$,*

$$\inf_{w^L \in \bar{V}_i^L} \|w - w^L\|_{V_i} \le ch_L \|w\|_{\mathcal{W}_i}.$$

The proof can be found in [25]. We employ the FE spaces V_0^L and $\bar{V}_i^L$ in the places of V^L and V_i^L in the backward Euler approximation (3.1). We denote the FE solution as $\bar{u}_{0,m}^L$ and $\bar{u}_{i,m}^L$, and $z_{0,m}^L$ and $z_{i,m}^L$ by $\bar{z}_{0,m}^L$ and $\bar{z}_{i,m}^L$ respectively. We have the following result.

Theorem 3.4 *Assume that $u_0 \in C^2([0,T],H) \cap C([0,T],W^{2,p}(D)) \cap H^1((0,T),H^1(D))$, $u_i \in C([0,T],\mathcal{W}_i)$ for $i = 1,\ldots,n$, and $\|g - g^L\|_H \le ch_L^{p/(2(p-1))}$, then*

$$\|\bar{z}_{0,M}\|_H^2 + \Delta t \sum_{m=1}^{M} (\|\bar{z}_{0,m}\|_V^p + \sum_{i=1}^{n} \|\bar{z}_{i,m}\|_{V_i}^p) \le c(h_L^{p/(p-1)} + (\Delta t)^2).$$

Proof We bound the right hand side of (3.3). As $u_i \in C([0,T],\mathcal{W}_i)$, we can choose $\tilde{u}_{i,m} \in V_i^L$ for $m = 1,\ldots,M$ and a constant c independent of m such that

$$\|(u_i - \tilde{u}_i)_m\|_{V_i} \le ch_L \|u_i(t_m)\|_{\mathcal{W}_i} \le ch_L.$$

Let $I^L(u_0)(t) \in V_0^L$ be the interpolation operator whose value at each node equals the value of $u_0(t)$ (notes $u_0 \in C([0,T],W^{2,p}(D)) \subset C([0,T],C(\bar{D}))$). We have

$$\|u_0(t) - I^L(u_0)(t)\|_V \le ch_L \|u_0(t)\|_{W^{2,p}(D)} \le ch_L.$$

Let $\tilde{u}_0(t) = I^L(u_0)(t)$, we have

$$\|(u_0 - \tilde{u}_0)_m\|_V \le ch_L$$

where c is indepedent of m. With $\tilde{u}_0(t) = I^L(u_0)(t)$, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \tilde{u}_0}{\partial t} \right\|_H \le ch_L \left\| \frac{\partial u_0}{\partial t} \right\|_{H^1(D)}.$$

Using the procedure of [18]

$$\begin{aligned}
\left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2 &= \left\| \int_{m\Delta t}^{(m+1)\Delta t} \frac{\partial (u_0 - \tilde{u}_0)}{\partial t}(t)dt \right\|_H^2 (\Delta t)^{-2} \\
&\le \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial (u_0 - \tilde{u}_0)}{\partial t}(t) \right\|_H dt \right)^2 (\Delta t)^{-2} \\
&\le ch_L^2 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{H^1(D)} dt \right)^2 (\Delta t)^{-2} \\
&\le ch_L^2 \left(\int_{m\Delta t}^{(m+1)\Delta t} \left\| \frac{\partial u_0}{\partial t}(t) \right\|_{H^1(D)}^2 dt \right) (\Delta t)^{-1}.
\end{aligned}$$

Therefore

$$\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|_H^2 \le ch_L^2.$$

We then get the conclusion. □

3.1.3 Backward Euler for sparse tensor product FE spaces

For $p = 2$, we define the sparse tensor product FE spaces as in [26]. We define the following orthogonal projections

$$P^{l0} : L^2(D) \to V^l, \quad P_{\#}^{l0} : L^2(Y) \to V_{\#}^l, \quad P_{\#}^{l1} : H_{\#}^1(Y) \to V_{\#}^l$$

in the norm of $L^2(D)$, $L^2(Y)$ and $H^1_\#(Y)$ respectively. The increment spaces are defined as

$$W^l = (P^{l0} - P^{(l-1)0})V^l, \;\; W^{l0}_\# = (P^{l0}_\# - P^{(l-1)0}_\#)V^l_\#, \;\; W^{l1}_\# = (P^{l1}_\# - P^{(l-1)1}_\#)V^l_\#$$

with the convention that $W^0 = V^0$, $W^{00}_\# = V^0_\#$ and $W^{01}_\# = V^0_\#$. We then have

$$V^l = \bigoplus_{0\le l'\le l} W^{l'}, \;\; V^l_\# = \bigoplus_{0\le l'\le l} W^{l'0}_\#, \;\; V^l_\# = \bigoplus_{0\le l'\le l} W^{l'1}_\#.$$

The full tensor product space $\bar{V}^L_i$ can be written as

$$\bar{V}^L_i = \bigoplus_{\substack{0\le l_k\le L\\ k=0,1,\ldots,i}} W^{l_0}\otimes W^{l_1 0}_\# \otimes \ldots \otimes W^{l_{i-1}0}_\# \otimes W^{l_i 1}_\#.$$

We define the sparse tensor product space $\hat{V}^L_i$ as

$$\hat{V}^L_i = \bigoplus_{0\le l_0+l_1+\ldots+l_i\le L} W^{l_0}\otimes W^{l_1 0}_\# \otimes \ldots \otimes W^{l_{i-1}0}_\# \otimes W^{l_i 1}_\#.$$

For $p > d$, then $W^{1,p}(D) \subset C(D)$ and $W^{1,p}_\#(Y_i) \subset C_\#(Y_i)$. Let $\mathcal{S}^l$ be the set of nodes of the triangulation $\mathcal{T}^l$. We have that $\mathcal{S}^l \subset \mathcal{S}^{l+1}$. Following [25], we consider the basis of V^l that consists of functions ϕ^l_x for $x \in \mathcal{S}^l$ such that $\phi^l_x(x) = 1$ and $\phi^l_x(x') = 0$ where $x' \in \mathcal{S}^l, x' \neq x$. For continuous functions w in D, we define the interpolation operator $I^l : C(D) \to V^l$ as

$$I^l w = \sum_{x\in\mathcal{S}^l} w(x)\phi^l_x. \tag{3.6}$$

Similarly, we define the set $\mathcal{S}^l_\#$ of triangulation nodes of $V^l_\#$, and the basis function $\phi^l_{\#y}$ which equals 1 at $y \in \mathcal{S}^l_\#$ and 0 at other nodes. We define the interpolation operator $I^l_\# : C_\#(Y) \to V^l_\#$ as

$$I^l_\# w = \sum_{y\in\mathcal{S}^l_\#} w(y)\phi^l_{\#y}. \tag{3.7}$$

We define the subspaces $W^l \subset V^l$ and $W^l_\# \subset V^l_\#$ as

$$W^l = (I^l - I^{l-1})C(\bar{D}) \;\; \text{and} \;\; W^l_\# = (I^l_\# - I^{l-1}_\#)C_\#(\bar{Y})$$

with $W^0 = V^0$ and $W^0_\# = V^0_\#$. The space W^l constains the linear combinations of basis functions ϕ^l_x of V^l where $x \in \mathcal{S}^l \setminus \mathcal{S}^{l-1}$; and the space $W^l_\#$ contains the linear combinations of basis functions $\phi^l_{\#y}$ for $y \in \mathcal{S}^l_\# \setminus \mathcal{S}^{l-1}_\#$. We then have that

$$V^l = \bigoplus_{0\le l'\le l} W^{l'}, \quad V^l_\# = \bigoplus_{0\le l'\le l} W^{l'}_\#.$$

The full tensor product FE space $\bar{V}^L_i$ is of the form

$$\bar{V}^L_i = \bigoplus_{\substack{0\le l_j\le L\\ j=0,\ldots,i}} W^{l_0}\otimes W^{l_1}_\# \otimes \ldots \otimes W^{l_i}_\#.$$

The sparse tensor product FE space $\hat{V}^L_i$ is defined as

$$\hat{V}^L_i = \bigoplus_{0\le l_0+l_1+\ldots+l_i\le L} W^{l_0}\otimes W^{l_1}_\# \otimes \ldots \otimes W^{l_i}_\#. \tag{3.8}$$

To quantify the approximating properties of the spaces $\hat{V}^L_i$, we define the regularity spaces $\hat{\mathcal{W}}_i$ of functions $w(x, y_1, \ldots, y_i)$ which are Y_j periodic with respect to y_j for $j = 0, \ldots, i$ such that for all $\alpha_j \in \mathbb{R}^d$

$(j = 0, \ldots, i-1)$ with $|\alpha_j| \le 1$ and $\alpha_i \in \mathbb{R}^d$ with $|\alpha_i| \le 2$, we have $\partial^{\sum_{j=0}^i |\alpha_j|} w/(\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \ldots \partial^{\alpha_i} y_i) \in L^p(D \times Y_1 \times \ldots \times Y_i)$. In other words, $\hat{\mathcal{W}}_i = W^{1,p}(D, W^{1,p}_\#(Y_1, \ldots, W^{1,p}_\#(Y_{i-1}, W^{2,p}_\#(Y_i)))) \cong W^{1,p}(D) \otimes W^{1,p}_\#(Y_1) \otimes \ldots \otimes W^{1,p}_\#(Y_{i-1}) \otimes W^{2,p}_\#(Y_i)$. This space is equipped with the norm

$$\|w\|_{\hat{\mathcal{W}}_i} = \sum_{\substack{0 \le |\alpha_i| \le 2 \\ 0 \le |\alpha_0|, \ldots, |\alpha_{i-1}| \le 1}} \left\| \frac{\partial^{\sum_{j=0}^i |\alpha_j|} w}{\partial^{\alpha_0} x \partial^{\alpha_1} y_1 \ldots \partial^{\alpha_i} y_i} \right\|_{L^p(D \times Y_1 \times \ldots \times Y_i)}.$$

We have the following approximation properties

Lemma 3.5 *For $w \in \hat{\mathcal{W}}_i$, when $p = 2$*

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{L^p(D \times Y_1 \times \ldots \times Y_i)} \le c L^{i/2} h_L \|w\|_{\hat{\mathcal{W}}_i};$$

and when $p > d$

$$\inf_{w^L \in \hat{V}_i^L} \|w - w^L\|_{L^p(D \times Y_1 \times \ldots \times Y_i)} \le c L^i h_L \|w\|_{\hat{\mathcal{W}}_i}.$$

The proofs for these are presented in [26] and [25]. We employ the sparse tensor product FEs for the backward Euler approximating problem (3.1), i.e. we let $V_i^L = \hat{V}_i^L$ for $i = 1, \ldots, n$. We denote the solution as $\hat{u}^L_{0,m}$ and $\hat{u}^L_{i,m}$. We denote $z^L_{0,m}$ by $\hat{z}^L_{0,m}$ and $z^L_{i,m}$ by $\hat{z}^L_{i,m}$. We then have:

Theorem 3.6 *Assume that $u_0 \in C^2([0,T], H) \cap C([0,T], W^{2,p}(D)) \cap H^1([0,T], H^1(D))$, $u_i \in C([0,T], \hat{\mathcal{W}}_i)$ for $i = 1, \ldots, n$ and $\|g - g^L\|_H \le c L^{np/(2(p-1))} h_L^{p/(2(p-1))}$. Then*

$$\|\hat{z}^L_{0,M}\|_H^2 + \Delta t \sum_{m=1}^M (\|\hat{z}^L_{0,m}\|_V^2 + \sum_{i=1}^n \|\hat{z}^L_{i,m}\|_{V_i}^2) \le c(L^n h_L^2 + (\Delta t)^2)$$

when $p = 2$; and

$$\|\hat{z}^L_{0,M}\|_H^2 + \Delta t \sum_{m=1}^M (\|\hat{z}^L_{0,m}\|_V^p + \sum_{i=1}^n \|\hat{z}^L_{i,m}\|_{V_i}^p) \le c(L^{np/(p-1)} h_L^{p/(p-1)} + (\Delta t)^{p/(p-1)})$$

when $p > d$.

The proof of this theorem is similar to that for Theorem 3.4.

Remark 3.7 *The requirement of $p > d$ is to ensure that $W^{1,p}(D) \subset C(D)$ and $W^{1,p}_\#(Y) \subset C_\#(Y)$. If u_i is smoother than $W^{1,p}(D)$ with respect to x and $W^{1,p}_\#(Y_i)$ with respect to y_i so that they are continuous with respect to x and y_i, then we can remove this requirement.*

Remark 3.8 *The dimension of the full tensor product FE space $\bar{V}_i^L$ is $O(2^{idL})$ which is very large when L is large. The dimension of the sparse tensor product FE space $\hat{V}_i^L$ is $O(L^i 2^{dL})$ which is essentially optimal.*

3.2 Crank-Nicholson method

We use the Crank-Nicholson discretizing scheme to solve problem (2.5) in this section. We first consider the scheme for general FE spaces, and prove the convergence of the scheme. We then use the full tensor product FEs and sparse tensor product FEs for the Crank-Nicholson scheme, and deduce the error of convergence.

3.2.1 Crank-Nicholson method for general FE spaces

We employ the partitian $0 \le t_1 \le \ldots \le t_M = T$ as in the previous section. The discretized problem is: For $m = 1, \ldots, M$, find $U^L_{0,m} \in V^L$ and $U^L_{i,m} \in V_i^L$ for $i = 1, \ldots, n$ such that

$$\begin{aligned}
&\left\langle \frac{U^L_{0,m+1} - U^L_{0,m}}{\Delta t}, \phi_0 \right\rangle_H \\
&+ \int_D \int_{\mathbf{Y}} A\left(t_{m+1/2}, x, \boldsymbol{y}, \frac{1}{2}((\nabla U^L_{0,m} + \nabla U^L_{0,m+1}) + \sum_{i=1}^n (\nabla_{y_i} U^L_{i,m} + \nabla_{y_i} U^L_{i,m+1}))\right) \\
&\qquad \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i} \phi_i) dy dx \\
&= \int_D f(t_{m+1/2}) \phi_0 dx
\end{aligned} \tag{3.9}$$

for all $\phi_0 \in V^L$ and $\phi_i \in V_i^L$. We then have:

Proposition 3.9 *Problem* (3.9) *has a unique solution.*

The proof of this proposition is similar to that of Proposition 3.1.

We have the following approximation result. Let

$$Z^L_{0,m} = u_0(t_m) - U^L_{0,m},\ Z^L_{i,m} = u_i(t_m) - U^L_{i,m}\ Z^L_{0,m+1/2} = \frac{1}{2}(Z^L_{0,m} + Z^L_{0,m+1}),\ Z^L_{i,m+1/2} = \frac{1}{2}(Z^L_{i,m} + Z^L_{i,m+1}).$$

Theorem 3.10 *Assume that $u_0 \in C^3([0,T], H) \cap C^2([0,T], V)$, $u_i \in C^2([0,T], V_i)$. Then*

$$\begin{aligned}
&\|Z^L_{0,M}\|^2_H + \Delta t \sum_{m=0}^{M-1} (\|Z^L_{0,m+1/2}\|^p_V + \sum_{i=1}^n \|Z^L_{i,m+1/2}\|^p_{V_i}) \\
&\le c\Delta t \left(\sum_{m=0}^{M-1} (\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V^{p/(p-1)} + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i}^{p/(p-1)}) \right. \\
&\qquad + \sum_{m=0}^{M-1} (\|(u_0 - \tilde{u}_0)_{m+1/2}\|^p_V + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^p_{V_i}) \\
&\qquad \left. + \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|^2_H \right) \\
&\qquad + \max_{m=1,\ldots,M} \|(u_0 - \tilde{u}_0)_{m-1/2}\|^2_H + \|g - g^L\|^2_H + c(\Delta t)^{2p/(p-1)}.
\end{aligned} \tag{3.10}$$

for all $\{\tilde{u}_{0,m},\ m = 0, \ldots, M\} \subset V^L$ and $\{\tilde{u}_{i,m},\ m = 1, \ldots, M\} \subset V_i^L$ for $i = 1, \ldots, n$ where

$$(u_i - \tilde{u}_i)_{m+1/2} = \frac{1}{2}[(u_i(t_m) - \tilde{u}_{i,m}) + (u_i(t_{m+1}) - \tilde{u}_{i,m+1})]$$

for $i = 0, 1, \ldots, n$.

Proof Let $\rho_{0,m} = \frac{1}{\Delta t}(u_0(t_{m+1}) - u_0(t_m)) - \frac{\partial u_0}{\partial t}(t_{m+1/2})$, $\zeta_{0,m} = \frac{1}{2}(u_0(t_{m+1}) + u_0(t_m)) - u_0(t_{m+1/2})$, $\zeta_{i,m} = \frac{1}{2}(u_i(t_{m+1}) + u_i(t_m)) - u_i(t_{m+1/2})$. Since $u_0 \in C^3([0,T], H) \cap C^2([0,T], V)$, $u_i \in C^2([0,T], V_i)$, we deduce that

$$\|\rho_{0,m}\|_H \le c(\Delta t)^2,\ \|\zeta_{0,m}\|_V \le c(\Delta t)^2,\ \|\zeta_{i,m}\|_{V_i} \le c(\Delta t)^2$$

where the constant c does not depend on m. From (2.5) and (3.9) considered at $t = t_{m+1/2}$ we deduce that

$$
\begin{aligned}
&\left\langle \frac{Z^L_{0,m+1}-Z^L_{0,m}}{\Delta t}, \phi_0 \right\rangle_H - \langle \rho_{0,m}, \phi_0\rangle_H \\
&+ \int_D \int_{\mathbf{Y}} \Bigg(A(t_{m+1/2}, x, \boldsymbol{y}, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) + \sum_{i=1}^n \frac{1}{2}(\nabla_{y_i} u_{i,m} + \nabla_{y_i} u_{i,m+1}) - \nabla \zeta_{0,m} - \sum_{i=1}^n \nabla_{y_i}\zeta_{i,m}) \\
&\qquad - A(t_{m+1/2}, x, \boldsymbol{y}, \frac{1}{2}(\nabla U^L_{0,m} + \nabla U^L_{0,m+1}) + \sum_{i=1}^n \frac{1}{2}(\nabla_{y_i} U^L_{i,m} + \nabla_{y_i} U^L_{i,m+1})) \Bigg) \\
&\qquad\qquad \cdot (\nabla \phi_0 + \sum_{i=1}^n \nabla_{y_i}\phi_i) d\boldsymbol{y} dx \\
&= 0. \qquad (3.11)
\end{aligned}
$$

Consider

$$
\begin{aligned}
I = &\left\langle \frac{Z^L_{0,m+1}-Z^L_{0,m}}{\Delta t}, \frac{Z^L_{0,m+1}+Z^L_{0,m}}{2} \right\rangle_H \qquad (3.12) \\
&+ \int_D \int_{\mathbf{Y}} \Bigg(A\left(t_{m+1/2}, x, \boldsymbol{y}, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) + \frac{1}{2}\sum_{i=1}^n (\nabla_y u_{i,m} + \nabla_y u_{i,m+1}) \right) - \\
&\qquad A\left(t_{m+1/2}, x, \boldsymbol{y}, \frac{1}{2}(\nabla U^L_{0,m} + \nabla U^L_{0,m+1}) + \frac{1}{2}\sum_{i=1}^n (\nabla_{y_i} U^L_{i,m} + \nabla_{y_i} U^L_{i,m+1}) \right) \Bigg) \\
&\qquad\qquad \cdot \left(\nabla \frac{Z^L_{0,m+1}+Z^L_{0,m}}{2} + \sum_{i=1}^n \nabla_{y_i} \frac{Z^L_{i,m+1}+Z^L_{i,m}}{2} \right) d\boldsymbol{y} dx.
\end{aligned}
$$

For $\{\tilde{u}_{0,m},\ m=0,\ldots,M\} \subset V^L$ and $\{\tilde{u}_{i,m},\ m=1,\ldots,M\} \subset V_i^L$, we have

$$
\begin{aligned}
I \quad = \quad &\left\langle \frac{Z^L_{0,m+1}-Z^L_{0,m}}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2} \right\rangle_H + \left\langle \frac{Z^L_{0,m+1}-Z^L_{0,m}}{\Delta t}, (\tilde{u}_0 - U^L_0)_{m+1/2} \right\rangle_H \\
+ \quad &\int_D \int_{\mathbf{Y}} \Bigg(A(t_{m+1/2}, x, \boldsymbol{y}, \frac{1}{2}(\nabla u_{0,m} + \nabla u_{0,m+1}) + \frac{1}{2}\sum_{i=1}^n (\nabla_{y_i} u_{i,m} + \nabla_{y_i} u_{i,m+1})) - \\
&A(t_{m+1/2}, x, \boldsymbol{y}, \frac{1}{2}(\nabla U^L_{0,m} + \nabla U^L_{0,m+1}) + \frac{1}{2}\sum_{i=1}^n (\nabla_{y_i} U^L_{i,m} + \nabla_{y_i} U^L_{i,m+1})) \Bigg) \\
&\cdot \left((\nabla (u_0 - \tilde{u}_0)_{m+1/2} + \sum_{i=1}^n \nabla_{y_i} (u_i - \tilde{u}_i)_{m+1/2}) + (\nabla (\tilde{u}_0 - U^L_0)_{m+1/2} + \sum_{i=1}^n \nabla_{y_i} (\tilde{u}_i - U^L_i)_{m+1/2}) \right) d\boldsymbol{y} dx.
\end{aligned}
$$

From (3.11) we have

$$
\begin{aligned}
I &= \left\langle \frac{Z^L_{0,m+1}-Z^L_{0,m}}{\Delta t}, (u_0-\tilde u_0)_{m+1/2}\right\rangle_H \\
&+ \int_D\int_{\mathbf{Y}}\Bigg(A(t_{m+1/2},x,\boldsymbol{y},\frac{1}{2}(\nabla u_{0,m}+\nabla u_{0,m+1})+\frac{1}{2}\sum_{i=1}^n(\nabla_{y_i}u_{i,m}+\nabla_{y_i}u_{i,m+1}))- \\
&\quad A(t_{m+1/2},x,\boldsymbol{y},\frac{1}{2}(\nabla U^L_{0,m}+\nabla U^L_{0,m+1})+\frac{1}{2}\sum_{i=1}^n(\nabla_{y_i}U^L_{i,m}+\nabla_{y_i}U^L_{i,m+1}))\Bigg) \\
&\quad \cdot\left((\nabla(u_0-\tilde u_0)_{m+1/2}+\sum_{i=1}^n\nabla_{y_i}(u_i-\tilde u_i)_{m+1/2})\right)d\boldsymbol{y}dx \\
&\quad +\langle\rho_{0,m},(\tilde u_0-U^L_0)_{m+1/2}\rangle_H \\
&\quad -\int_D\int_{\mathbf{Y}}\Bigg(A(t_{m+1/2},x,\boldsymbol{y},\frac{1}{2}(\nabla u_{0,m}+\nabla u_{0,m+1})+\frac{1}{2}\sum_{i=1}^n(\nabla_{y_i}u_{i,m}+\nabla_{y_i}u_{i,m+1})-\nabla\zeta_{0,m}-\sum_{i=1}^n\nabla_{y_i}\zeta_{i,m}) \\
&\quad -A(t_{m+1/2},x,\boldsymbol{y},\frac{1}{2}(\nabla u_{0,m}+\nabla u_{0,m+1})+\frac{1}{2}\sum_{i=1}^n(\nabla_{y_i}u_{i,m}+\nabla_{y_i}u_{i,m+1}))\Bigg) \\
&\quad \cdot(\nabla(\tilde u_0-U^L_0)_{m+1/2}+\sum_{i=1}^n\nabla_{y_i}(\tilde u_i-U^L_i)_{m+1/2})dydx.
\end{aligned}
$$

We note that $(\tilde u_0-U^L_0)_{m+1/2}=(\tilde u_0-u_0)_{m+1/2}+Z^L_{0,m+1/2}$ and $(\tilde u_i-U^L_i)_{m+1/2}=(\tilde u_i-u_i)_{m+1/2}+Z^L_{i,m+1/2}$. For a positive constant $\delta>0$, using Young inequality, we have

$$
\begin{aligned}
I &\le \left\langle \frac{Z^L_{0,m+1}-Z^L_{0,m}}{\Delta t}, (u_0-\tilde u_0)_{m+1/2}\right\rangle_H \\
&\quad +\delta\|Z^L_{0,m+1/2}\|^p_V+\delta\sum_{i=1}^n\|Z^L_{i,m+1/2}\|^p_{V_i}+c\|(u_0-\tilde u_0)_{m+1/2}\|^q_V+c\sum_{i=1}^n\|(u_i-\tilde u_i)_{m+1/2}\|^q_{V_i} \\
&\quad +c\|(u_0-\tilde u_0)_{m+1/2}\|^p_V+c\sum_{i=1}^n\|(u_i-\tilde u_i)_{m+1/2}\|^p_{V_i} \\
&\quad +c\|\rho_{0,m}\|^q_H+\delta\|Z^L_{0,m+1/2}\|^p_H+c\|(u_0-\tilde u_0)_{m+1/2}\|^p_H \\
&\quad +c\|\zeta_{0,m}\|^q_V+c\sum_{i=1}^n\|\zeta_{i,m}\|^q_{V_i}+\delta\|Z^L_{0,m+1/2}\|^p_V+\delta\sum_{i=1}^n\|Z^L_{i,m+1/2}\|^p_{V_i} \\
&\quad +c\|(u_0-\tilde u_0)_{m+1/2}\|^p_V+c\sum_{i=1}^n\|(u_i-\tilde u_i)_{m+1/2}\|^p_{V_i} \\
&\le \left\langle \frac{Z^L_{0,m+1}-Z^L_{0,m}}{\Delta t}, (u_0-\tilde u_0)_{m+1/2}\right\rangle_H+c\|(u_0-\tilde u_0)_{m+1/2}\|^q_V+c\sum_{i=1}^n\|(u_i-\tilde u_i)_{m+1/2}\|^q_{V_i} \\
&\quad +c\|(u_0-\tilde u_0)_{m+1/2}\|^p_V+c\sum_{i=1}^n\|(u_i-\tilde u_i)_{m+1/2}\|^p_{V_i} \\
&\quad +c(\Delta t)^{2q}+\delta\|Z^L_{0,m+1/2}\|^p_V+\delta\sum_{i=1}^n\|Z^L_{i,m+1/2}\|^p_{V_i}
\end{aligned}
$$

where we have used the fact that $\|w\|_H\le\|w\|_V$ for all $w\in V$. From (2.1), we have

$$
I\ge\frac{1}{2\Delta t}(\|Z^L_{0,m+1}\|^2_H-\|Z^L_{i,m}\|^2_H)+c\|Z^L_{0,m+1/2}\|^p_V+c\sum_{i=1}^n\|Z^L_{i,m+1/2}\|^p_{V_i}.
$$

Choosing δ sufficiently small, we have

$$\frac{1}{2\Delta t}(\|Z^L_{0,m+1}\|^2_H - \|Z^L_{0,m}\|^2_H) + c(\|Z^L_{0,m+1/2}\|^p_V + \sum_{i=1}^n \|Z^L_{i,m+1/2}\|^p_{V_i})$$
$$\leq c\left\langle \frac{Z^L_{0,m+1} - Z^L_{0,m}}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2}\right\rangle_H + c\|(u_0 - \tilde{u}_0)_{m+1/2}\|^q_V + c\sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^q_{V_i}$$
$$+c\|(u_0 - \tilde{u}_0)_{m+1/2}\|^p_V + c\sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^p_{V_i} + c(\Delta t)^{2q}.$$

Fixing an integer $P \leq M$, taking the sum for $m = 0, \dots, P-1$, we have

$$\|Z^L_{0,P}\|^2_H - \|z^L_{0,0}\|^2_H + c\Delta t \sum_{m=0}^{P-1}(\|Z^L_{0,m+1/2}\|^p_V + \sum_{i=1}^n \|Z^L_{i,m+1/2}\|^p_{V_i})$$
$$\leq c\Delta t \sum_{m=0}^{P-1}\left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|^q_V + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^q_{V_i}\right)$$
$$+c\Delta t \sum_{m=0}^{P-1}\left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|^p_V + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^p_{V_i}\right)$$
$$cP(\Delta t)^{2q+1} + 2\Delta t \sum_{m=0}^{P-1}\left\langle \frac{Z^L_{0,m+1} - Z^L_{0,m}}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2}\right\rangle_H. \tag{3.13}$$

We note that

$$\Delta t \sum_{m=0}^{P-1}\left\langle \frac{Z^L_{0,m+1} - Z^L_{0,m}}{\Delta t}, (u_0 - \tilde{u}_0)_{m+1/2}\right\rangle_H$$
$$= \langle Z^L_{0,P}, (u_0 - \tilde{u}_0)_{P-1/2}\rangle_H - \langle Z^L_{0,0}, (u_0 - \tilde{u}_0)_{1/2}\rangle_H$$
$$+\Delta t \sum_{m=1}^{P-1}\left\langle \frac{Z^L_{0,m}}{\Delta t}, (u_0 - \tilde{u}_0)_{m-1/2} - (u_0 - \tilde{u}_0)_{m+1/2}\right\rangle_H$$
$$\leq \delta\|Z^L_{0,P}\|^2_H + c\|(u_0 - \tilde{u}_0)_{P-1/2}\|^2_H + \|Z^L_{0,0}\|^2_H + \|(u_0 - \tilde{u}_0)_{1/2}\|^2_H$$
$$+\delta\Delta t \sum_{m=1}^{P-1}\|Z^L_{0,m}\|^2_H + c\Delta t \sum_{m=1}^{P-1}\left\|\frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t}\right\|^2_H$$

where we have employed the Cauchy-Schwartz inequality; δ is a constant which we choose to be small. From the fact that $\Delta t \sum_{m=1}^{P-1}\|Z^L_{0,m}\|^2_H \leq T \max_{m=0,\dots,M}\|Z^L_{0,m}\|^2_H$ and (3.13), we have

$$\begin{aligned}\|Z^L_{0,P}\|^2_H \quad \leq \quad & c\Delta t \sum_{m=0}^{P-1}\left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|^q_V + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^q_{V_i}\right)\\
&+c\Delta t \sum_{m=0}^{P-1}\left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|^p_V + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^p_{V_i}\right)\\
&+c(\Delta t)^{2q} + c\|(u_0 - \tilde{u}_0)_{P-1/2}\|^2_H + c\|Z^L_{0,0}\|^2_H + \|(u_0 - \tilde{u}_0)_{1/2}\|^2_H\\
&+c\Delta t \sum_{m=1}^{P-1}\left\|\frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t}\right\|^2_H + \delta T \max_{m=0,\dots,M}\|Z^L_{0,m}\|^2_H\end{aligned}$$

for all $P = 1, \dots, M$. Choosing δ sufficiently small, we have

$$\begin{aligned}
\max_{m=0,\dots,M} \|Z^L_{0,m}\|^2_H \quad \leq \quad & c\Delta t \sum_{m=0}^{M-1} \left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|^q_V + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^q_{V_i} \right) \\
& + c\Delta t \sum_{m=0}^{M-1} \left(\|(u_0 - \tilde{u}_0)_{m+1/2}\|^p_V + \sum_{i=1}^n \|(u_i - \tilde{u}_i)_{m+1/2}\|^p_{V_i} \right) \\
& + c(\Delta t)^{2q} + c \max_{m=1,\dots,M} \|(u_0 - \tilde{u}_0)_{m-1/2}\|^2_H + c\|Z^L_{0,0}\|^2_H \\
& + c\Delta t \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|^2_H .
\end{aligned}$$

From this, we get the conclusion. □

3.2.2 Crank-Nicholson method for full tensor product FE spaces

We deduce the error for the Crank-Nicholson method where we use the full tensor product FE space $\bar{V}^L_i$ defined in (3.5) for V_i in Subsection 3.2.1. We denote the solution as $\bar{U}^L_{0,m}$, $\bar{U}^L_{i,m}$. We denote $Z^L_{0,m+1/2}$ and $Z^L_{i,m+1/2}$ by $\bar{Z}^L_{0,m+1/2}$ and $\bar{Z}^L_{i,m+1/2}$ respectively. We then have the following result.

Theorem 3.11 *Assume that* $u_0 \in C^3([0,T], H) \cap C^2([0,T], V) \cap C([0,T], W^{2,p}(D) \cap H^1((0,T), H^1(D))$, $u_i \in C^2([0,T], V_i) \cap C([0,T], \mathcal{W}_i)$. *Then*

$$\|\bar{Z}^L_{0,M}\|^2_H + \Delta t \sum_{m=0}^{M-1} (\|\bar{Z}^L_{0,m+1/2}\|^p_V + \sum_{i=1}^n \|\bar{Z}^L_{i,m+1/2}\|^p_{V_i}) \leq c(h_L^{p/(p-1)} + (\Delta t)^{2p/(p-1)}).$$

Proof Since $u_i \in C([0,T], \mathcal{W}_i)$, we choose $\tilde{u}_{i,m} \in \bar{V}^L_i$ for $m = i, \dots, M$ such that

$$\|(u_i - \tilde{u}_i)_{m+1/2}\|_{V_i} \leq ch_L(\|u_i(t_m)\|_{\mathcal{W}_i} + \|u_i(t_{m+1})\|_{\mathcal{W}_i}) \leq ch_L,$$

where c is independent of t. Define the interpolation $I^L(u_0)(t) \in V^L$ such that the value of $I^L(u_0)(t)$ at each node equals the value of $u_0(t)$. We have

$$\|u_0(t) - I^L(u_0)(t)\|_V \leq ch_L \|u_0(t)\|_{W^{2,p}(D)} \leq ch_L.$$

Choosing $\tilde{u}_0(t) = I^L(u_0)(t)$, we have

$$\|(u_0 - \tilde{u}_0)_{m+1/2}\|_V \leq ch_L$$

where c does not depend on m. We then have

$$\begin{aligned}
& \sum_{m=1}^{M-1} \left\| \frac{(u_0 - \tilde{u}_0)_{m+1/2} - (u_0 - \tilde{u}_0)_{m-1/2}}{\Delta t} \right\|^2_H \\
& \leq c \sum_{m=1}^{M-1} \left(\left\| \frac{(u_0 - \tilde{u}_0)_{m+1} - (u_0 - \tilde{u}_0)_m}{\Delta t} \right\|^2_H + \left\| \frac{(u_0 - \tilde{u}_0)_m - (u_0 - \tilde{u}_0)_{m-1}}{\Delta t} \right\|^2_H \right).
\end{aligned}$$

With $\tilde{u}_0(t) = I^L(u_0)(t)$, we have

$$\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \tilde{u}_0}{\partial t} \right\|_H \leq ch_L \left\| \frac{\partial u_0}{\partial t} \right\|_{H^1(D)} .$$

Thus

$$\begin{aligned}\left\|\frac{(u_0-\tilde u_0)_{m+1}-(u_0-\tilde u_0)_m}{\Delta t}\right\|_H^2 &= \left\|\int_{m\Delta t}^{(m+1)\Delta t}\frac{\partial(u_0-\tilde u_0)}{\partial t}(t)dt\right\|_H^2(\Delta t)^{-2}\\ &\le \left(\int_{m\Delta t}^{(m+1)\Delta t}\left\|\frac{\partial(u_0-\tilde u_0)}{\partial t}(t)\right\|_H dt\right)^2(\Delta t)^{-2}\\ &\le ch_L^2\left(\int_{m\Delta t}^{(m+1)\Delta t}\left\|\frac{\partial u_0}{\partial t}(t)\right\|_{H^1(D)}dt\right)^2(\Delta t)^{-2}\\ &\le ch_L^2\left(\int_{m\Delta t}^{(m+1)\Delta t}\left\|\frac{\partial u_0}{\partial t}(t)\right\|_{H^1(D)}^2 dt\right)(\Delta t)^{-1}.\end{aligned}$$

Therefore

$$\Delta t\sum_{m=1}^{M-1}\left\|\frac{(u_0-\tilde u_0)_{m+1/2}-(u_0-\tilde u_0)_{m-1/2}}{\Delta t}\right\|_H^2\le ch_L^2.$$

We then get the conclusion. □

3.2.3 Crank-Nicholson method for sparse tensor product FE spaces

Using the sparse tensor product FEs, i.e. we use the sparse tensor product FE space $\hat V_i^L$ defined in (3.8) for V_i in Subsection 3.2.1, for $i=1,\dots,n$, we denote the solution as $\hat U_{0,m}^L$ and $\hat U_{i,m}^L$, and $Z_{0,m+1/2}^L$ and $Z_{i,m+1/2}^L$ by $\hat Z_{0,m+1/2}^L$ and $\hat Z_{i,m+1/2}^L$ respectively. We then have the following result.

Theorem 3.12 *Assume that*

$$\begin{aligned}u_0&\in C^3([0,T],H)\cap C^2([0,T],V)\cap C([0,T],W^{2,p}(D))\cap H^1([0,T],H^1(D)),\\ u_i&\in C^2([0,T],V_i)\cap C([0,T],\hat{\mathcal W}_i)\end{aligned}$$

for $i=1,\dots,n$. Then

$$\|\hat Z_{0,M}^L\|_H^2+\Delta t\sum_{m=0}^{M-1}(\|\hat Z_{0,m+1/2}^L\|_V^2+\sum_{i=1}^n\|\hat Z_{i,m+1/2}^L\|_{V_i}^2)\le c\big(L^nh_L^2+(\Delta t)^4\big) \tag{3.14}$$

when $p=2$; and when $p>d$

$$\|\hat Z_{0,M}^L\|_H^2+\Delta t\sum_{m=0}^{M-1}(\|\hat Z_{0,m+1/2}^L\|_V^p+\sum_{i=1}^n\|\hat Z_{i,m+1/2}^L\|_{V_i}^p)\le c\big((L^nh_L)^{p/(p-1)}+(\Delta t)^{2p/(p-1)}\big). \tag{3.15}$$

The proof of this theorem is similar to that of Theorem 3.11.

4 Numerical correctors

We derive numerical correctors in this section. First we derive the homogenized equation from (2.5). Let $\phi_0=0$ and $\phi_i=0$ for $i=1,\dots,n-1$ in (2.5), we have

$$\int_D\int_{\mathbf Y}A(t,x,\boldsymbol y,\nabla u_0+\nabla_{y_1}u_1+\ldots+\nabla_{y_n}u_n)\cdot\nabla_{y_n}\phi_n dy_n\ldots dy_1dx=0.$$

For each vector $\xi\in\mathbb R^d$, we denote by $N^n(t,x,\boldsymbol y_{n-1},y_n,\xi)\in W_\#^{1,p}(Y_n)/\mathbb R$ the solution of the problem

$$\nabla_{y_n}\cdot A(t,x,\boldsymbol y_{n-1},y_n,\xi+\nabla_{y_n}N^n(t,x,\boldsymbol y,\xi))=0.$$

We then have $u_n = N^n(t, x, \boldsymbol{y}, \nabla u_0 + \nabla_{y_1} u_1 + \ldots + \nabla_{y_{n-1}} u_{n-1})$. The $(n-1)$th homogenized operator is determined as

$$A^{n-1}(t, x, \boldsymbol{y}_{n-1}, \xi) = \int_{Y_n} A(t, x, \boldsymbol{y}_{n-1}, y_n, \xi + \nabla_{y_n} N(t, x, \boldsymbol{y}_{n-1}, y_n, \xi)) dy_n.$$

It can be shown that A^{n-1} satisfies the monotone and local Lipchitzness conditions similar to those of (2.1) and (2.2) (see, e.g., [11], [17] and [9]). Inductively, let $A^n(t, x, \boldsymbol{y}, \xi) = A(t, x, \boldsymbol{y}, \xi)$. Let $N^i(t, x, \boldsymbol{y}_{i-1}, y_i, \xi) \in W^{1,p}_{\#}(Y_i)/\mathbb{R}$ as a function of y_i be the solution of the problem

$$\nabla_{y_i} \cdot A^i(t, x, \boldsymbol{y}_{i-1}, y_i, \xi + \nabla_{y_i} N^i(t, x, \boldsymbol{y}_{i-1}, y_i, \xi) = 0.$$

The $(i-1)$th homogenized operator is defined as

$$A^{i-1}(t, x, \boldsymbol{y}_{i-1}, \xi) = \int_{Y_i} A^i(t, x, \boldsymbol{y}_{i-1}, y_i, \xi + \nabla_{y_i} N^i(t, x, \boldsymbol{y}_{i-1}, y_i, \xi)) dy_i$$

which satisfies the monotone and local Lipschitzness conditions similar to (2.1) and (2.2). The homogenized equation is:

$$\langle \frac{\partial u_0}{\partial t}, \phi_0 \rangle_H + \int_D A^0(t, x, \nabla u_0) \cdot \nabla u_0 dx = \int_D f \phi dx, \quad \forall \phi \in V. \tag{4.1}$$

4.1 Two scale problems

For problems of two scales, we can deduce an explicit homogenization error in terms of the microscopic scale. We then derive a numerical corrector with an error that is the sum of the FE error and the homogenization error. Let $N(t, x, y, \xi)$ belong to $W^{1,p}_{\#}(Y)/\mathbb{R}$ as a function of y and satisfy the cell problem

$$\nabla_y \cdot A(t, x, y, \xi + \nabla_y N(t, x, y, \xi)) = 0. \tag{4.2}$$

The homogenized opertor is determined by

$$A^0(t, x, \xi) = \int_Y A(t, x, y, \xi + \nabla_y N(x, y, \xi)) dy. \tag{4.3}$$

We have the following homogenization result:

Proposition 4.1 *Assume that $u_0 \in C([0, T], C^2(\bar{D}))$, $u_1 \in C([0, T], C^1(\bar{D}, C^1(\bar{Y})))$, we then have*

$$\|\nabla u^\varepsilon - [\nabla u_0 + \nabla_y u_1(\cdot, \cdot, \frac{\cdot}{\varepsilon})]\|_{L^p((0,T)\times D)} \le c\varepsilon^{1/(p(p-1))}.$$

Proof Let

$$u_1^\varepsilon(t, x) = u_0(t, x) + \varepsilon u_1(t, x, \frac{x}{\varepsilon}).$$

We have from the fact that $A \in C^1(\bar{D} \times \bar{Y} \times \mathbb{R}^d)$ and (2.2) that

$$|A(t, x, \frac{x}{\varepsilon}, \nabla u_0(t, x) + \nabla_y u_1(t, x, \frac{x}{\varepsilon}) - A(t, x, \frac{x}{\varepsilon}, \nabla u_1^\varepsilon(t, x))| \le c\varepsilon. \tag{4.4}$$

For $i = 1, \ldots, d$, we define the function

$$g_i(t, x, y) = A_i(t, x, y, \nabla u_0(x) + \nabla_y u_1(t, x, y)) - A_i^0(t, x, \nabla u_0(t, x)).$$

From (4.2) we have

$$\frac{\partial}{\partial y_i} g_i(t, x, y) = 0,$$

and

$$\int_Y g_i(t, x, y) dy = 0.$$

From [29], there are functions $\alpha_{ij}(t,x,y)$ for $i,j=1,\dots,d$ such that $\alpha_{ij}=-\alpha_{ji}$ and

$$g_i(t,x,y)=\frac{\partial}{\partial y_j}\alpha_{ij}(t,x,y).$$

As $g_i\in C^1([0,T]\times\bar D, C(\bar Y))$, we have $\alpha_{ij}\in C^1([0,T]\times\bar D, C^1(\bar Y))$. We then have

$$A_i\left(t,x,\frac{x}{\varepsilon},\nabla u_0(t,x)+\nabla_y u_1\left(t,x,\frac{x}{\varepsilon}\right)\right)-A_i^0(t,x,\nabla u_0(t,x))=\varepsilon\frac{d}{dx_j}\alpha_{ij}\left(t,x,\frac{x}{\varepsilon}\right)-\varepsilon\frac{\partial}{\partial x_j}\alpha_{ij}\left(t,x,\frac{x}{\varepsilon}\right),$$

where $\frac{d}{dx_j}$ denotes the partial derivative of $\alpha_{ij}(t,x,\frac{x}{\varepsilon})$ as a function of x only. Thus for any functions $\phi\in W_0^{1,p}(D)$ we have

$$\begin{aligned}&\int_D\left(A_i\left(t,x,\frac{x}{\varepsilon},\nabla u_0(t,x)+\nabla_y u_1\left(t,x,\frac{x}{\varepsilon}\right)\right)-A_i^0(t,x,\nabla u_0(t,x))\right)\frac{\partial\phi}{\partial x_i}(x)dx\\&=-\varepsilon\int_D\alpha_{ij}\left(t,x,\frac{x}{\varepsilon}\right)\frac{\partial^2\phi}{\partial x_i\partial x_j}dx-\varepsilon\int_D\frac{\partial}{\partial x_j}\alpha_{ij}\left(t,x,\frac{x}{\varepsilon}\right)\frac{\partial\phi}{\partial x_i}dx=-\varepsilon\int_D\frac{\partial}{\partial x_j}\alpha_{ij}\left(t,x,\frac{x}{\varepsilon}\right)\frac{\partial\phi}{\partial x_i}dx\end{aligned}$$

due to $\alpha_{ij}(t,x,y)=-\alpha_{ji}(t,x,y)$. From this and (4.4) we have

$$\left\|\left(\frac{\partial u_1^\varepsilon}{\partial t}(t)-\nabla\cdot(A^\varepsilon(t,\cdot,\nabla u_1^\varepsilon)\right)-\left(\frac{\partial u_0}{\partial t}(t)-A^0(t,\cdot,\nabla u_0)\right)\right\|_{W^{-1,q}(D)}\le c\varepsilon. \tag{4.5}$$

Let $\tau^\varepsilon\in\mathcal{D}(D)$ be such that $\tau^\varepsilon(x)=1$ outside an ε neighbourhood of ∂D and $\varepsilon|\nabla_x\tau^\varepsilon(x)|\le c$ where c is independent of ε. Let $\delta>0$ be sufficiently large. We consider the function

$$w_1^\varepsilon(t,x)=u_0(t,x)+\varepsilon\tau^\varepsilon(x)u_1(t,x,x/\varepsilon).$$

We note that

$$\nabla(u_1^\varepsilon(t,x)-w_1^\varepsilon(t,x))=-\varepsilon\nabla\tau^\varepsilon(x)u_1\left(t,x,\frac{x}{\varepsilon}\right)+\varepsilon(1-\tau^\varepsilon(x))\nabla_x u_1\left(t,x,\frac{x}{\varepsilon}\right)+(1-\tau^\varepsilon(x))\nabla_y u_1\left(t,x,\frac{x}{\varepsilon}\right).$$

As the support of $\nabla(u_1^\varepsilon-w_1^\varepsilon)$ is in an ε neighbourhood of ∂D, we deduce that

$$\|u_1^\varepsilon-w_1^\varepsilon\|_{W^{1,p}(D)}\le c\varepsilon^{1/p}. \tag{4.6}$$

From (2.2), we have

$$\|A^\varepsilon(t,\cdot,\nabla u_1^\varepsilon(\cdot))-A^\varepsilon(t,\cdot,\nabla w_1^\varepsilon(\cdot))\|_{L^q(D)}\le c\varepsilon^{1/p}.$$

Therefore

$$\left\|\left(\frac{\partial u_1^\varepsilon}{\partial t}(t)-\nabla\cdot(A^\varepsilon(t,\cdot,\nabla u_1^\varepsilon)\right)-\left(\frac{\partial w_1^\varepsilon}{\partial t}(t)-A^\varepsilon(t,\cdot,\nabla w_1^\varepsilon)\right)\right\|_{W^{-1,q}(D)}\le c\varepsilon^{1/p}.$$

From this and (4.5), we have

$$\left\|\left(\frac{\partial u^\varepsilon}{\partial t}-\nabla\cdot(A^\varepsilon(t,\cdot,\nabla u^\varepsilon)\right)-\left(\frac{\partial w_1^\varepsilon}{\partial t}-A^\varepsilon(t,\cdot,\nabla w_1^\varepsilon)\right)\right\|_{W^{-1,q}(D)}\le c\varepsilon^{1/p}. \tag{4.7}$$

Therefore

$$\begin{aligned}&\int_0^T\left\langle\frac{\partial}{\partial t}(u^\varepsilon-w_1^\varepsilon),(u^\varepsilon-w_1^\varepsilon)\right\rangle_H dt+\int_0^T\int_D(A^\varepsilon(t,\cdot,\nabla u^\varepsilon)-A^\varepsilon(t,\cdot,\nabla w_1^\varepsilon))\cdot(\nabla u^\varepsilon-\nabla w_1^\varepsilon)dxdt\\&\le c\varepsilon^{1/p}\int_0^T\|u^\varepsilon-w_1^\varepsilon\|_V dt\\&\le c\varepsilon^{1/p}\left(\int_0^T\|u^\varepsilon-w_1^\varepsilon\|_V^p dt\right)^{1/p}\end{aligned}$$

Using (2.1), we have

$$\|u^\varepsilon(T) - w_1^\varepsilon(T)\|_H^2 + \|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|^p_{L^p((0,T)\times D)} \le c\varepsilon^{1/p}\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p((0,T)\times D)} + \|u^\varepsilon(0) - w_1^\varepsilon(0)\|_H^2.$$

As $u_1 \in C([0,T]\times\bar D\times\bar Y)$ and $u^\varepsilon(0) = u_0(0) = g$, we have $\|u^\varepsilon(0) - w_1^\varepsilon(0)\|_H \le c\varepsilon$. Therefore

$$\|\nabla u^\varepsilon - \nabla w_1^\varepsilon\|_{L^p((0,T)\times D)} \le c\varepsilon^{1/(p(p-1))}.$$

From (4.6), we have

$$\|\nabla u^\varepsilon - \nabla u_1^\varepsilon\|_{L^p((0,T)\times D)} \le c\varepsilon^{1/(p(p-1))}.$$

From this we get the conclusion. □

To construct a numerical corrector, we employ the following operator. For $\Phi \in L^1(D\times Y)$ we define

$$\mathcal{U}^\varepsilon(\Phi)(x) = \int_Y \Phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon z, \Big\{\frac{x}{\varepsilon}\Big\}\Big)dz, \tag{4.8}$$

where $[x/\varepsilon]$ denotes the "integer" part of x/ε with respect to the unit cube Y and $\{x/\varepsilon\} = x/\varepsilon - [x/\varepsilon]$. The operator $\mathcal{U}^\varepsilon$ satisfies:

$$\int_{D^\varepsilon} \mathcal{U}^\varepsilon(\Phi)(x)dx = \int_{D\times Y} \Phi(x,y)dydx, \tag{4.9}$$

for all $\Phi \in L^1(D\times Y)$, where D^ε is the 2ε neighbourhood of D; Φ is regarded as 0 when x is outside D. The proof of this proposition is quite straightforward; we refer to [16] for details.

We then have the following corrector result. For the solution of the backward Euler approximation using full tensor product FEs, let $\bar u_0^L : (0,T) \to V$ and $\bar u_1^L : (0,T) \to V_1$ be defined as

$$\bar u_0^L(t) = \frac{1}{2}(\bar u^L_{0,m} + \bar u^L_{0,m+1}),\ \bar u_1(t) = \frac{1}{2}(\bar u^L_{1,m} + \bar u^L_{1,m+1}) \ \text{ for } t\in[t_m, t_{m+1}).$$

Theorem 4.2 *Assume that $u_0 \in C^2([0,T],V)$ and $u_1 \in C^2([0,T],V_1)\cap C([0,T],C^1(\bar D\times\bar Y))$ and the hypothesis of Theorem 3.4 holds. For the solution of the full tensor product FE backward Euler approximation in* (3.4)*, we have*

$$\|\nabla u^\varepsilon - [\nabla \bar u_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \bar u_1^L)(\cdot)]\|_{L^p((0,T)\times D)} \le c(\varepsilon^{1/(p(p-1))} + h_L^{1/(p-1)} + (\Delta t)^{1/(p-1)}).$$

Proof Using the midpoint rule, we have

$$\begin{aligned}\int_0^T \|\nabla u_0(t) - \nabla \bar u_0^L(t)\|^p_{L^p(D)} &= \sum_{m=0}^{M-1}\int_{m\Delta t}^{(m+1)\Delta t} \|\nabla u_0(t) - \nabla\bar u_0^L(t)\|^p_{L^p(D)}dt\\ &\le \sum_{m=0}^{M-1}(\Delta t\|\nabla u_0(t_{m+1/2}) - \nabla\bar u_0^L(t_{m+1/2}))\|^p_{L^p(D)} + c(\Delta t)^3)\end{aligned}$$

where the constant c is independent of Δt and $t_{m+1/2} = t_m + \frac{1}{2}\Delta t$. We have

$$\Big\|\frac{1}{2}(\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \nabla u_0(t_{m+1/2})\Big\|_{L^p(D)} \le c(\Delta t)^2.$$

From this we deduce

$$\begin{aligned}&\int_0^T \|\nabla u_0(t) - \nabla\bar u_0^L(t)\|^p_{L^p(D)}\\ \le\ & \sum_{m=0}^{M-1}\Big(\Delta t\|\frac{1}{2}(\nabla u_0(t_m) + \nabla u_0(t_{m+1})) - \frac{1}{2}(\nabla\bar u^L_{0,m} + \nabla\bar u^L_{0,m+1})\|^p_{L^p(D)} + c(\Delta t)^3\Big)\\ \le\ & c\Delta t\sum_{m=0}^{M-1}(\|\nabla\bar z^L_{0,m}\|^p_{L^p(D)} + \|\nabla\bar z^L_{0,m+1}\|^p_{L^p(D)}) + c(\Delta t)^2\\ \le\ & c((\Delta t)^{p/(p-1)} + h_L^{p/(p-1)}).\end{aligned}$$

By the same argument, we have

$$\int_0^T \|\nabla_y u_1(t) - \nabla_y \bar{u}_1^L(t)\|^p_{L^p(D\times Y)} dt \le c((\Delta t)^{p/(p-1)} + h_L^{p/(p-1)}).$$

From (4.9) we deduce

$$\|\mathcal{U}^\varepsilon(\nabla_y u_1 - \nabla_y \bar{u}_1^L)\|_{L^p((0,T)\times D)} \le \|\nabla_y u_1 - \nabla_y \bar{u}_1^L\|_{L^p((0,T)\times D\times Y)}.$$

Further, as $u_1 \in C([0,T], C^1(\bar{D}\times\bar{Y}))$,

$$|\nabla_y u_1(t,x,\frac{x}{\varepsilon}) - \mathcal{U}^\varepsilon(\nabla_y u_1)(t,x)| \le c\varepsilon.$$

Thus

$$\begin{array}{rl} & \|\nabla u^\varepsilon - [\nabla \bar{u}_0^L + \mathcal{U}^\varepsilon(\nabla_y \bar{u}_1^L)]\|_{L^p((0,T)\times D)} \\ \le & \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}^\varepsilon(\nabla_y u_1)]\|_{L^p((0,T)\times D)} \\ & +\|\nabla u_0 - \nabla \bar{u}_0^L\|_{L^p((0,T)\times D)} + \|\mathcal{U}^\varepsilon(\nabla_y u_1) - \mathcal{U}^\varepsilon(\nabla \bar{u}_1^L)\|_{L^p((0,T)\times D)} \\ \le & c(\varepsilon^{1/(p(p-1))} + (\Delta t)^{1/(p-1)} + h_L^{1/(p-1)}). \end{array}$$

□

Similarly, for the solution of the backward Euler scheme using the sparse tensor product FEs, we define $\hat{u}_0^L : [0,T] \to V$ and $\hat{u}_1^L : [0,T] \to V_1$ as

$$\hat{u}_0^L(t) = \frac{1}{2}(\hat{u}_{0,m}^L + \hat{u}_{0,m+1}^L),\ \hat{u}_1(t) = \frac{1}{2}(\hat{u}_{1,m}^L + \hat{u}_{1,m+1}^L)\ \text{ for } t \in [t_m, t_{m+1}).$$

Theorem 4.3 *Assume that $u_0 \in C^2([0,T],V)$ and $u_1 \in C^2([0,T],V_1) \cap C([0,T],C^1(\bar{D}\times\bar{Y}))$ and the hypothesis of Theorem 3.6 holds. For the solution of the sparse tensor product FE backward Euler approximation in* (3.1)*, we have*

$$\|\nabla u^\varepsilon - [\nabla \hat{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(\cdot)]\|_{L^2((0,T)\times D)} \le c(\varepsilon^{1/2} + L^{1/2}h_L + \Delta t)$$

when $p = 2$; and when $p > d$

$$\|\nabla u^\varepsilon - [\nabla \hat{u}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{u}_1^L)(\cdot)]\|_{L^p((0,T)\times D)} \le c(\varepsilon^{1/(p(p-1))} + (Lh_L)^{1/(p-1)} + (\Delta t)^{1/(p-1)}).$$

The results for the Crank-Nicholson scheme are similar. For the full tensor product FEs, we define $\bar{U}_0^L : [0,T] \to V$ and $\bar{U}_1^L : [0,T] \to V_1$ as

$$\bar{U}_0^L(t) = \frac{1}{2}(\bar{U}_{0,m}^L + \bar{U}_{0,m+1}^L),\ \bar{U}_1^L(t) = \frac{1}{2}(\bar{U}_{1,m}^L + \bar{U}_{1,m+1}^L)\ \text{ for } t \in [t_m, t_{m+1}).$$

Theorem 4.4 *Assume that $u_0 \in C^2([0,T],V)$ and $u_1 \in C^2([0,T],V_1) \cap C([0,T],C^1(\bar{D}\times\bar{Y}))$ and the hypothesis of Theorem 3.11 holds. For the solution of the full tensor product FE Crank-Nicholson approximation in* (3.4)*, we have*

$$\|\nabla u^\varepsilon - [\nabla \bar{U}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \bar{U}_1^L)(\cdot)]\|_{L^p((0,T)\times D)} \le c(\varepsilon^{1/(p(p-1))} + h_L^{1/(p-1)} + (\Delta t)^{2/p}).$$

Similarly, for the solution of the Crank-Nicholson scheme using the sparse tensor product FEs, we define $\hat{U}_0^L : [0,T] \to V$ and $\hat{U}_1^L : [0,T] \to V_1$ as

$$\hat{U}_0^L(t) = \frac{1}{2}(\hat{U}_{0,m}^L + \hat{U}_{0,m+1}^L),\ \hat{U}_1^L(t) = \frac{1}{2}(\hat{U}_{1,m}^L + \hat{U}_{1,m+1}^L)\ \text{ for } t \in [t_m, t_{m+1}).$$

We then have the following corrector result.

Theorem 4.5 *Assume that $u_0 \in C^2([0,T],V)$ and $u_1 \in C^2([0,T],V_1) \cap C([0,T],C^1(\bar{D}\times\bar{Y}))$ and the hypothesis of Theorem 3.12 holds. For the solution of the sparse tensor product FE Crank-Nicholson approximation in* (3.1)*, we have*

$$\|\nabla u^\varepsilon - [\nabla \hat{U}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{U}_1^L)(\cdot)]\|_{L^2((0,T)\times D)} \le c(\varepsilon^{1/2} + L^{1/2}h_L + \Delta t)$$

when $p = 2$; and when $p > d$

$$\|\nabla u^\varepsilon - [\nabla \hat{U}_0^L(\cdot) + \mathcal{U}^\varepsilon(\nabla_y \hat{U}_1^L)(\cdot)]\|_{L^p((0,T)\times D)} \le c(\varepsilon^{1/(p(p-1))} + (Lh_L)^{1/(p-1)} + (\Delta t)^{2/p}).$$

4.2 Multiscale problems

For general problems with more than one microscopic scale, a homogenization error in terms of the microscopic scales is not available. However, we can establish a numerical corrector from the FE solutions. We assume that $\varepsilon_{i-1}/\varepsilon_i$ is an integer for all $i = 2, \ldots, n$. We first define the operator $\mathcal{T}_n^\varepsilon : L^1(D) \to L^1(D \times \mathbf{Y})$ as:

$$\mathcal{T}_n^\varepsilon(\phi)(x, \boldsymbol{y}) = \phi\Big(\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_2\Big[\frac{y_1}{\varepsilon_2/\varepsilon_1}\Big] + \ldots + \varepsilon_n\Big[\frac{y_{n-1}}{\varepsilon_n/\varepsilon_{n-1}}\Big] + \varepsilon_n y_n\Big) \tag{4.10}$$

where the function ϕ is understood as 0 outside D, and $[\cdot]$ denotes the "integer" part with respect to Y. Let D^ε be the 2ε neighbourhood of D. For all functions $\phi \in L^1(D)$ which are understood as 0 outside D, we have

$$\int_D \phi dx = \int_{D^\varepsilon}\int_{\mathbf{Y}} \mathcal{T}_n^\varepsilon(\phi) d\boldsymbol{y} dx\ . \tag{4.11}$$

We can also show that

$$\mathcal{T}_n^\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u_0 + \nabla_{y_1} u_1 + \ldots + \nabla_{y_n} u_n \text{ in } L^p(D \times \mathbf{Y}). \tag{4.12}$$

We define the operator $\mathcal{U}_n^\varepsilon : L^1(D \times \mathbf{Y}) \to L^1(D)$ as

$$\begin{aligned}\mathcal{U}_n^\varepsilon(\Phi)(x) = \int_{Y_1}\cdots\int_{Y_n} \Phi\Big(&\varepsilon_1\Big[\frac{x}{\varepsilon_1}\Big] + \varepsilon_1 t_1, \frac{\varepsilon_2}{\varepsilon_1}\Big[\frac{\varepsilon_1}{\varepsilon_2}\Big\{\frac{x}{\varepsilon_1}\Big\}\Big] + \frac{\varepsilon_2}{\varepsilon_1} t_2, \ldots,\\ &\frac{\varepsilon_n}{\varepsilon_{n-1}}\Big[\frac{\varepsilon_{n-1}}{\varepsilon_n}\Big\{\frac{x}{\varepsilon_{n-1}}\Big\}\Big] + \frac{\varepsilon_n}{\varepsilon_{n-1}} t_n, \Big\{\frac{x}{\varepsilon_n}\Big\}\Big) dt_n \ldots dt_1\end{aligned} \tag{4.13}$$

where $\{\cdot\} = \cdot - [\cdot]$; the function Φ is understood as 0 outside D. We have $\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\Phi)) = \Phi\ \forall\, \Phi \in L^1(D)$. Further,

$$\int_{D^\varepsilon} \mathcal{U}_n^\varepsilon(\Phi)(x) dx = \int_D\int_{\mathbf{Y}} \Phi d\boldsymbol{y} dx \tag{4.14}$$

where Φ is understood as 0 outside D. The proofs of these facts can be found in [16].

We have the following corrector results.

Proposition 4.6 *We have*

$$\lim_{\varepsilon\to 0} \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i} u_i)]\|_{L^p((0,T)\times D)} = 0.$$

Proof As u^ε is uniformly bounded in $L^p((0,T); V)$, from (2.2), $A(t, x, \frac{x}{\varepsilon}, \ldots, \frac{x}{\varepsilon_n}, \nabla u^\varepsilon)$ is bounded in $L^q((0,T) \times D)$. Let $\chi \in L^q((0,T) \times D \times \mathbf{Y})$ be the $(n+1)$-scale convergence limit of a subsequence of $A(t, x, \frac{x}{\varepsilon}, \ldots, \frac{x}{\varepsilon_n}, \nabla u^\varepsilon)$. From (2.5), we have that

$$\int_0^T\int_D\int_{\mathbf{Y}} \chi(\nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) d\boldsymbol{y} dx = \int_0^T\int_D\int_{\mathbf{Y}} A(t, x, \boldsymbol{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)\cdot(\nabla v_0 + \sum_{i=1}^n \nabla_{y_i} v_i) d\boldsymbol{y} dx dt. \tag{4.15}$$

for all $v_0 \in V$ and $v_i \in V_i$. We consider:

$$\begin{aligned} I = \int_0^T \Big\langle \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t}, u^\varepsilon - u_0\Big\rangle_H + \int_0^T\int_D\int_{\mathbf{Y}} \Big(\mathcal{T}^\varepsilon(A(t, x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}, \nabla u^\varepsilon)) - A(t, x, \boldsymbol{y}, \nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)\Big)\\ \cdot(\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0 + \sum_{i=1}^n \nabla_{y_i} u_i)) d\boldsymbol{y} dx dt.\end{aligned}$$

From (2.3), (2.5), (4.15) we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} I \\
&= \lim_{\varepsilon\to 0}\int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t}, u^\varepsilon\right\rangle_H - \left\langle \frac{\partial u_0}{\partial t}, u_0\right\rangle dt + \int_0^T\int_D A(t,x,\frac{x}{\varepsilon},\dots,\frac{x}{\varepsilon_n},\nabla u^\varepsilon)\cdot\nabla u^\varepsilon dxdt - \\
&\quad \int_0^T\int_D\int_{\mathbf{Y}} A(t,x,\boldsymbol{y},\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i)\cdot(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i)d\boldsymbol{y}dxdt \\
&= 0.
\end{aligned}$$

Using the smoothness of A, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} I = \\
&\lim_{\varepsilon\to 0}\int_0^T \left\langle \frac{\partial u^\varepsilon}{\partial t}-\frac{\partial u_0}{\partial t}, u^\varepsilon - u_0\right\rangle_H + \int_0^T\int_D\int_{\mathbf{Y}}\left(A(t,x,\boldsymbol{y},\mathcal{T}^\varepsilon(\nabla u^\varepsilon))) - A(t,x,\boldsymbol{y},\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i)\right) \\
&\cdot(\mathcal{T}^\varepsilon(\nabla u^\varepsilon) - (\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i))d\boldsymbol{y}dxdt.
\end{aligned}$$

From (2.1) and $u^\varepsilon(0)=u_0(0)=g$, we have

$$\lim_{\varepsilon\to 0}\|u^\varepsilon(T)-u_0(T)\|_H + \|\mathcal{T}^\varepsilon(\nabla u^\varepsilon)-(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i)\|_{L^p((0,T)\times D\times\mathbf{Y})} = 0.$$

From (4.13), we have $(\mathcal{U}^\varepsilon(\Phi)(t,x))^p \le \mathcal{U}^\varepsilon(\Phi^p)(t,x)$. From (4.14), we have

$$\|\mathcal{U}^\varepsilon(\Phi)\|^p_{L^p((0,T)\times D)} \le \|\mathcal{U}^\varepsilon(\Phi^p)\|_{L^1((0,T)\times D))} \le \|\Phi\|^p_{L^p((0,T)\times D\times\mathbf{Y})}. \tag{4.16}$$

We therefore have

$$\|\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon)-(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i))\|_{L^p((0,T)\times D)} \le \|\mathcal{T}^\varepsilon(\nabla u^\varepsilon)-(\nabla u_0+\sum_{i=1}^n\nabla_{y_i}u_i)\|_{L^p((0,T)\times D\times\mathbf{Y})}\to 0$$

when $\varepsilon\to 0$. Using $\mathcal{U}^\varepsilon(\mathcal{T}^\varepsilon(\nabla u^\varepsilon))=\nabla u^\varepsilon$, we get the conclusion. □

We construct the numerical correctors from the finite element solutions of problem (2.5). We present the results for the backward Euler approximations. The results for the Crank-Nicholson approximations are similar. For the full tensor product FE approximations, let $\bar u_0 : (0,T)\to V$ and $\bar u_i : (0,T)\to V_i$ be defined as

$$\bar u_0^L(t) = \frac{1}{2}(\bar u^L_{0,m}+\bar u^L_{0,m+1}),\ \ \bar u_i(t) = \frac{1}{2}(\bar u^L_{i,m}+\bar u^L_{i,m+1})\ \text{ for } t\in[t_m,t_{m+1}).$$

Theorem 4.7 *Assume that the hypothesis of Theorem 3.4 hold. For the solution of the approximating problems* (3.1) *using the full tensor product FE approximation for u_i, we have*

$$\lim_{\substack{L\to\infty\\ \varepsilon\to 0}}\|\nabla u^\varepsilon - [\nabla\bar u_0^L + \mathcal{U}_n^\varepsilon(\sum_{i=1}^n\nabla_{y_i}\bar u_i^L)\|_{L^p((0,T)\times D)} = 0.$$

Proof As in the proof of Theorem 4.2

$$\int_0^T \|\nabla u_0(t)-\nabla\bar u_0^L(t)\|^p_{L^p(D)} \le c((\Delta t)^{p/(p-1)} + h_L^{p/(p-1)})$$

and for all $i=1,\dots,n$,

$$\int_0^T \|\nabla_{y_i}u_i(t)-\nabla_{y_i}\bar u_i^L(t)\|^p_{L^p(D\times\mathbf{Y})}dt \le c((\Delta t)^{p/(p-1)} + h_L^{p/(p-1)}).$$

From (4.14) we deduce

$$\|\mathcal{U}^\varepsilon(\nabla_{y_i}u_i - \nabla_{y_i}\bar{u}_i^L)\|_{L^p((0,T)\times D)} \leq \|\nabla_{y_i}u_i - \nabla_{y_i}\bar{u}_i^L\|_{L^p((0,T)\times D\times \mathbf{Y})}.$$

Thus

$$\begin{aligned}
& \|\nabla u^\varepsilon - [\nabla \bar{u}_0^L + \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i}\bar{u}_i^L)]\|_{L^p((0,T)\times D)} \\
\leq\ & \|\nabla u^\varepsilon - [\nabla u_0 + \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i}u_i)]\|_{L^p((0,T)\times D)} \\
& + \|\nabla u_0 - \nabla \bar{u}_0^L\|_{L^p((0,T)\times D)} + \|\mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i}u_i) - \mathcal{U}_n^\varepsilon(\sum_{i=1}^n \nabla_{y_i}\bar{u}_i^L)\|_{L^p((0,T)\times D)}
\end{aligned}$$

which converges to 0 when $\varepsilon \to 0$ and $L \to \infty$. □

Similarly, for the sparse tensor product FE approximation, we have:

Theorem 4.8 *Assume that the hypothesis of Theorem 3.6 hold. For the solution of the approximating problem* (3.1) *using the sparse tensor product FE approximation for u_i for $i = 1, \ldots, n$, we have*

$$\lim_{\substack{L\to\infty \\ \varepsilon\to 0}} \|\nabla u^\varepsilon - [\nabla \hat{u}_0^L + \mathcal{U}^\varepsilon(\sum_{i=1}^n \nabla_{y_i}\hat{u}_i^L)\|_{L^p((0,T)\times D)} = 0.$$

5 Numerical examples

We present some one dimensional and two dimensional numerical results in this section to illustrate the sparse tensor product FE method.

We first consider a problem on the domain $D = (0, 1)$ where the monotone nonlinear function

$$\begin{aligned}
A(t, x, y, \xi) &= \xi + \frac{1}{2}\sin(x\sin(2\pi y)\xi) - t^2 x\cos(2\pi y) \\
&\quad - \frac{1}{2}\sin(t^2 x\sin(2\pi y)(2x - 1 + x\cos(2\pi y)))
\end{aligned} \tag{5.1}$$

and $f(t, x) = -2t^2 + 2t(x^2 - x)$. In this case $p = 2$. The multiscale homogenized problem has the exact solution

$$u_0(t, x) = t^2(x^2 - x)$$

and

$$u_1(t, x, y) = \frac{t^2}{2\pi}x\sin(2\pi y).$$

We apply the backward Euler and Crank-Nicholson methods with sparse tensor product FE spaces to the multiscale homogenized problem. We use the Broyden's method ([10]) and the Polak-Ribière method ([14]) at each time step to solve the simultaneous nonlinear problems.

For the backward Euler method with sparse tensor product FE spaces, for the meshsize $h_L = O(2^{-L})$, we use a timestep of $\Delta t = 1/2^L$. We plot the errors $\|u_0 - \hat{u}_0^L\|_{H_0^1((D))}$ and $\|u_1 - \hat{u}_1^L\|_{L^2(D, H^1_\#(Y)/\mathbb{R})}$ in Figures 1 and 2 respectively at $t = 1$. The numerical results show that the errors are $O(\Delta t) + O(h_L)$. When these errors hold for all t_m, we get the errors estimate as in Theorem 3.6. This result supports the theoretical finding.

In Figures 3 and 4, we plot the numerical errors for the Crank-Nicholson method on sparse tensor product FE spaces for the same problem. For the Crank-Nicholson method, we choose a timestep of $\Delta t = 1/(\lceil 2^{(L/2)} \rceil)$. The numerical results show that the errors are $O((\Delta t)^2) + O(h_L)$. This result supports the theoretical finding in Theorem 3.12.

We then consider the multiscale monotone problem for $p = 4$ on $D = (0, 1)$ with the monotone function

$$\begin{aligned}
A(t, x, y, \xi) &= \xi + \frac{1}{10}(2 + \sin(2\pi y))\xi^3 - t^2 x\cos(2\pi y) \\
&\quad - \frac{t^6}{10}(2 + \sin(2\pi y))(2x - 1 + x\cos(2\pi y))^3
\end{aligned} \tag{5.2}$$

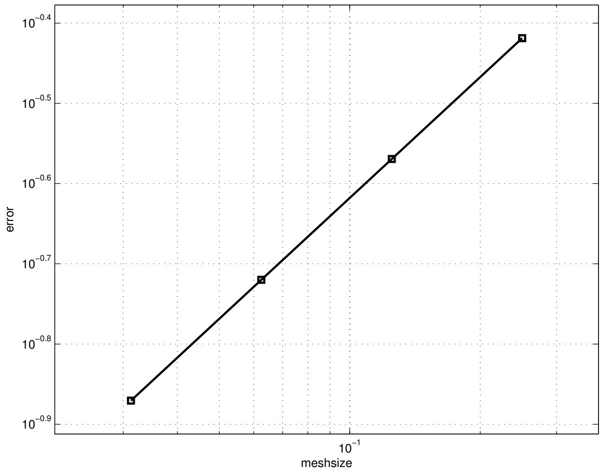

Figure 1: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for one dimensional problem (5.1) at $t = 1$ for backward Euler method.

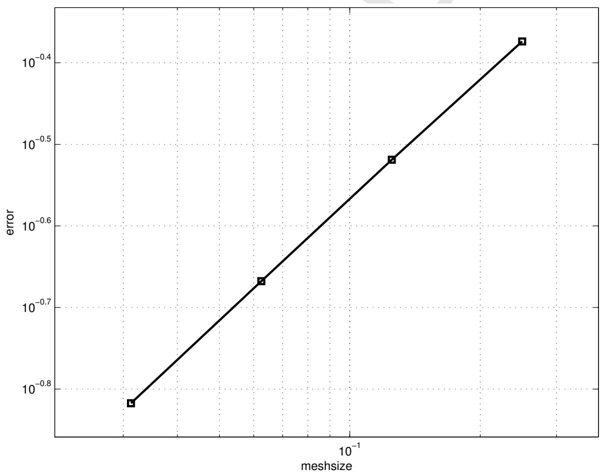

Figure 2: The error $\|u_1 - \hat{u}_1^L\|_{L^2(D,H^1_\#(Y)/\mathbb{R})}$ versus the mesh size h for one dimensional problem (5.1) at $t = 1$ for backward Euler method.

and $f(t, x) = -2t^2 + 2t(x^2 - x)$. The multiscale homogenized problem has the exact solution

$$u_0(t, x) = t^2(x^2 - x)$$

and

$$u_1(t, x, y) = \frac{t^2}{2\pi} x \sin(2\pi y).$$

We plot the numerical errors $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}((D))}$ and $\|u_1 - \hat{u}_1^L\|_{L^4(D,W^{1,4}_\#(Y)/\mathbb{R})}$ for $t = 1$ in Figures 5 and 6 respectively for the backward Euler method, using sparse tensor product FE spaces. The numerical results show that the error behaves like $O(\Delta t) + O(h_L)$.

The numerical results for the Crank-Nicholson scheme are plotted in Figures 7 and 8. The numerical results show that the error behaves like $O((\Delta t)^2) + O(h_L)$.

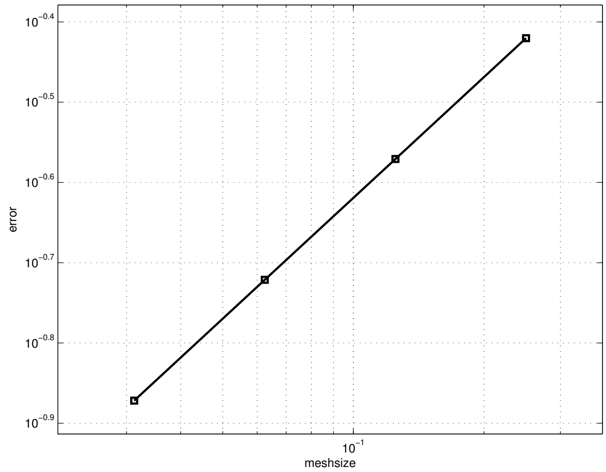

Figure 3: The error $\|u_0 - \hat{u}_0^L\|_{H_0^1(D)}$ versus the mesh size h for one dimensional problem (5.1) at $t = 1$ for Crank-Nicholson method.

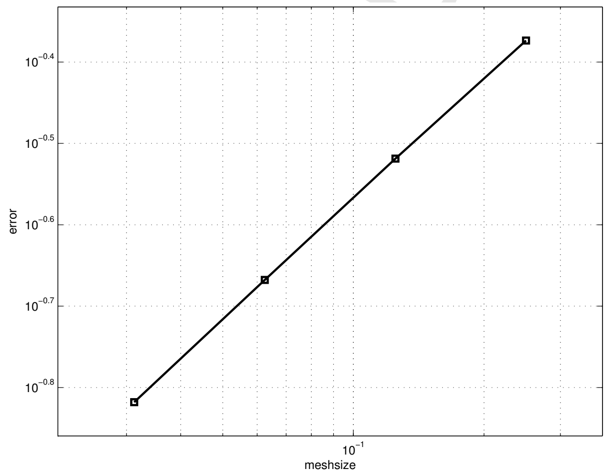

Figure 4: The error $\|u_1 - \hat{u}_1^L\|_{L^2(D,H^1_\#(Y)/\mathbb{R})}$ versus the mesh size h for one dimensional problem (5.1) at $t = 1$ for Crank-Nicholson method.

For the two dimensional domain $D = (0,1)\times(0,1)$, we consider the problem for $p = 4$ with coefficient

$$\begin{aligned} & A(t,x,y,\xi) \\ = \; & \xi' + \xi'' + \frac{1}{10}(2+\sin(2\pi y''))\xi'^3 + \frac{1}{10}(2+\sin(2\pi y'))\xi''^3 \\ & -t^2(x'+x'')\big(\cos(2\pi y')\sin(2\pi y'') + \sin(2\pi y')\cos(2\pi y'')\big) \\ & -\frac{t^6}{10}(2+\sin(2\pi y''))\big((1-2x')(x''-x''^2) \\ & \qquad\qquad + (x'+x'')\cos(2\pi y')\sin(2\pi y'')\big)^3 \\ & -\frac{t^6}{10}(2+\sin(2\pi y'))\big((x'-x'^2)(1-2x'') \\ & \qquad\qquad + (x'+x'')\sin(2\pi y')\cos(2\pi y'')\big)^3, \end{aligned} \tag{5.3}$$

and

$$f(t,x) = 2t^2(x'+x''-x'^2-x''^2) + 2t(x'-x'^2)(x''-x''^2)$$

with $x = (x', x'') \in D$, $y = (y', y'') \in Y$ and $\xi = (\xi', \xi'') \in \mathbb{R}^2$. The multiscale homogenized problem has

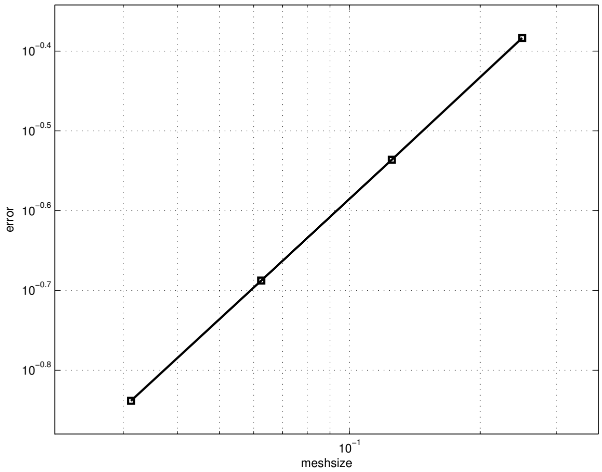

Figure 5: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for one dimensional problem (5.2) at $t = 1$ for backward Euler method.

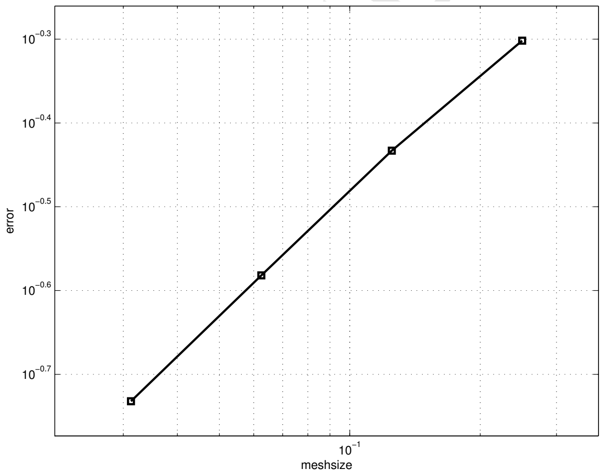

Figure 6: The error $\|u_1 - \hat{u}_1^L\|_{L^4(D,W_\#^{1,4}(Y)/\mathbb{R})}$ versus the mesh size h for one dimensional problem (5.2) at $t = 1$ for backward Euler method.

the exact solution

$$u_0(t,x) = t^2(x' - x'^2)(x'' - x''^2)$$

and

$$u_1(t,x,y) = \frac{t^2}{2\pi}(x' + x'')\sin(2\pi y')\sin(2\pi y'')$$

For $t = 1$, we plot the errors $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ and $\|u_1 - \hat{u}_1^L\|_{L^4(D,W_\#^{1,4}(Y)/\mathbb{R})}$ for the backward Euler method with sparse tensor product FE spaces in Figures 9 and 10 respectively. The errors for the Crank-Nicholson method for sparse tensor product FE spaces are plotted in Figures 11 and 12. Once more the results show that the error behaves like $O(\Delta t) + O(h_L)$ and $O((\Delta t)^2) + O(h_L)$ for the backward Euler and Crank-Nicholson method respectively.

The results for the one dimensional and two dimensional cases for $p = 4$ show that the observed numerical errors are of the optimal orders of Δt and h_L which are far better than what we can show theoretically. This is in agreement with the well known facts for numerical solutions of monotone problems (see [25]).

We now consider a one dimensional three scale problem on $D = (0,1)$ for $p = 4$ where the nonlinear

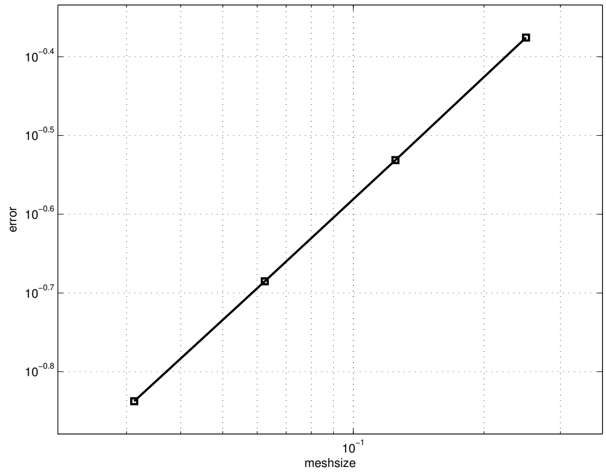

Figure 7: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for one dimensional problem (5.2) at $t = 1$ for Crank-Nicholson method.

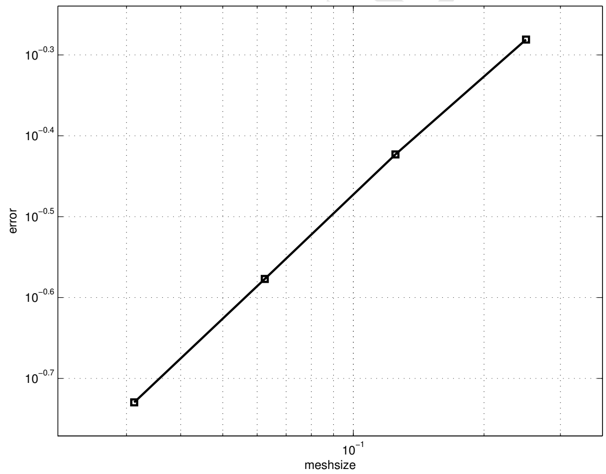

Figure 8: The error $\|u_1 - \hat{u}_1^L\|_{L^4(D,W_\#^{1,4}(Y)/\mathbb{R})}$ versus the mesh size h for one dimensional problem (5.2) at $t = 1$ for Crank-Nicholson method.

monotone function

$$
\begin{aligned}
A(t,x,y_1,y_2,\xi) = \xi + \frac{1}{10}(4+\sin(2\pi y_1)+\sin(2\pi y_2))\xi^3 \\
t^2(x\cos(2\pi y_1) + x\sin(2\pi y_1)\cos(2\pi y_2)) \\
-\frac{t^6}{10}(4+\sin(2\pi y_1)+\sin(2\pi y_2))(2x-1+x\cos(2\pi y_1)+x\sin(2\pi y_1)\cos(2\pi y_2))^3 \qquad (5.4)
\end{aligned}
$$

and $f = 2t(x^2 - x) - 2t^2$. The multiscale homogenized equation has the solution

$$u_0(t,x) = t^2(x^2 - x), \quad u_1(t,x,y_1) = \frac{t^2}{2\pi}x\sin(2\pi y_1)$$

and

$$u_2(t,x,y_1,y_2) = \frac{t^2}{2\pi}x\sin(2\pi y_1)\sin(2\pi y_2).$$

We solve the multiscale homogenized equation with both the backward Euler and the Crank- Nicholson methods using sparse tensor FE product spaces. The results are as proved theoretically. In Figures 13,

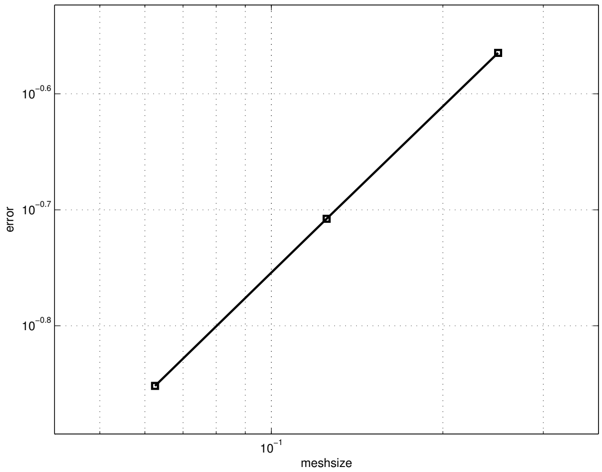

Figure 9: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for two dimensional problem (5.3) at $t = 1$ for backward Euler method.

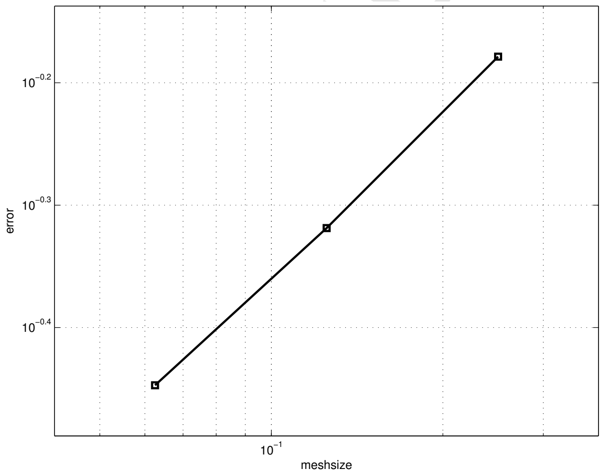

Figure 10: The error $\|u_1 - \hat{u}_1^L\|_{L^4(D,W_\#^{1,4}(Y)/\mathbb{R})}$ versus the mesh size h for two dimensional problem (5.3) at $t = 1$ for backward Euler method.

14 and 15 we present the numerical results for the backward Euler method. The figures show that the errors are as predicted.

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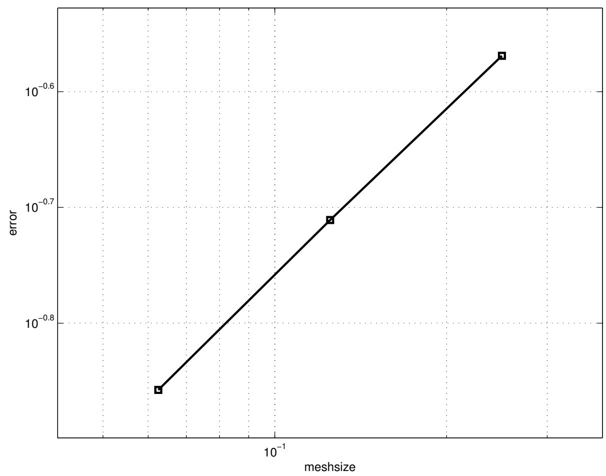

Figure 11: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for two dimensional problem (5.3) at $t = 1$ for Crank-Nicholson method.

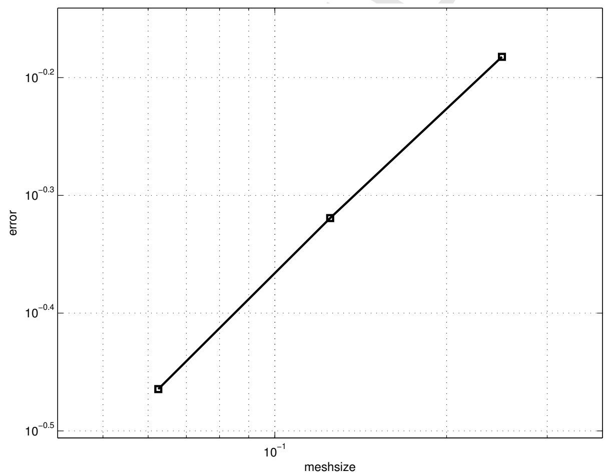

Figure 12: The error $\|u_1 - \hat{u}_1^L\|_{L^4(D,W_{\#}^{1,4}(Y)/\mathbb{R})}$ versus the mesh size h for two dimensional problem (5.3) at $t = 1$ for Crank-Nicholson method.

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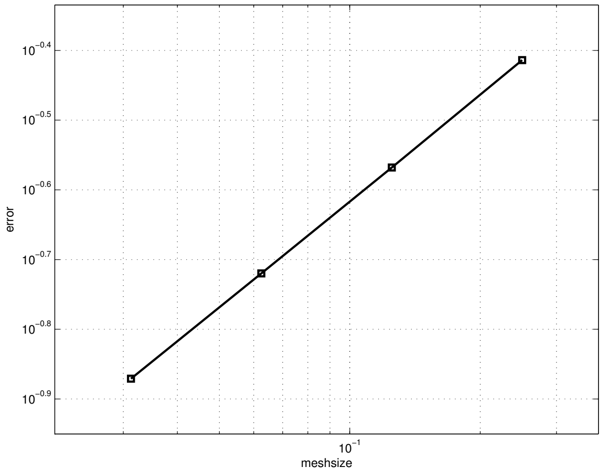

Figure 13: The error $\|u_0 - \hat{u}_0^L\|_{W_0^{1,4}(D)}$ versus the mesh size h for one dimensional three scale problem (5.4) at $t = 1$ for Crank-Nicholson method.

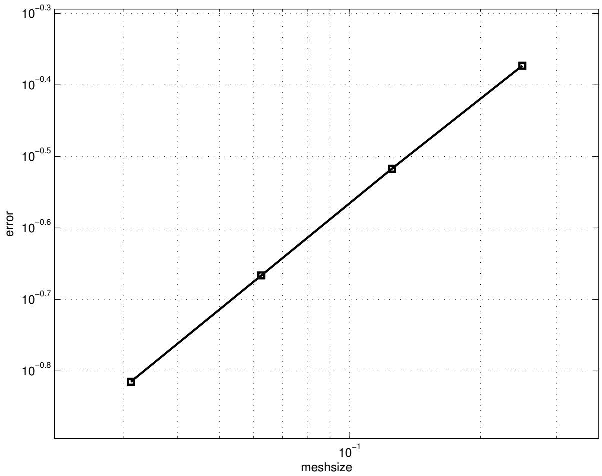

Figure 14: The error $\|u_1 - \hat{u}_1^L\|_{L^4(D,W_\#^{1,4}(Y)/\mathbb{R})}$ versus the mesh size h for one dimensional three scale problem (5.4) at $t = 1$ for Crank-Nicholson method.

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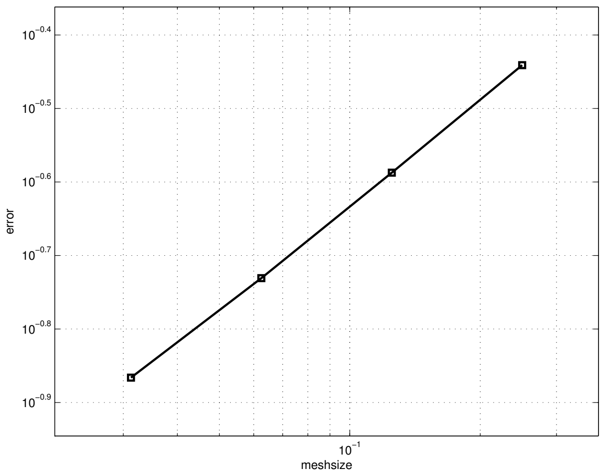

Figure 15: The error $\|u_2 - \hat{u}_2^L\|_{L^4(D,L^4(Y_1,W^{1,4}_{\#}(Y_2)/\mathbb{R}))}$ versus the mesh size h for one dimensional three scale problem (5.4) at $t = 1$ for Crank-Nicholson method.

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