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On penalty method for unilateral contact problem with non-monotone contact condition[☆]

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ABSTRACT

In this paper, we consider a penalty based numerical method to solve a model contact problem with unilateral constraint that is described by a constrained stationary hemivariational inequality. The penalty technique is applied to approximately enforce the constraint condition, and a corresponding numerical method using finite elements is introduced. We show the convergence of the penalty based numerical solutions to the solution of the constrained hemivariational inequality as both the mesh-size and the penalty parameter approach zero.

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1. Introduction

It is important to study contact problems due to their wide applications in industry and engineering. The mathematical formulations of the problems are usually stated in the form of variational inequalities and hemivariational inequalities. Comprehensive references on variational inequalities in contact mechanics include [1,2]. When material properties and contact conditions involve non-monotone and possibly multi-valued relations, contact problems are formulated as more complicated hemivariational inequalities. A comprehensive reference on hemivariational inequalities in contact mechanics is [3].

Hemivariational inequalities were first introduced in early 1980s by Panagiotopoulos in the context of applications in engineering problems involving non-monotone and possibly multi-valued constitutive or interface laws for deformable bodies. Studies of hemivariational inequalities can be found in several comprehensive references, e.g., [4,5], and more recently, [3]. The book [6] is devoted to the finite element approximations of hemivariational inequalities, where convergence of the numerical methods is discussed. In the recent years, there have been efforts to derive error estimates. In the literature, [7] represents the first paper that provides an optimal order error estimate for the linear finite element

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method in solving hemivariational or variational–hemivariational inequalities. The idea of the derivation technique in [7] was adopted by various authors for deriving optimal order error estimates for the linear finite element method of a few individual hemivariational or variational–hemivariational inequalities. More recently, general frameworks were developed on convergence and error estimation for hemivariational or variational–hemivariational inequalities, see [8,9] for internal numerical approximations of general hemivariational and variational–hemivariational inequalities, and [10] for both internal and external numerical approximations of general hemivariational and variational–hemivariational inequalities. In these recent papers, convergence is shown for numerical solutions by internal or external approximation schemes under minimal solution regularity condition, Céa type inequalities are derived that serve as the starting point for error estimation, and optimal order error estimates for the linear finite element solutions are derived for hemivariational and variational–hemivariational inequalities arising in contact mechanics.

Penalty methods have been used as an approximation tool to treat constraints in variational inequalities [1,11,12] and in proving solution existence [13,14]. In a penalty method, the constraint is approximately satisfied through the inclusion of a penalty term with a small penalty parameter δ and the constraint is enforced by taking the limit $\delta \to 0$. In this paper, we take a unilateral contact problem with a non-monotone contact condition as an example to show the convergence of a penalty based finite element method as both the penalty parameter and the finite element mesh-size tend to zero. We introduce the contact problem and its weak formulation in Section 2. In Section 3, we present a numerical method for the contact problem based on a finite element discretization of the corresponding penalty approximation. In Section 4, we prove the convergence of the penalty based numerical method.

2. The contact problem with unilateral constraint

In this section, we introduce the classical formulation of the model contact problem, list assumptions on the data, and present the corresponding weak formulation.

The contact problem concerns the deformation of an elastic material subject to the action of body and boundary forces, is clamped on part of its surface and is in contact with a foundation on another part of its surface. We denote by Ω the reference configuration of the elastic material, and assume Ω is an open, bounded, connected set in $\mathbb{R}^d$ ($d = 2$ or 3) with a Lipschitz boundary $\Gamma = \partial\Omega$. To describe the boundary conditions, we split the boundary Γ into disjoint, measurable parts Γ_1, Γ_2 and Γ_3. Here, $\mathrm{meas}\,(\Gamma_1) > 0$ and $\mathrm{meas}\,(\Gamma_3) > 0$, whereas Γ_2 is allowed to be empty.

To describe the contact problem, we need some notations. We use $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ for the displacement field of the elastic material and use $\boldsymbol{v}\colon \Omega \to \mathbb{R}^d$ for an arbitrary virtual displacement. We only consider small deformations and thus will use the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)/2$. The linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ and the stress tensor $\boldsymbol{\sigma}$ are $\mathbb{S}^d$-valued functions defined on $\overline{\Omega}$. The space $\mathbb{S}^d$ consists of all second order symmetric tensors on $\mathbb{R}^d$. We use the symbols "$\cdot$" and "$\|\cdot\|$" for the canonical inner product and norm over both the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$. Since the boundary Γ is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ is defined a.e. on Γ. For a vector field $\boldsymbol{v}$, its normal and tangential components on the boundary are $v_\nu := \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau := \boldsymbol{v} - v_\nu\boldsymbol{\nu}$, respectively. For the stress field $\boldsymbol{\sigma}$, its normal and tangential components on the boundary are $\sigma_\nu := (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively.

We now list the relations for the classical formulation of the contact problem. The constitutive law of the elastic material is

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{2.1}$$

where $\mathcal{F}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the elasticity operator. We assume the body is under the action of a body force of density $\boldsymbol{f}_0$ and put down the equilibrium equation

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega. \tag{2.2}$$

The material is fixed on Γ_1, and so

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1. \tag{2.3}$$

If Γ_2 is non-empty, then the material is subject to a surface traction of density $\boldsymbol{f}_2$:

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2. \tag{2.4}$$

Over Γ_3, we consider a frictionless unilateral contact condition:

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } \Gamma_3, \tag{2.5}$$

$$u_\nu \le g,\ \ \sigma_\nu + \xi_\nu \le 0,\ \ (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0,\ \ \xi_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_3. \tag{2.6}$$

The equality (2.5) reflects the fact that the contact is frictionless. The condition (2.6) models the contact between the elastic material and a foundation that consists of two parts: a rigid body and a layer of deformable material on its surface. The thickness of the layer is described by the non-negative function g. Penetration of the elastic material to the foundation is allowed but is limited by the unilateral constraint $u_\nu \le g$. At points where $u_\nu < g$, the contact is described by a multi-valued normal compliance condition $-\sigma_\nu = \xi_\nu \in \partial j_\nu(u_\nu)$, ∂j_ν being the Clarke subdifferential of a potential functional j_ν that will be assumed to be locally Lipschitz continuous. We recall that for a locally Lipschitz continuous function $\psi : X \to \mathbb{R}$ defined on a normed space X, the generalized (Clarke) directional derivative of ψ at $x \in X$ in the direction $v \in X$ is

$$\psi^0(x; v) := \limsup_{y\to x,\, \lambda\downarrow 0} \frac{\psi(y+\lambda v)-\psi(y)}{\lambda}.$$

The generalized subdifferential of ψ at x is a subset of the dual space X^* given by

$$\partial\psi(x) := \{\, \zeta \in X^* \mid \psi^0(x; v) \ge \langle \zeta, v\rangle_{X^*\times X}\ \forall\, v \in X \,\}.$$

Properties of the Clarke subdifferential can be found in the books [3,4,15,16]. In particular, we will use the relation

$$\psi^0(x; v) = \max\{\langle \zeta, v\rangle_{X^*\times X} \mid \zeta \in \partial\psi(x)\}, \tag{2.7}$$

as well as the upper semi-continuity of ψ^0:

$$x_n \to x \text{ and } v_n \to v \quad \Longrightarrow \quad \limsup_{n\to\infty} \psi^0(x_n; v_n) \le \psi^0(x; v). \tag{2.8}$$

The classical formulation of the contact problem consists of the relations (2.1)–(2.6). For the weak formulation of the contact problem, we need to make assumptions on the data and introduce function spaces. We assume the elasticity operator $\mathcal{F}: \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ has the following properties:

$$\left\{\begin{array}{l}
\text{(a) there exists } L_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega,\\
\qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|;\\
\text{(b) there exists } m_{\mathcal{F}} > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega,\\
\qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2;\\
\text{(c) } \mathcal{F}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d;\\
\text{(d) } \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{0} \text{ for a.e. } \boldsymbol{x} \in \Omega.
\end{array}\right. \tag{2.9}$$

If $\mathcal{F}$ is a linear operator, (2.1) represents the constitutive law of linearly elastic materials,

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\boldsymbol{u}),$$

where σ_{ij} are the components of the stress tensor $\boldsymbol{\sigma}$ and a_{ijkl} are the components of the elasticity tensor $\mathcal{F}$. Clearly, assumption (2.9) is satisfied in this particular case, if all the components a_{ijkl} belong to $L^\infty(\Omega)$ and satisfy the usual properties of symmetry and ellipticity:

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad 1 \le i, j, k, l \le d$$

and there exists $m_{\mathcal{F}} > 0$ such that

$$a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \ge m_{\mathcal{F}} \|\boldsymbol{\varepsilon}\|^2 \quad \forall\, \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathbb{S}^d.$$

In particular, for an isotropic linearized elastic material, the constitutive law is

$$\boldsymbol{\sigma} = \lambda\, \mathrm{tr}\boldsymbol{\varepsilon}(\boldsymbol{u}) \boldsymbol{I}_d + 2\,\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}),$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients, and $\boldsymbol{I}_d \in \mathbb{S}^d$ is the identity tensor.

An example of a nonlinear elastic constitutive law of the form (2.1) is

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \beta\,(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{P}_K \boldsymbol{\varepsilon}(\boldsymbol{u})). \tag{2.10}$$

Here $\mathcal{E}$ is a linear or nonlinear operator which satisfies (2.9), $\beta > 0$, K is a closed convex subset of $\mathbb{S}^d$ such that $\boldsymbol{0} \in K$ and $\mathcal{P}_K : \mathbb{S}^d \to K$ denotes the projection operator. The corresponding elasticity operator is nonlinear and is given by

$$\mathcal{F}\boldsymbol{\varepsilon} = \mathcal{E}\boldsymbol{\varepsilon} + \beta\,(\boldsymbol{\varepsilon} - \mathcal{P}_K \boldsymbol{\varepsilon}). \tag{2.11}$$

Usually the set K is defined by

$$K = \{\, \boldsymbol{\varepsilon} \in \mathbb{S}^d \mid \mathcal{F}(\boldsymbol{\varepsilon}) \le 0 \,\} \tag{2.12}$$

where $\mathcal{F} : \mathbb{S}^d \to \mathbb{R}$ is a convex continuous function such that $\mathcal{F}(\boldsymbol{0}) < 0$.

Another example of nonlinear elasticity operators satisfying the conditions (2.9) is provided by a Hencky material, see [2] for details. For the Hencky material, the constitutive law is

$$\boldsymbol{\sigma} = k_0\,(\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I}_d + \psi(\|\boldsymbol{\varepsilon}^D(\boldsymbol{u})\|^2)\,\boldsymbol{\varepsilon}^D(\boldsymbol{u}),$$

so that the elasticity operator is

$$\mathcal{F}(\boldsymbol{\varepsilon}) = k_0\,(\mathrm{tr}\,\boldsymbol{\varepsilon})\boldsymbol{I}_d + \psi(\|\boldsymbol{\varepsilon}^D\|^2)\,\boldsymbol{\varepsilon}^D. \tag{2.13}$$

Here, $k_0 > 0$ is a material coefficient, $\boldsymbol{I}_d$ is the identity matrix of order d, $\psi : \mathbb{R} \to \mathbb{R}$ is a constitutive function and $\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon}^D(\boldsymbol{u})$ denotes the deviatoric part of $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$, defined by

$$\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon} - \frac{1}{d}(\mathrm{tr}\,\boldsymbol{\varepsilon})\boldsymbol{I}_d. \tag{2.14}$$

The function ψ is assumed to be piecewise continuously differentiable, and there exist positive constants c_1, c_2, d_1 and d_2, such that

$$\psi(\xi) \le d_1, \quad -c_1 \le \psi'(\xi) \le 0, \quad c_2 \le \psi(\xi) + 2\,\psi'(\xi)\xi \le d_2$$

for all $\xi \ge 0$. The conditions (2.9) are satisfied for the elasticity operator defined in (2.13), see for instance [2, p. 125].

On the densities of the body force and the surface traction, we assume

$$\boldsymbol{f}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \boldsymbol{f}_2 \in L^2(\Gamma_2; \mathbb{R}^d). \tag{2.15}$$

On the thickness function g, we assume

$$g \in L^2(\Gamma_3), \quad g(\boldsymbol{x}) \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{2.16}$$

On the potential functional j_ν, we assume

$$\left\{\begin{array}{l} \text{(a)}\, j_\nu(\cdot, r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } \bar{e} \in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot, \bar{e}(\cdot)) \in L^1(\Gamma_3); \\ \text{(b)}\, j_\nu(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \boldsymbol{x} \in \Gamma_3; \\ \text{(c)}\, |\partial j_\nu(\boldsymbol{x}, r)| \le \bar{c}_0 + \bar{c}_1|r| \text{ for a.e. } \boldsymbol{x} \in \Gamma_3\ \forall\, r \in \mathbb{R} \text{ with } \bar{c}_0,\ \bar{c}_1 \ge 0; \\ \text{(d)}\, j_\nu^0(\boldsymbol{x}, r_1; r_2 - r_1) + j_\nu^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_{j_\nu}|r_1 - r_2|^2 \\ \qquad \text{for a.e. } \boldsymbol{x} \in \Gamma_3,\ \text{all } r_1, r_2 \in \mathbb{R} \text{ with } \alpha_{j_\nu} \ge 0. \end{array}\right. \tag{2.17}$$

Here, the relation

$$|\partial j_\nu(\boldsymbol{x}, r)| \le \bar{c}_0 + \bar{c}_1|r|$$

means

$$|\xi| \le \bar{c}_0 + \bar{c}_1|r| \quad \forall\, \xi \in \partial j_\nu(\boldsymbol{x}, r).$$

Let $a \ge 0$, $b > 0$ and consider the function $j_\nu : \mathbb{R} \to \mathbb{R}$ given by

$$j_\nu(r) = \left\{\begin{array}{ll} 0 & \text{if } r < 0, \\ \dfrac{a + e^{-b}}{2b}\, r^2 & \text{if } 0 \le r \le b, \\ ar - e^{-r} + \dfrac{(b+2)e^{-b} - ab}{2} & \text{if } r > b. \end{array}\right. \tag{2.18}$$

It is easy to see that the function j_ν satisfies assumption (2.17) with $\bar{c}_0 = a + e^{-b}$ and $\bar{c}_1 = 0$. Moreover,

$$\partial j_\nu(r) = \left\{\begin{array}{ll} 0 & \text{if } r < 0, \\ \dfrac{a + e^{-b}}{b}\, r & \text{if } 0 \le r \le b, \\ e^{-r} + a & \text{if } r > b, \end{array}\right. \tag{2.19}$$

for all $r \in \mathbb{R}$. In this case the condition $-\sigma_\nu = \xi_\nu \in \partial j_\nu(u_\nu)$ models the contact with an elastic layer, with softening. The softening effect consists in the fact that, when the penetration reach the limit b, then the reactive force decreases. Additional examples of contact conditions of this form in which the function j_ν satisfies condition (2.17), together with the corresponding mechanical interpretations, can be found in [3,17].

Let

$$V = \left\{ \boldsymbol{v} = (v_i) \in H^1(\Omega; \mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1 \right\}. \tag{2.20}$$

The displacement will be sought from the following subset of the space V:

$$U := \{ \boldsymbol{v} \in V \mid v_\nu \le g \text{ on } \Gamma_3 \}. \tag{2.21}$$

Note that since $\mathrm{meas}\,(\Gamma_1) > 0$, V is a Hilbert space with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad \boldsymbol{u}, \boldsymbol{v} \in V.$$

We denote the trace of a function $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d)$ on Γ by the same symbol $\boldsymbol{v}$. We use the space $Q = L^2(\Omega; \mathbb{S}^d)$ for the stress and strain fields; Q is a Hilbert space with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q := \int_\Omega \sigma_{ij}(\boldsymbol{x})\, \tau_{ij}(\boldsymbol{x})\, dx, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in Q.$$

To simplify the notation, we define $\boldsymbol{f} \in V^*$ by

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^* \times V} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^d)} + (\boldsymbol{f}_2, \boldsymbol{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\, \boldsymbol{v} \in V. \tag{2.22}$$

By a standard approach (cf. [2,3]), the following weak formulation of the contact problem (2.1)–(2.6) can be derived.

Problem (P)**.** Find a displacement field $\boldsymbol{u} \in U$ such that

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\, ds \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u} \rangle_{V^* \times V} \quad \forall\, \boldsymbol{v} \in U. \tag{2.23}$$

It is known (cf. [8,14]) that Problem (P) has a unique solution $\boldsymbol{u} \in U$ under the stated assumptions (2.9)–(2.17) and the smallness condition

$$\alpha_{j_\nu} < \lambda_{1\nu,V}\, m_{\mathcal{F}}, \tag{2.24}$$

where $\lambda_{1\nu,V} > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\boldsymbol{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = \lambda \int_{\Gamma_3} u_\nu v_\nu ds \quad \forall\, \boldsymbol{v} \in V.$$

3. A penalty based numerical method

We now introduce a penalty based numerical method of Problem (P). For a small parameter $\delta > 0$, known as the penalty parameter, we consider the following penalty approximation of Problem (P).

Problem (P_δ)**.** Find a displacement field $\boldsymbol{u}_\delta \in V$ such that

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_\delta)\, dx + \frac{1}{\delta} \int_{\Gamma_3} (u_{\delta,\nu} - g)_+ (v_\nu - u_{\delta,\nu})\, ds$$
$$+ \int_{\Gamma_3} j_\nu^0(u_{\delta,\nu}; v_\nu - u_{\delta,\nu})\, ds \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\delta \rangle_{V^* \times V} \quad \forall\, \boldsymbol{v} \in V. \tag{3.1}$$

We use the finite element method for the numerical solution of Problem (P_δ). For brevity, assume Ω is a polygonal/polyhedral domain and express the three parts of the boundary, Γ_k, $1 \le k \le 3$, as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_k} = \cup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \le k \le 3.$$

Let $\{\mathcal{T}^h\}_{h>0}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{k,i}$, $1 \le i \le i_k$, $1 \le k \le 3$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{k,i}$ has a positive measure with respect to $\Gamma_{k,i}$, then the side/face lies entirely in $\Gamma_{k,i}$. Construct the linear element space corresponding to $\mathcal{T}^h = \{T\}$:

$$V^h = \left\{ \boldsymbol{v}^h \in C(\overline{\Omega})^d \mid \boldsymbol{v}^h|_T \in \mathbb{P}_1(T)^d,\ T \in \mathcal{T}^h,\ \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \Gamma_1 \right\},$$

where $\mathbb{P}_1(T)$ is the space of linear functions defined on T.

We note the following property of the finite element spaces:

$$\forall\, \boldsymbol{v} \in V,\ \exists\, \boldsymbol{v}^h \in V^h \text{ s.t. } \lim_{h \to 0} \|\boldsymbol{v}^h - \boldsymbol{v}\|_V = 0. \tag{3.2}$$

This property can be shown by the standard finite element interpolation error estimates for smooth functions (cf. [18–20]) combined with a density argument, noting that smooth functions are dense in V.

A further finite element approximation property will be needed:

$$\forall\, \boldsymbol{v} \in U,\ \exists\, \boldsymbol{v}^h \in V^h \cap U \text{ s.t. } \lim_{h \to 0} \|\boldsymbol{v}^h - \boldsymbol{v}\|_V = 0. \tag{3.3}$$

One set of sufficient conditions for (3.3) are (i) $C^\infty(\overline{\Omega})^d \cap U$ is dense in U and (ii) the function g is concave. A sketch of this statement is as follows. Let $\boldsymbol{v} \in U$ and let $\varepsilon > 0$ be arbitrarily small. Then there exists $\boldsymbol{v}_\varepsilon \in C^\infty(\overline{\Omega})^d \cap U$ such that $\|\boldsymbol{v}_\varepsilon - \boldsymbol{v}\|_V \le \varepsilon/2$. Let $\boldsymbol{v}_\varepsilon^h \in V^h$ be the finite element interpolant of $\boldsymbol{v}_\varepsilon$; $\boldsymbol{v}_\varepsilon^h \in V^h \cap U$ since $\boldsymbol{v}_\varepsilon \in U$ and g is concave. By the finite element interpolation theory, $\|\boldsymbol{v}_\varepsilon^h - \boldsymbol{v}_\varepsilon\|_V \le c\, h\, \|\boldsymbol{v}_\varepsilon\|_{H^2(\Omega)^d}$. Let $h > 0$ be sufficiently small so that $c\, h\, \|\boldsymbol{v}_\varepsilon\|_{H^2(\Omega)^d} \le \varepsilon/2$. Then, $\|\boldsymbol{v}_\varepsilon^h - \boldsymbol{v}\|_V \le \varepsilon$.

We comment that in [21] the following density result is shown: Assume Ω is a Lipschitz planar domain whose boundary is split into three mutually disjoint parts $\overline{\Gamma_1}$, $\overline{\Gamma_2}$, and $\overline{\Gamma_3}$, such that $\overline{\Gamma_i} \cap \overline{\Gamma_j}$ consists of a finite number of points for $1 \le i < j \le 3$. Then $C^\infty(\overline{\Omega})^d \cap U$ is dense in U in the case $g = 0$. In particular, with our assumptions on Ω and its boundary splitting, this density result can be applied. In addition, the proof method in [21] can be extended to cover the 3D case of a polyhedral domain Ω. Moreover, for g a non-zero constant, $\overline{\Gamma_1} \cap \overline{\Gamma_3} = \emptyset$ and Γ_3 a line segment in 2D or a polygon in 3D, it is easy to see that the density statement of $C^\infty(\overline{\Omega})^d \cap U$ in U carries over and (3.3) holds.

The penalty based numerical method for Problem (P) is as follows.

Problem (P^h_δ)**.** Find a displacement field $\boldsymbol{u}^h_\delta \in V^h$ such that

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}^h_\delta)\, dx + \frac{1}{\delta} \int_{\Gamma_3} (u^h_{\delta,\nu} - g)_+ (v^h_\nu - u^h_{\delta,\nu})\, ds$$
$$+ \int_{\Gamma_3} j^0_\nu(u^h_{\delta,\nu}; v^h_\nu - u^h_{\delta,\nu})\, ds \ge \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}^h_\delta \rangle_{V^*\times V} \quad \forall\, \boldsymbol{v}^h \in V^h. \tag{3.4}$$

We will focus on the convergence of the numerical method as both the penalty parameter and the finite element mesh-size tend to zero:

$$\boldsymbol{u}^h_\delta \to \boldsymbol{u} \quad \text{in } V, \ \text{ as } h, \delta \to 0.$$

As a preparation, we first prove a uniform boundedness property of the numerical solutions.

Proposition 3.1. *Let $\boldsymbol{u}^h_\delta \in V^h$ be the solution of Problem (P^h_δ). Then for some constant $M > 0$ independent of h and δ, we have $\|\boldsymbol{u}^h_\delta\|_V \le M$.*

Proof. By (2.9)(b) and (2.9)(d), we have

$$m_{\mathcal{F}} \|\boldsymbol{u}^h_\delta\|_V^2 \le \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta)\, dx.$$

From (3.4) with $\boldsymbol{v}^h = \boldsymbol{0}$,

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta)\, dx \le -\frac{1}{\delta} \int_{\Gamma_3} (u^h_{\delta,\nu} - g)_+ u^h_{\delta,\nu} ds + \int_{\Gamma_3} j^0_\nu(u^h_{\delta,\nu}; -u^h_{\delta,\nu})\, ds + \langle \boldsymbol{f}, \boldsymbol{u}^h_\delta \rangle_{V^*\times V}.$$

Thus,

$$m_{\mathcal{F}} \|\boldsymbol{u}^h_\delta\|_V^2 \le -\frac{1}{\delta} \int_{\Gamma_3} (u^h_{\delta,\nu} - g)_+ u^h_{\delta,\nu} ds + \int_{\Gamma_3} j^0_\nu(u^h_{\delta,\nu}; -u^h_{\delta,\nu})\, ds + \langle \boldsymbol{f}, \boldsymbol{u}^h_\delta \rangle_{V^*\times V}. \tag{3.5}$$

Note that

$$-\frac{1}{\delta} \int_{\Gamma_3} (u^h_{\delta,\nu} - g)_+ u^h_{\delta,\nu} ds \le 0 \tag{3.6}$$

since

$$-(u^h_{\delta,\nu} - g)_+ u^h_{\delta,\nu} = -(u^h_{\delta,\nu} - g)_+ (u^h_{\delta,\nu} - g) - (u^h_{\delta,\nu} - g)_+ g \le 0.$$

By (2.17)(d),

$$j^0_\nu(u^h_{\delta,\nu}; -u^h_{\delta,\nu}) \le \alpha_{j_\nu} |u^h_{\delta,\nu}|^2 - j^0_\nu(0; u^h_{\delta,\nu}),$$

and by (2.17)(c) and (2.7),

$$|j^0_\nu(0; u^h_{\delta,\nu})| \le \overline{c}_0 |u^h_{\delta,\nu}|;$$

thus,

$$j^0_\nu(u^h_{\delta,\nu}; -u^h_{\delta,\nu}) \le \alpha_{j_\nu} |u^h_{\delta,\nu}|^2 + \overline{c}_0 |u^h_{\delta,\nu}|. \tag{3.7}$$

Moreover,

$$\langle \boldsymbol{f}, \boldsymbol{u}^h_\delta \rangle_{V^*\times V} \le c \left(\|\boldsymbol{f}_0\|_{L^2(\Omega)^d} + \|\boldsymbol{f}_2\|_{L^2(\Gamma_2)^d} \right) \|\boldsymbol{u}^h_\delta\|_V. \tag{3.8}$$

Using (3.6)–(3.8) in (3.5), we find that

$$\left(m_{\mathcal{F}} - \alpha_{j_\nu} \lambda^{-1}_{1\nu,V} - \varepsilon \right) \|\boldsymbol{u}^h_\delta\|_V \le c.$$

Therefore, $\{\|\boldsymbol{u}^h_\delta\|_V\}_{h,\delta>0}$ is uniformly bounded with respect to h and δ. ■

4. Convergence of the numerical method

We now prove the convergence of the numerical solution of Problem (P^h_δ) to that of Problem (P) as the penalty parameter δ and the meshsize h tend to zero.

Theorem 4.1. *Assume (2.9)–(2.21), (2.24), and (3.3). Then,*

$$\boldsymbol{u}^h_\delta \to \boldsymbol{u} \quad \text{in } V \text{ as } h, \delta \to 0. \tag{4.1}$$

Proof. By Proposition 3.1, $\{\boldsymbol{u}_\delta^h\}_{h,\delta>0}$ is bounded in V. Since V is a Hilbert space, and the trace operator $V \ni \boldsymbol{v} \mapsto v_\nu \in L^2(\Gamma_3)$ is compact, we can find a sequence of $\{\boldsymbol{u}_\delta^h\}_{h,\delta>0}$, again denoted as $\{u_\delta^h\}_{h,\delta>0}$, and an element $\boldsymbol{w} \in V$ such that

$$\boldsymbol{u}_\delta^h \rightharpoonup \boldsymbol{w} \quad \text{in } V, \quad u_{\delta,\nu}^h \to w_\nu \quad \text{in } L^2(\Gamma_3). \tag{4.2}$$

Let us first show that $\boldsymbol{w} \in U$. By (3.4), for any $\boldsymbol{v}^h \in V^h$,

$$\frac{1}{\delta}\int_{\Gamma_3}(u_{\delta,\nu}^h - g)_+(u_{\delta,\nu}^h - v_\nu^h)\,ds \le \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta^h))\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{u}_\delta^h)\,dx + \int_{\Gamma_3} j_\nu^0(u_{\delta,\nu}^h; v_\nu^h - u_{\delta,\nu}^h)\,ds$$
$$- \langle \boldsymbol{f}, \boldsymbol{v}^h - \boldsymbol{u}_\delta^h\rangle_{V^*\times V}.$$

Fix $\boldsymbol{v}^h \in V^h$. Since $\{\|\boldsymbol{u}_\delta^h\|_V\}_{h,\delta\to 0}$ is bounded, there is a constant $c(\boldsymbol{v}^h)$, dependent on $\boldsymbol{v}^h$ but independent of δ, such that

$$\frac{1}{\delta}\int_{\Gamma_3}(u_{\delta,\nu}^h - g)_+(u_{\delta,\nu}^h - v_\nu^h)\,ds \le c(v^h).$$

Then, we deduce that

$$\limsup_{\delta\to 0}\int_{\Gamma_3}(u_{\delta,\nu}^h - g)_+(u_{\delta,\nu}^h - v_\nu^h)\,ds \le 0 \quad \forall\, \boldsymbol{v}^h \in V^h. \tag{4.3}$$

For any fixed $\boldsymbol{v} \in V$, by (3.2), there exists $\boldsymbol{v}^h \in V^h$ such that $\boldsymbol{v}^h \to \boldsymbol{v}$ in V. Since $\{\|u_{\delta\,\nu}^h\|_{L^2(\Gamma_3)}\}_{h,\delta>0}$ is bounded, we derive from (4.3) that

$$\limsup_{\delta\to 0}\int_{\Gamma_3}(u_{\delta,\nu}^h - g)_+(u_{\delta,\nu}^h - v_\nu)\,ds \le 0 \quad \forall\, \boldsymbol{v} \in V.$$

Since $u_{\delta,\nu}^h \to w_\nu$ in $L^2(\Gamma_3)$, we derive from the above that

$$\int_{\Gamma_3}(w_\nu - g)_+(w_\nu - v_\nu)\,ds \le 0 \quad \forall\, \boldsymbol{v} \in V.$$

Therefore,

$$\int_{\Gamma_3}(w_\nu - g)_+ v_\nu ds = 0 \quad \forall\, \boldsymbol{v} \in V$$

and then

$$(w_\nu - g)_+ = 0 \quad \text{a.e. on } \Gamma_3,$$

i.e., $w_\nu \le g$ a.e. on Γ_3. Thus, $\boldsymbol{w} \in U$.

Let us then prove the strong convergence

$$\boldsymbol{u}_\delta^h \to \boldsymbol{w} \quad \text{in } V.$$

By (3.3), there exists $\boldsymbol{w}^h \in V^h \cap U$ such that

$$\|\boldsymbol{w}^h - \boldsymbol{w}\|_V \to 0 \quad \text{as } h \to 0.$$

We take $\boldsymbol{v}^h = \boldsymbol{w}^h$ in (3.4) to get

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta^h))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta^h - \boldsymbol{w}^h)\,dx \le \frac{1}{\delta}\int_{\Gamma_3}(u_{\delta,\nu}^h - g)_+(w_\nu^h - u_{\delta,\nu}^h)\,ds + \int_{\Gamma_3} j_\nu^0(u_{\delta,\nu}^h; w_\nu^h - u_{\delta,\nu}^h)\,ds$$
$$- \langle \boldsymbol{f}, \boldsymbol{w}^h - \boldsymbol{u}_\delta^h\rangle_{V^*\times V}.$$

Since $w_\nu^h \le g$ on Γ_3,

$$\frac{1}{\delta}\int_{\Gamma_3}(u_{\delta,\nu}^h - g)_+(w_\nu^h - u_{\delta,\nu}^h)\,ds \le -\frac{1}{\delta}\int_{\Gamma_3}(u_{\delta,\nu}^h - g)_+(u_{\delta,\nu}^h - g)\,ds \le 0.$$

Hence,

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta^h))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta^h - \boldsymbol{w}^h)\,dx \le \int_{\Gamma_3} j_\nu^0(u_{\delta,\nu}^h; w_\nu^h - u_{\delta,\nu}^h)\,ds - \langle \boldsymbol{f}, \boldsymbol{w}^h - \boldsymbol{u}_\delta^h\rangle_{V^*\times V}. \tag{4.4}$$

By (2.17)(c) and (2.7),

$$|j_\nu^0(u_{\delta,\nu}^h; w_\nu^h - u_{\delta,\nu}^h)| \le \left(\bar{c}_0 + \bar{c}_1|u_{\delta,\nu}^h|\right)|w_\nu^h - u_{\delta,\nu}^h|.$$

Then,

$$\int_{\Gamma_3} j_\nu^0(u_{\delta,\nu}^h; w_\nu^h - u_{\delta,\nu}^h)\,ds \le c\left(1 + \|u_{\delta,\nu}^h\|_{L^2(\Gamma_3)}\right)\|w_\nu^h - u_{\delta,\nu}^h\|_{L^2(\Gamma_3)}$$

and so

$$\limsup_{h,\delta\to 0}\int_{\Gamma_3} j_\nu^0(u^h_{\delta,\nu};\, w^h_\nu - u^h_{\delta,\nu})\,ds \le 0.$$

Note that $\boldsymbol{w}^h - \boldsymbol{u}^h_\delta \rightharpoonup \boldsymbol{0}$ in V, implying

$$\lim_{h,\delta\to 0}\langle \boldsymbol{f}, \boldsymbol{w}^h - \boldsymbol{u}^h_\delta\rangle_{V^*\times V} = 0.$$

Hence, from (4.4), we derive that

$$\limsup_{h,\delta\to 0}\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta - \boldsymbol{w}^h)\,dx \le 0.$$

Since $\{\|\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta))\|_Q\}_{h,\delta>0}$ is bounded and $\boldsymbol{w}^h \to \boldsymbol{w}$ in V, the previous inequality implies

$$\limsup_{h,\delta\to 0}\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta - \boldsymbol{w})\,dx \le 0. \tag{4.5}$$

Apply the condition (2.9)(b),

$$\begin{aligned} m_{\mathcal{F}}\|\boldsymbol{u}^h_\delta - \boldsymbol{w}\|_V^2 &\le \int_\Omega \big(\mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta)) - \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{w}))\big)\cdot\big(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta) - \boldsymbol{\varepsilon}(\boldsymbol{w})\big)\,dx \\ &= \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta - \boldsymbol{w})\,dx - \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{w}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta - \boldsymbol{w})\,dx. \end{aligned}$$

By (4.5) and the weak convergence $\boldsymbol{u}^h_\delta \rightharpoonup \boldsymbol{w}$ in V, we find from the above inequality that

$$\limsup_{h,\delta\to 0} m_{\mathcal{F}}\|\boldsymbol{u}^h_\delta - \boldsymbol{w}\|_V^2 \le 0.$$

Therefore,

$$\|\boldsymbol{u}^h_\delta - \boldsymbol{w}\|_V \to 0 \quad \text{as } h,\,\delta \to 0,$$

and we have shown the strong convergence.

Finally, we show that the limit $\boldsymbol{w}$ is the solution of Problem (P). We fix an arbitrary element $\boldsymbol{v}^{h'} \in V^{h'} \cap U$. We consider (3.4) in a space $V^h \supset V^{h'}$ with $\boldsymbol{v}^h = \boldsymbol{v}^{h'}$:

$$\begin{aligned} \int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta - \boldsymbol{v}^{h'})\,dx \le \frac{1}{\delta}\int_{\Gamma_3}(u^h_{\delta,\nu} - g)_+(v^{h'}_\nu - u^h_{\delta,\nu})\,ds + \int_{\Gamma_3} j_\nu^0(u^h_{\delta,\nu};\, v^{h'}_\nu - u^h_{\delta,\nu})\,ds \\ - \langle \boldsymbol{f}, \boldsymbol{v}^{h'} - \boldsymbol{u}^h_\delta\rangle_{V^*\times V}. \end{aligned}$$

Since $\boldsymbol{v}^{h'} \in V^{h'} \cap U$,

$$\int_{\Gamma_3}(u^h_{\delta,\nu} - g)_+(v^{h'}_\nu - u^h_{\delta,\nu})\,ds \le -\int_{\Gamma_3}(u^h_{\delta,\nu} - g)_+(u^h_{\delta,\nu} - g)\,ds \le 0.$$

Thus,

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta - \boldsymbol{v}^{h'})\,dx \le \int_{\Gamma_3} j_\nu^0(u^h_{\delta,\nu};\, v^{h'}_\nu - u^h_{\delta,\nu})\,ds - \langle \boldsymbol{f}, \boldsymbol{v}^{h'} - \boldsymbol{u}^h_\delta\rangle_{V^*\times V}. \tag{4.6}$$

By the upper semi-continuity property (2.8),

$$\begin{aligned} \limsup_{h,\delta\to 0}\int_{\Gamma_3} j_\nu^0(u^h_{\delta,\nu};\, v^{h'}_\nu - u^h_{\delta,\nu})\,ds &\le \int_{\Gamma_3}\limsup_{h,\delta\to 0} j_\nu^0(u^h_{\delta,\nu};\, v^{h'}_\nu - u^h_{\delta,\nu})\,ds \\ &\le \int_{\Gamma_3} j_\nu^0(w_\nu;\, v^{h'}_\nu - w_\nu)\,ds. \end{aligned}$$

We take the upper limit as $h \to 0$ and $\delta \to 0$ along the subsequence of the spaces $V^h \supset V^{h'}$ in (4.6) to obtain

$$\limsup_{h,\delta\to 0}\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta))\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}^h_\delta - \boldsymbol{v}^{h'})\,dx \le \int_{\Gamma_3} j_\nu^0(w_\nu;\, v^{h'}_\nu - w_\nu)\,ds - \langle \boldsymbol{f}, \boldsymbol{v}^{h'} - \boldsymbol{w}\rangle_{V^*\times V}. \tag{4.7}$$

Since $\boldsymbol{u}^h_\delta$ converges to $\boldsymbol{w}$ strongly, we deduce from (4.7) that

$$\int_\Omega \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{w}))\cdot\boldsymbol{\varepsilon}(\boldsymbol{w} - \boldsymbol{v}^{h'})\,dx \le \int_{\Gamma_3} j_\nu^0(w_\nu;\, v^{h'}_\nu - w_\nu)\,ds - \langle \boldsymbol{f}, \boldsymbol{v}^{h'} - \boldsymbol{w}\rangle_{V^*\times V}.$$

The element $\boldsymbol{v}^{h'} \in V^{h'} \cap U$ being arbitrary, we use the density of $\{V^{h'} \cap U\}$ in U to obtain

$$\int_{\Omega} \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{w})) \cdot \boldsymbol{\varepsilon}(\boldsymbol{w} - \boldsymbol{v})\, dx \leq \int_{\Gamma_3} j_{\nu}^{0}(w_{\nu}; v_{\nu} - w_{\nu})\, ds - \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{w} \rangle_{V^* \times V} \quad \forall\, \boldsymbol{v} \in U. \tag{4.8}$$

There, $w = u$ is the unique solution of Problem (P). Since the solution of Problem (P) is unique, we conclude that the entire family $\{\boldsymbol{u}_{\delta}^{h}\}_{h,\delta>0}$ converges to $\boldsymbol{u}$ in V. ■

References

[1] N. Kikuchi, J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, SIAM, Philadelphia, 1988.
[2] W. Han, M. Sofonea, Quasistatic contact problems in viscoelasticity and viscoplasticity, in: Studies in Advanced Mathematics, vol. 30, Americal Mathematical Society, RI–International Press, Providence, Somerville, MA, 2002.
[3] S. Migórski, A. Ochal, M. Sofonea, Nonlinear inclusions and hemivariational inequalities. Models and analysis of contact problems, in: Advances in Mechanics and Mathematics, vol. 26, Springer, New York, 2013.
[4] Z. Naniewicz, P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.
[5] P.D. Panagiotopoulos, Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer-Verlag, Berlin, 1993.
[6] J. Haslinger, M. Miettinen, P.D. Panagiotopoulos, Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999.
[7] W. Han, S. Migórski, M. Sofonea, A class of variational-hemivariational inequalities with applications to frictional contact problems, SIAM J. Math. Anal. 46 (2014) 3891–3912.
[8] W. Han, M. Sofonea, M. Barboteu, Numerical analysis of elliptic hemivariational inequalities, SIAM J. Numer. Anal. 55 (2017) 640–663.
[9] W. Han, M. Sofonea, D. Danan, Numerical analysis of stationary variational-hemivariational inequalities, Numer. Math. 139 (2018) 563–592.
[10] W. Han, Numerical analysis of stationary variational-hemivariational inequalities with applications in contact mechanics, Math. Mech. Solids 23 (2018) 279–293.
[11] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer-Verlag, New York, 1984.
[12] R. Glowinski, J.-L. Lions, R. Trémolières, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.
[13] M. Sofonea, F. Patrulescu, Penalization of history-dependent variational inequalities, European J. Appl. Math. 25 (2014) 155–176.
[14] S. Migórski, A. Ochal, M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, J. Elasticity 127 (2017) 151–178.
[15] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, Interscience, New York, 1983.
[16] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.
[17] M. Sofonea, S. Migórski, Variational-hemivariational inequalities with applications, in: Pure and Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton-London, 2018.
[18] K. Atkinson, W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, third ed., Springer-Verlag, New York, 2009.
[19] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, third ed., Springer-Verlag, New York, 2008.
[20] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.
[21] I. Hlaváček, J. Lovíšek, A finite element analysis for the Signorini problem in plane elastostatics, Apl. Mat. 22 (1977) 215–227.