Journal of Computational and Applied Mathematics 223 (2009) 953–966

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A computer-assisted proof for the Kolmogorov flows of incompressible viscous fluid

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ARTICLE INFO

Article history:
Received 5 February 2007
Received in revised form 12 March 2008

MSC:
35Q30
65N15
76D03

Keywords:
Kolmogorov flows
Computer-assisted proof
Fixed-point theorem

ABSTRACT

A computer-assisted proof of non-trivial steady-state solutions for the Kolmogorov flows is described. The method is based on the infinite-dimensional fixed-point theorem using Newton-like operator. This paper also proposes a numerical verification algorithm which generates automatically on a computer a set including the exact non-trivial solution with local uniqueness. All discussed numerical results take into account the effects of rounding errors in the floating point computations.

1. Introduction

Consider the following Navier–Stokes equations:

$$u_t + uu_x + vu_y = \nu\Delta u - \frac{1}{\rho}p_x + \gamma\sin\left(\frac{\pi y}{b}\right), \tag{1}$$

$$v_t + uv_x + vv_y = \nu\Delta v - \frac{1}{\rho}p_y, \tag{2}$$

$$u_x + v_y = 0, \tag{3}$$

where (u, v), ρ, p and ν are velocity vector, mass density, pressure and kinematic viscosity, respectively and γ is a constant representing the strength of the sinusoidal outer force. Also $*_\xi := \partial/\partial\xi (\xi = t, x, y)$ and $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$. The flow region is a rectangle $[-a, a] \times [-b, b]$ and the periodic boundary conditions are imposed in both directions. The aspect ratio is denoted by $\alpha := b/a$.

The above Eqs. (1)–(3) describe the Navier–Stokes flows in a two-dimensional flat torus under a special driving force proposed in [1,7] and have a basic solution which is written as

$$(u, v, p) = (k\sin(\pi y/b), 0, d),$$

where $k := b^2\gamma/(\pi^2\nu)$ and d is any constant. It is known that non-trivial solutions bifurcate from the basic solution at a certain Reynolds number, which is defined below, if and only if $0 < \alpha < 1$ [1]. Okamoto–Shoji [7] computed numerically bifurcation diagrams with the Reynolds number as a bifurcation parameter varying the aspect ratio as a splitting parameter.

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doi:10.1016/j.cam.2008.03.034

They also strongly suggested stability of the bifurcating solutions for all $0 < \alpha < 1$. Nagatou [3] took a new approach to this stability problem by employing the theory of verified computation and showed that the stability of the bifurcating solutions is mathematical rigorously assured for the cases of $\alpha = 0.4$, 0.7 and 0.8. However, theoretical approach to the non-trivial solutions of the Eqs. (1)–(3) has not been showed up to now.

The aim of this paper is to propose a method to prove the existence and the local uniqueness of the steady-state solutions of the Navier–Stokes Eqs. (1)–(3) for a given Reynolds number and aspect ratio by a computer-assisted proof.

In the previous results [11,6], the author considered Rayleigh–Bénard heat convection model which is known as the Oberbeck–Boussinesq approximations and proposed an approach to prove the existence of the steady-state solutions. In [11,6], the equation is decomposed into a finite-dimensional part and an infinite-dimensional error part, and if both the parts lead to contraction maps under suitable assumptions, an infinite-dimensional fixed-point theorem implies the existence of the solution in a certain function set. In the self-validating process in computer, Newton-like iteration is executed for the finite-dimensional part, and the computation comes down to solving interval linear systems. However, the method adopted Schauder's fixed-point theorem and the local uniqueness is not assured.

On the other hand, Yamamoto [12] have proposed a method to prove the existence and the local uniqueness of solutions to infinite-dimensional fixed-point equations using computer. However, the algorithm needs a special form of the given finite-dimensional set and it turned out that there is a possibility that the verification algorithm come to an end unsuccessfully even if very fine approximate subspaces are used.

Therefore, this paper will take an alternative verification method using norm estimates in the Newton-like iteration. Note that our verification theorem can be described as a more general form and one may apply it to many kinds of differential equations and integral equations which can be transformed into fixed-point equations. We will discuss them in the forthcoming papers.

We admit that our study in this paper has some restrictions (a driving force, two-dimensional rectangle region, boundary condition, etc.), however, we believe that our idea, not our results themselves, will pave the way to a tool to study the global bifurcation structure for partial differential equations arising in more practical, or even industrial problems.

The contents of this paper are as follows. The Navier–Stokes equations are transformed into a non-dimensional form and the function spaces are defined in Section 2. Constructive a priori error estimates for the linearized problems are described in Section 3, which are needed in numerical computations. A fixed-point formulation and an existence theorem using Newton-like iteration is considered in Section 4. A computable verification condition is given in Section 5. Numerical results which prove the existence of steady-state solutions are described in Section 6. All numerical results discussed take into account the effects of rounding errors in the floating point computations.

2. Non-dimensionalization and function spaces

The letter $\mathbf{T}_\alpha$ denotes the rectangular region $(-\pi/\alpha, \pi/\alpha) \times (-\pi, \pi)$ for a given aspect ratio $0 < \alpha < 1$. Introducing the stream function ϕ satisfying $u = \phi_y$ and $v = -\phi_x$ so that $u_x + v_y = 0$, the Eqs. (1)–(3) can be rewritten as

$$(\Delta\phi)_t - \nu\Delta^2\phi - J(\phi, \Delta\phi) = \frac{\gamma\pi}{b}\cos\left(\frac{\pi y}{b}\right) \tag{4}$$

by cross-differentiating Eqs. (1) and (2) and eliminating the pressure p. Here J is a bilinear form defined by

$$J(u, v) := u_x v_y - u_y v_x. \tag{5}$$

Eq. (4) is non-dimensionalized using change of variables

$$(x', y') = \left(\frac{\pi x}{b}, \frac{\pi y}{b}\right), \qquad t' = \frac{\gamma b}{\nu\pi}t, \qquad \phi'(t', x', y') = \frac{\nu\pi^3}{\gamma b^3}\phi(t, x, y)$$

and the Reynolds number $R := \frac{\gamma b^3}{\nu^2\pi^3}$. After dropping the primes, an equation

$$(\Delta\phi)_t - \frac{1}{R}\Delta^2\phi - J(\phi, \Delta\phi) = \frac{1}{R}\cos(y) \tag{6}$$

is obtained.

We shall find *steady-state solutions*, where $(\Delta\phi)_t$ is equated to 0 in Eq. (6) in the region $\mathbf{T}_\alpha$, namely consider the following non-linear problem:

$$\Delta^2\phi = -RJ(\phi, \Delta\phi) - \cos(y) \quad \text{in } \mathbf{T}_\alpha. \tag{7}$$

Assume that the stream function ϕ is periodic in x and y, and the symmetric condition $\phi(x, y) = \phi(-x, -y)$ [3], then Eq. (7) has a trivial solution $\phi = -\cos(y)$ for any $R > 0$. The aim of this paper is to verify the existence of non-trivial solutions by a computer.

From the assumptions imposed above, the solutions of Eq. (7) should be obtained in the following function space $X^k \subset H^k(\Omega)$ $(k \geq 0)$ such that

$$X^k := X_0^k \oplus X_1^k \oplus X_2^k \oplus \cdots, \tag{8}$$

$$X_0^k := \left\{ \sum_{n=1}^{\infty} a_n \cos(ny) \,\middle|\, a_n \in \mathbb{R}, \sum_{n=1}^{\infty} n^{2k} a_n^2 < \infty \right\}, \tag{9}$$

$$X_m^k := \left\{ \sum_{n=-\infty}^{\infty} a_n \cos(m\alpha x + ny) \,|\, a_n \in \mathbb{R}, \sum_{n=-\infty}^{\infty} ((\alpha m)^{2k} + n^{2k}) a_n^2 < \infty \right\}, \quad m \geq 1, \tag{10}$$

especially

$$X := X^3.$$

For all $\psi \in X^k$ can be represented by

$$\psi = \sum_{(m,n)\in Q} A_{mn} \cos(m\alpha x + ny), \quad A_{mn} \in \mathbb{R}, \tag{11}$$

where

$$Q := \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} \,\middle|\, \begin{array}{l} \text{“}m = 0 \text{ and } 1 \leq n \leq \infty\text{” or} \\ \text{“}1 \leq m \leq \infty \text{ and } -\infty \leq n \leq \infty\text{”} \end{array} \right\}, \tag{12}$$

and it is noted that

$$(\cos(m\alpha x + ny), \cos(k\alpha x + ly))_{L^2} = \begin{cases} \dfrac{2\pi^2}{\alpha} & \text{if } k = m \text{ and } l = n \\ 0 & \text{else} \end{cases}$$

holds for any $(m, n), (k, l) \in Q$, where $(\cdot, \cdot)_{L^2}$ means the usual L^2-inner product in $\mathbf{T}_\alpha$.

3. Approximate subspace and norm estimates

Let X_N be the finite-dimensional subspace of X, which depends on a non-negative integer parameter N, defined by

$$X_N := \left\{ \sum_{(m,n)\in Q_N} A_{mn} \cos(m\alpha x + ny) \,|\, A_{mn} \in \mathbb{R} \right\}, \tag{13}$$

where

$$Q_N := \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} \,\middle|\, \begin{array}{l} \text{“}m = 0 \text{ and } 1 \leq n \leq N\text{” or} \\ \text{“}1 \leq m \leq N \text{ and } -N \leq n \leq N\text{”} \end{array} \right\}. \tag{14}$$

Also let X_* denote the orthogonal complement of X_N in X such that $X = X_N \oplus X_*$, then for any $\psi_* \in X_*$ can be represented by

$$\psi_* = \sum_{(m,n)\in Q_*} A_{mn} \cos(m\alpha x + ny), \qquad A_{mn} \in \mathbb{R}, \tag{15}$$

where

$$Q_* := Q - Q_N \tag{16}$$

$$= \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} \,\middle|\, \begin{array}{l} \text{“}0 \leq m \leq N \text{ and } N+1 \leq n \leq \infty\text{” or} \\ \text{“}1 \leq m \leq N \text{ and } -\infty \leq n \leq -N-1\text{” or} \\ \text{“}N+1 \leq m \leq \infty \text{ and } -\infty \leq n \leq \infty\text{”} \end{array} \right\}. \tag{17}$$

Now, we define the norm of X as

$$\|\phi\|_X := |\phi|_{H^3(\Omega)} = \sqrt{\|\phi_{xxx}\|^2_{L^2(\mathbf{T}_\alpha)} + 3\|\phi_{xxy}\|^2_{L^2(\mathbf{T}_\alpha)} + 3\|\phi_{xyy}\|^2_{L^2(\mathbf{T}_\alpha)} + \|\phi_{yyy}\|^2_{L^2(\mathbf{T}_\alpha)}}.$$

Here, using the property:

$$\|\phi\|_X^2 = \frac{2\pi^2}{\alpha} \sum_{(m,n)\in Q} (\alpha^2 m^2 + n^2)^3 A_{mn}^2$$

for $\phi = \sum_{(m,n)\in Q} A_{mn} \cos(m\alpha x + ny) \in X$, the following norm estimates hold.

Lemma 3.1. *For any $\psi \in X$ and $\psi_* \in X_*$, it can be checked that*

$$\begin{aligned}
&\|\psi\|_{L^2(\mathbf{T}_\alpha)} \le \alpha^{-3}\|\psi\|_X, && \|\psi_*\|_{L^2(\mathbf{T}_\alpha)} \le C_1\|\psi_*\|_X,\\
&\|\psi_x\|_{L^2(\mathbf{T}_\alpha)} \le \alpha^{-2}\|\psi\|_X, && \|(\psi_*)_x\|_{L^2(\mathbf{T}_\alpha)} \le C_2\|\psi_*\|_X,\\
&\|\psi_y\|_{L^2(\mathbf{T}_\alpha)} \le C_3\|\psi\|_X, && \|(\psi_*)_y\|_{L^2(\mathbf{T}_\alpha)} \le C_4\|\psi_*\|_X,\\
&\|\nabla\psi\|_{L^2(\mathbf{T}_\alpha)} \le \alpha^{-2}\|\psi\|_X, && \|\nabla\psi_*\|_{L^2(\mathbf{T}_\alpha)} \le C_2\|\psi_*\|_X,\\
&\|\nabla\psi_x\|_{L^2(\mathbf{T}_\alpha)} \le \alpha^{-1}\|\psi\|_X, && \|\nabla(\psi_*)_x\|_{L^2(\mathbf{T}_\alpha)} \le C_5\|\psi_*\|_X,\\
&\|\nabla\psi_y\|_{L^2(\mathbf{T}_\alpha)} \le C_6\|\psi\|_X, && \|\nabla(\psi_*)_y\|_{L^2(\mathbf{T}_\alpha)} \le C_7\|\psi_*\|_X,\\
&\|\Delta\psi\|_{L^2(\mathbf{T}_\alpha)} \le \alpha^{-1}\|\psi\|_X, && \|\Delta\psi_*\|_{L^2(\mathbf{T}_\alpha)} \le C_5\|\psi_*\|_X,\\
&\|\Delta\psi_x\|_{L^2(\mathbf{T}_\alpha)} \le \|\psi\|_X, && \|\Delta(\psi_*)_x\|_{L^2(\mathbf{T}_\alpha)} \le \|\psi_*\|_X,\\
&\|\Delta\psi_y\|_{L^2(\mathbf{T}_\alpha)} \le \|\psi\|_X, && \|\Delta(\psi_*)_y\|_{L^2(\mathbf{T}_\alpha)} \le \|\psi_*\|_X,
\end{aligned}$$

where

$$\begin{aligned}
&C_1 = \frac{1}{\alpha^3(N+1)^3}, \qquad C_2 = \frac{1}{\alpha^2(N+1)^2},\\
&C_3 = \max\left\{1, \frac{2\sqrt{3}}{9\alpha^2}\right\}, \qquad C_4 = \max\left\{\frac{1}{(N+1)^2}, \frac{2\sqrt{3}}{9\alpha^2(N+1)^2}\right\},\\
&C_5 = \frac{1}{\alpha(N+1)}, \qquad C_6 = \max\left\{1, \frac{1}{2\alpha}\right\},\\
&C_7 = \max\left\{\frac{1}{N+1}, \frac{1}{2\alpha(N+1)}\right\}.
\end{aligned}$$

Proof. We show the construction of C_5. The other estimates are quite similar. For ψ_* represented by Eq. (15),

$$\begin{aligned}
\|\nabla(\psi_*)_x\|^2_{L^2(\mathbf{T}_\alpha)} &= \|(\psi_*)_{xx}\|^2_{L^2(\mathbf{T}_\alpha)} + \|(\psi_*)_{xy}\|^2_{L^2(\mathbf{T}_\alpha)}\\
&= \frac{2\pi^2}{\alpha}\sum_{(m,n)\in Q_*}\alpha^2 m^2(\alpha^2 m^2 + n^2)A^2_{mn}\\
&\le \max_{(m,n)\in Q_*}\frac{\alpha^2 m^2}{(\alpha^2 m^2 + n^2)^2}\,\|\psi_*\|^2_X,
\end{aligned}$$

hence using $0 < \alpha < 1$,

$$\begin{aligned}
\max_{(m,n)\in Q_*}\frac{\alpha m}{\alpha^2 m^2 + n^2} &= \max\left\{\max_{0\le m\le N}\frac{\alpha m}{\alpha^2 m^2 + (N+1)^2}, \frac{1}{\alpha(N+1)}\right\}\\
&\le \max\left\{\frac{1}{2(N+1)}, \frac{1}{\alpha(N+1)}\right\}\\
&= \frac{1}{\alpha(N+1)}.
\end{aligned}$$

□

In actual calculations, L^∞-estimates proposed by Plum [8] are also needed.

Lemma 3.2 ([8]). *For $\psi \in X$, the following assertion holds true:*

$$\|\psi\|_{L^\infty(\mathbf{T}_\alpha)} \le C_8\|\psi\|_{L^2(\mathbf{T}_\alpha)} + C_9\|\nabla\psi\|_{L^2(\mathbf{T}_\alpha)} + C_{10}\|\Delta\psi\|_{L^2(\mathbf{T}_\alpha)}, \tag{18}$$

where $\|\cdot\|_{L^\infty(\mathbf{T}_\alpha)}$ is the sup-norm and

$$C_8 = \frac{\sqrt{\alpha}}{2\pi}, \qquad C_9 = \frac{1.1548}{\sqrt{3}}\sqrt{\frac{\alpha^2+1}{\alpha}}, \qquad C_{10} = \pi\frac{0.44722}{3}\sqrt{\frac{9\alpha^4+10\alpha^2+9}{5\alpha^3}}.$$

Lemmas 3.1 and 3.2 imply L^∞-estimates immediately.

Lemma 3.3. *For $\psi \in X$ and $\psi_* \in X_*$, the following estimates hold:*

$$\begin{aligned}
&\|\psi\|_{L^\infty(\mathbf{T}_\alpha)} \le C_{11}\|\psi\|_X, && \|\psi_*\|_{L^\infty(\mathbf{T}_\alpha)} \le C_{12}\|\psi\|_X,\\
&\|\psi_x\|_{L^\infty(\mathbf{T}_\alpha)} \le C_{13}\|\psi\|_X, && \|(\psi_*)_x\|_{L^\infty(\mathbf{T}_\alpha)} \le C_{14}\|\psi_*\|_X,\\
&\|\psi_y\|_{L^\infty(\mathbf{T}_\alpha)} \le C_{15}\|\psi\|_X, && \|(\psi_*)_y\|_{L^\infty(\mathbf{T}_\alpha)} \le C_{16}\|\psi_*\|_X,
\end{aligned}$$

where

$$C_{11} = \alpha^{-3}C_8 + \alpha^{-2}C_9 + \alpha^{-1}C_{10}, \qquad C_{12} = C_1C_8 + C_2C_9 + C_5C_{10},$$
$$C_{13} = \alpha^{-2}C_8 + \alpha^{-1}C_9 + C_{10}, \qquad C_{14} = C_2C_8 + C_5C_9 + C_{10},$$
$$C_{15} = C_3C_8 + C_6C_9 + C_{10}, \qquad C_{16} = C_4C_8 + C_7C_9 + C_{10}.$$

Moreover, some "inverse"-order estimates are required (proofs are similar as that of Lemma 3.1).

Lemma 3.4. *For $\psi_N \in X_N$ the following estimates hold:*

$$\|\psi_N\|_X \le C_{17}\|\Delta\psi_N\|_{L^2(\mathbf{T}_\alpha)},$$
$$\|(\psi_N)_x\|_X \le C_{18}\|\Delta\psi_N\|_{L^2(\mathbf{T}_\alpha)},$$
$$\|(\psi_N)_y\|_X \le C_{19}\|\Delta\psi_N\|_{L^2(\mathbf{T}_\alpha)},$$

where

$$C_{17} = N\sqrt{1+\alpha^2}, \qquad C_{18} = \alpha N^2\sqrt{1+\alpha^2}, \qquad C_{19} = N^2\sqrt{1+\alpha^2}.$$

4. Fixed-point formulation and error estimates

The bilinear form J defined by Eq. (5) has the following properties.

$$(J(u,v), w)_{L^2} = (J(w,u), v)_{L^2} = -(J(u,w), v)_{L^2}, \quad u, v, w \in X^2, \tag{19}$$

$$J(u,v) \in X^0, \quad u, v \in X^1. \tag{20}$$

Denote an approximate solution of Eq. (7) by $\phi_N \in X_N$ which is obtained by an appropriate method. Then setting

$$\psi := \phi - \phi_N \tag{21}$$

and

$$f(\psi) := -RJ(\phi_N + \psi, \Delta\phi_N + \Delta\psi) - \cos(y) - \Delta^2\phi_N, \tag{22}$$

problem (7) is rewritten as the residual form to find $\psi \in X$ satisfying

$$\Delta^2\psi = f(\psi) \quad \text{in } \mathbf{T}_\alpha \tag{23}$$

in a weak sense. Note that ψ is expected to be small if ϕ_N is an accurate approximation. By virtue of the property (20) for J, f is the bounded continuous map from X to X^0.

Moreover, it is easily shown that for all $g \in X^0$, the linear problem $\Delta^2\xi = g$ has a unique solution $\xi \in X^4$. When this mapping is denoted by $\xi = Kg$, denoting the embedding from X^4 into X by $\mathcal{I}$ and $\Delta^{-2} := \mathcal{I}K$, the operator $\Delta^{-2} : X^0 \longrightarrow X$ is a compact map because of the compactness of the embedding $H^4(\mathbf{T}_\alpha) \hookrightarrow H^3(\mathbf{T}_\alpha)$ and the boundedness of K. Therefore, Eq. (23) is rewritten by a fixed-point equation:

$$\psi = F(\psi) \tag{24}$$

for the compact operator $F := \Delta^{-2}f$ on X.

Now, the H_0^2-projection $P_N : X \longrightarrow X_N$ is defined by

$$(\Delta(\psi - P_N\psi), \Delta\psi_N)_{L^2} = 0, \quad \forall\psi_N \in X_N. \tag{25}$$

Note that for $\psi = \sum_{(m,n)\in Q} A_{mn}\cos(m\alpha x + ny) \in X$ the projection coincides with truncation: $P_N\psi = \sum_{(m,n)\in Q_N} A_{mn}\cos(m\alpha x + ny) \in X_N$. From this fact, the following constructive a priori error estimate is derived.

Lemma 4.1. *For each $g \in X^0$, let $\xi \in X^4$ be the solution of $\Delta^2\xi = g$ and $P_N\xi \in X_N$ be the finite-dimensional approximation defined by Eq. (25), then*

$$\|\xi - P_N\xi\|_X \le C_5\|g\|_{L^2(\mathbf{T}_\alpha)}. \tag{26}$$

Proof. The estimate (26) is derived immediately from the stronger fact:

$$\begin{aligned}\|\xi_*\|_X^2 &\le \max_{(m,n)\in Q_*}\frac{1}{\alpha^2m^2+n^2}\,\frac{2\pi^2}{\alpha}\sum_{(m,n)\in Q_*}(\alpha^2m^2+n^2)^4A_{mn}^2\\ &\le C_5^2\|\Delta^2\xi_*\|_{L^2(\mathbf{T}_\alpha)}^2,\end{aligned}$$

for $\xi_* = \sum_{(m,n)\in Q_*} A_{mn}\cos(m\alpha x + ny)$, $A_{mn} \in \mathbb{R}$ satisfying $\Delta^2\xi_* \in X^0$. $\square$

Now, we apply the Newton-like method for non-linear elliptic problems proposed by Nakao [4,5] to the fixed-point Eq. (24). Using the projection P_N, the fixed-point problem $\psi = F(\psi)$ can be uniquely decomposed as the finite-dimensional (projection) part X_N and infinite-dimensional (error) part X_* as follows:

$$\begin{cases} P_N\psi = P_N F(\psi), \\ (I - P_N)\psi = (I - P_N)F(\psi), \end{cases} \tag{27}$$

where I is the identity map on X. Suppose that the restriction of the operator $P_N(I - F'[0]) : X \longrightarrow X_N$ to X_N has an inverse

$$[I - P_N F'[0]]_N^{-1} : X_N \longrightarrow X_N, \tag{28}$$

where $F'[\psi]$ denotes the Fréchet derivative of F at ψ. Note that this assumption is equivalent to the invertibility of a matrix, which can be checked numerically in actual verified computations (for example see Rump [9]). Applying the Newton-like method to the first term of Eq. (27), the operator $\mathcal{N} : X \longrightarrow X_N$ is defined by

$$\mathcal{N}(\psi) := P_N\psi - [I - P_N F'[0]]_N^{-1} P_N(\psi - F(\psi)),$$

and also the compact map $T : X \longrightarrow X$ is defined by

$$T(\psi) := \mathcal{N}(\psi) + (I - P_N)F(\psi).$$

Then under the invertibility assumption of the existence for $[I - P_N F'[0]]_N^{-1}$, two fixed-point problems:

$$\psi = T(\psi) \tag{29}$$

and Eq. (24) are equivalent. If the approximate solution ϕ_N is sufficiently good, the finite-dimensional part of T will possibly be a contraction. On the other hand, the magnitude of the infinite-dimensional part of T is expected to be small when the truncation numbers of X_N are taken to be sufficiently large, because of Lemma 4.1.

The question which we must consider next is to find a solution of Eq. (29) in a set U, referred to as a *candidate set*. Let the finite-dimensional part of the candidate set U_N and the infinite-dimensional part of the candidate set U_* be balls with radius $\gamma > 0$ and $\beta > 0$ such as

$$U_N := \{\psi_N \in X_N \mid \|\psi_N\|_X \le \gamma\}, \tag{30}$$

$$U_* := \{\psi_* \in X_* \mid \|\psi_*\|_X \le \beta\}, \tag{31}$$

respectively. The candidate set $U \subset X$ is defined by

$$U := U_N + U_*. \tag{32}$$

Now, the following verification condition is held.

Theorem 4.1. *Let $\mathcal{N}'$ be the Fréchet derivative of $\mathcal{N}$. For $Y_1, Y_2, Z_1(U)$ and $Z_2(U) > 0$ satisfying*

$$\|\mathcal{N}(0)\|_X \le Y_1, \tag{33}$$

$$\sup_{\psi_1,\psi_2 \in U} \|\mathcal{N}'[\psi_1](\psi_2)\|_X \le Z_1(U), \tag{34}$$

$$\|(I - P_N)F(0)\|_X \le Y_2, \tag{35}$$

$$\sup_{\psi_1,\psi_2 \in U} \|(I - P_N)F'[\psi_1](\psi_2)\|_X \le Z_2(U), \tag{36}$$

if it holds that

$$Y_1 + Z_1(U) < \gamma, \qquad Y_2 + Z_2(U) < \beta,$$

then there exists a fixed-point of F in

$$\hat{U} := \hat{U}_N + \hat{U}_*,$$

$$\hat{U}_N := \{\psi_N \in X_N \mid \|\psi_N\|_X \le Y_1 + Z_1(U)\},$$

$$\hat{U}_* := \{\psi_* \in X_* \mid \|\psi_*\|_X \le Y_2 + Z_2(U)\}.$$

Moreover, this fixed-point is unique within the set U.

Proof. From Banach's fixed-point theorem, it is sufficient to check that the two conditions:

$$T(U) \subset U, \tag{37}$$

and $\exists k < 1$ such that

$$\|T(u_1) - T(u_2)\| \le k\|u_1 - u_2\|, \quad \forall u_1, u_2 \in U \tag{38}$$

hold. Applying the mean value theorem [12], condition (37) can be shown by

$$\|\mathcal{N}(u)\|_X \le \|\mathcal{N}(0)\|_X + \sup_{s\in[0,1]} \|\mathcal{N}'[su](u)\|_X \le Y_1 + Z_1(U) \le \gamma,$$

$$\begin{aligned}\|(I-P_N)F(u)\|_X &\le \|(I-P_N)F(0)\|_X + \sup_{s\in[0,1]} \|(I-P_N)F'[su](u)\|_X \\ &\le Y_2 + Z_2(U) \le \alpha\end{aligned}$$

for any $u \in U$.

Next, define the norm $\|\cdot\|_U$ by

$$\|u\|_U := \max\left\{\frac{\|P_N u\|_X}{\gamma}, \frac{\|(I-P_N)u\|_X}{\alpha}\right\}, \tag{39}$$

then for any $u_1, u_2 \in U$, the condition (38) holds by

$$\begin{aligned}\|T(u_1) - T(u_2)\|_U &\le \sup_{w\in U} \|T'[w](u_1-u_2)\|_U \\ &\le \sup_{w\in U} \max\left\{\frac{\|\mathcal{N}'[w](u_1-u_2)\|_X}{\gamma}, \frac{\|(I-P_N)F'[w](u_1-u_2)\|_X}{\alpha}\right\} \\ &\le \max\left\{\frac{\gamma - Y_1}{\gamma}, \frac{\alpha - Y_2}{\alpha}\right\} \|u_1 - u_2\|_U. \quad \Box\end{aligned}$$

Note that in the References [2,12] the finite-dimensional part is taken to be a set of linear combinations of base functions with interval coefficients.

If we obtain a fixed-point $\psi \in X$ by Theorem 4.1, we can also assure the existence of a non-trivial solution $\phi = \phi_N + \psi \in X$ for (7) with the error bound

$$\|\phi - \phi_N\|_X \le Y_1 + Z_1(U) + Y_2 + Z_2(U).$$

Moreover, since ψ can be written as $\psi = \psi_N + \psi_*$, $\psi_N \in U_N$, $\psi_* \in U_*$, a L^∞-error estimate:

$$\|\phi - \phi_N\|_{L^\infty(\mathbf{T}_\alpha)} \le C_{11}(Y_1 + Z_1(U)) + C_{12}(Y_2 + Z_2(U)) \tag{40}$$

is obtained by Lemma 3.3.

5. Verification procedure

This section is devoted to the detailed estimation satisfying Eqs. (33)–(36).

5.1. Estimation of Y_1

Consider the computation of $Y_1 > 0$ such that $\|\mathcal{N}(0)\|_X \le Y_1$. Since $\mathcal{N}(0) = [I - P_N F'[0]]_N^{-1} P_N F(0)$, it holds that

$$P_N(I - F'[0])\mathcal{N}(0) = P_N F(0). \tag{41}$$

Let $M := \dim X_N$ and let $\psi_i (1 \le i \le M)$ be a basis of X_N, then $\mathcal{N}(0)$ can be represented as

$$\mathcal{N}(0) = \sum_{i=1}^{M} a_i \psi_i, \qquad \mathbf{a} = [a_i] \in \mathbb{R}^M.$$

By the definition of P_N, Eq. (41) is equivalent to

$$\sum_{j=1}^{M} (\Delta(I - F'[0])(\psi_j), \Delta\psi_i)_{L^2}\, a_j = (\Delta F(0), \Delta\psi_i)_{L^2}, \quad 1 \le i \le M. \tag{42}$$

Then by partial integration and the definition of F and F', the Eq. (42) is written as

$$\sum_{j=1}^{M} \left\{(\Delta\psi_j, \Delta\psi_i)_{L^2} - (f'[0](\psi_j), \psi_i)_{L^2}\right\} a_j = (f(0), \psi_i)_{L^2}, \quad 1 \le i \le M. \tag{43}$$

Here, defining a $M \times M$ matrix $G = [G_{ij}]$ by

$$G_{ij} := (\Delta\psi_j, \Delta\psi_i)_{L^2} - (f'[0](\psi_j), \psi_i)_{L^2}, \quad 1 \le i, j \le M, \tag{44}$$

and M-dimensional vector $\mathbf{r} = [r_i]$ by

$$r_i := (f(0), \psi_i)_{L^2}, \quad 1 \le i \le M.$$

the vector $\mathbf{a}$ is obtained by

$$\mathbf{a} = G^{-1}\mathbf{r}. \tag{45}$$

Therefore after some transformation of indices for $a_i \leftarrow a_{m,n}$, $(m,n) \in Q_N$, the norm $\|\mathcal{N}(0)\|_X$ can be estimated as

$$\|\mathcal{N}(0)\|_X^2 = \frac{2\pi^2}{\alpha} \sum_{(m,n)\in Q_N} (\alpha^2 m^2 + n^2)^3 a_{mn}^2,$$

and $Y_1 > 0$ is taken to be

$$\pi \sqrt{\frac{2}{\alpha} \sum_{(m,n)\in Q_N} (\alpha^2 m^2 + n^2)^3 a_{mn}^2} \le Y_1.$$

Note that all the computation procedures for $Y_1 > 0$ should take into account the effects of rounding errors. Here, $f(0)$ and $f'[0](\psi_j)$ can be computed using the approximate solution ϕ_N.

5.2. Estimation of $Z_1(U)$

Consider the computation of $Z_1(U) > 0$ satisfying $\sup_{\psi_1,\psi_2\in U} \|\mathcal{N}'[\psi_1](\psi_2)\|_X \le Z_1(U)$.
Let $M \times M$ diagonal matrices $D = [D_{ij}]$ and $H = [H_{ij}]$ be

$$D_{ij} := (\Delta\psi_j, \Delta\psi_i)_{L^2} \quad 1 \le i,j \le M, \tag{46}$$

$$\begin{aligned} H_{ij} \;:=\; & ((\psi_j)_{xxx}, (\psi_i)_{xxx})_{L^2} + 3((\psi_j)_{xxy}, (\psi_i)_{xxy})_{L^2} + 3((\psi_j)_{xyy}, (\psi_i)_{xyy})_{L^2} + ((\psi_j)_{yyy}, (\psi_i)_{yyy})_{L^2}, \\ & 1 \le i,j \le M. \end{aligned} \tag{47}$$

Since H is the positive diagonal, H can be decomposed as $H = H^{1/2}H^{1/2}$. Define now the upper bound of the Euclidean norm $\rho > 0$ by

$$\|H^{1/2}G^{-1}DH^{-1/2}\|_E \le \rho, \tag{48}$$

the following Lemma is obtained.

Lemma 5.1. *For each $w_N \in X_N$ and $v_N = [I - P_N F'[0]]_N^{-1} w_N \in X_N$, it holds that*

$$\|v_N\|_X \le \rho\, \|w_N\|_X.$$

Proof. Set $v_N, w_N \in X_N$ as

$$v_N = \sum_{i=1}^M v_i\psi_i, \qquad \mathbf{v} = [v_i] \in \mathbb{R}^M, \qquad w_N = \sum_{i=1}^M w_i\psi_i, \qquad \mathbf{w} = [w_i] \in \mathbb{R}^M.$$

Then the relation $P_N(I - F'[0])v_N = w_N$ can be expanded as

$$\sum_{j=1}^M \left\{(\Delta\psi_j, \Delta\psi_i)_{L^2} - (f'[0](\psi_j), \psi_i)_{L^2}\right\} v_j = \sum_{j=1}^M (\Delta\psi_j, \Delta\psi_i)_{L^2}\, w_j, \qquad 1 \le i \le M. \tag{49}$$

Eq. (49) is represented by the matrix and vector form by $\mathbf{v} = G^{-1}D\mathbf{w}$ and this implies

$$\begin{aligned} \|v_N\|_X &= \|H^{1/2}\mathbf{v}\|_E \\ &= \|H^{1/2}G^{-1}D\mathbf{w}\|_E \\ &\le \|H^{1/2}G^{-1}DH^{-1/2}\|_E\, \|H^{1/2}\mathbf{w}\|_E \\ &\le \rho\|w_N\|_X \end{aligned}$$

which proves the lemma. □

Now for any $\psi_1, \psi_2 \in U$,

$$\mathcal{N}'[\psi_1](\psi_2) = [I - P_N F'[0]]_N^{-1} P_N(F'[\psi_1](\psi_2) - F'[0](P_N\psi_2)),$$

then Lemma 5.1 leads

$$\|\mathcal{N}'(\psi_1)\psi_2\|_X \le \rho\, \|P_N(F'[\psi_1](\psi_2) - F'[0](P_N\psi_2))\|_X.$$

Therefore $Z_1(U) > 0$ can be decided satisfying

$$\rho \sup_{\psi_1,\psi_2\in U} \|P_N(F'[\psi_1](\psi_2) - F'[0](P_N\psi_2))\|_X \le Z_1(U). \tag{50}$$

We will discuss more concrete computation procedure for $Z_1(U)$ in the next subsection.

The estimation of ρ satisfying inequality (48) is generally reduced to a computation of the singular value of a matrix. Actually, when setting indices of Q_N by (m_i, n_i), and $M \times M$ diagonal matrix $\tilde{D}$ by

$$\tilde{D}_{ii} = \frac{1}{\sqrt{\alpha^2 m_i^2 + n_i^2}},$$

it is not difficult to check that

$$(H^{1/2}G^{-1}DH^{-1/2})^{-1} = \frac{\alpha}{2\pi^2}\tilde{D}G\tilde{D}^3.$$

Hence, we apply some computational algorithm with the result verification to estimate rigorous bounds for the smallest singular value (e.g. see Rump [10]).

5.3. Estimation of $Z_1(U)$ (detail)

This subsection is devoted for the detailed estimation of

$$\sup_{\psi_1,\psi_2\in U} \|P_N(F'[\psi_1](\psi_2) - F'[0](P_N\psi_2))\|_X$$

in Eq. (50). First, for obtained approximate solution ϕ_N, define computable upper bounds $\tau_1, \tau_2, \tau_3, \tau_4 > 0$ such that

$$\|(\phi_N)_x\|_{L^\infty(\mathbf{T}_\alpha)} \le \tau_1, \tag{51}$$

$$\|(\phi_N)_y\|_{L^\infty(\mathbf{T}_\alpha)} \le \tau_2, \tag{52}$$

$$\|\Delta(\phi_N)_x\|_{L^\infty(\mathbf{T}_\alpha)} \le \tau_3, \tag{53}$$

$$\|\Delta(\phi_N)_y\|_{L^\infty(\mathbf{T}_\alpha)} \le \tau_4. \tag{54}$$

For fixed $\psi_1, \psi_2 \in U$ can be decomposed as

$$\psi_1 = \psi_N^{(1)} + \psi_*^{(1)}, \qquad \psi_2 = \psi_N^{(2)} + \psi_*^{(2)}, \quad \psi_N^{(1)}, \psi_N^{(2)} \in U_N, \psi_*^{(1)}, \psi_*^{(2)} \in U_*.$$

Then the result

$$f'[\psi_1](\psi_2) - f'[0](P_N\psi_2) = -R\left(J(\phi_N, \Delta\psi_*^{(2)}) + J(\psi_*^{(2)}, \Delta\phi_N) + J(\psi_2, \Delta\psi_1) + J(\psi_1, \Delta\psi_2)\right)$$

implies

$$\begin{aligned}\|P_N(F'[\psi_1](\psi_2) - F'[0](P_N\psi_2))\|_X &= \|P_N\Delta^{-2}(f'[\psi_1](\psi_2) - f'[0](P_N\psi_2))\|_X\\ &\le \|P_N\Delta^{-2}\xi_1\|_X + \|P_N\Delta^{-2}\xi_2\|_X,\end{aligned}$$

where

$$\xi_1 := -RJ(\phi_N, \Delta\psi_*^{(2)}),$$

$$\xi_2 := -R\left(J(\psi_*^{(2)}, \Delta\phi_N) + J(\psi_2, \Delta\psi_1) + J(\psi_1, \Delta\psi_2)\right).$$

Here, term ξ_1 has to be estimated separately in order to obtain $O(1/N)$ described below.

5.3.1. Estimation of $\|P_N\Delta^{-2}\xi_1\|_X$

Setting $\psi_N = P_N\Delta^{-2}J(\phi_N, \Delta\psi_*^{(2)}) \in X_N$, from the definition of P_N, Eq. (19) and Lemma 3.1, it holds that

$$\begin{aligned}\|\Delta\psi_N\|^2_{L^2(\mathbf{T}_\alpha)} &= (J(\phi_N, \Delta\psi_*^{(2)}), \psi_N)_{L^2}\\ &= (\Delta J(\psi_N, \phi_N), \psi_*^{(2)})_{L^2}\\ &= (\Delta(I-P_N)J(\psi_N, \phi_N), \psi_*^{(2)})_{L^2}\\ &\le \|\Delta(I-P_N)J(\psi_N, \phi_N)\|_{L^2(\mathbf{T}_\alpha)}\,\|\psi_*^{(2)}\|_{L^2(\mathbf{T}_\alpha)}\\ &\le C_1C_5\|(I-P_N)J(\psi_N, \phi_N)\|_X\,\beta.\end{aligned}$$

And Lemma 3.4 assures that

$$\begin{aligned}\|(I-P_N)J(\psi_N, \phi_N)\|_X &\le \|J(\psi_N, \phi_N)\|_X\\ &\le \|(\psi_N)_x(\phi_N)_y\|_X + \|(\psi_N)_y(\phi_N)_x\|_X\\ &\le \|(\psi_N)_x\|_X\|(\phi_N)_y\|_{L^\infty(\mathbf{T}_\alpha)} + \|(\psi_N)_y\|_X\|(\phi_N)_x\|_{L^\infty(\mathbf{T}_\alpha)}\\ &\le (\tau_2C_{18} + \tau_1C_{19})\|\Delta\psi_N\|_{L^2(\mathbf{T}_\alpha)},\end{aligned}$$

then

$$\|\Delta\psi_N\|_{L^2(\mathbf{T}_\alpha)} \le C_1C_5(\tau_2C_{18} + \tau_1C_{19})\beta$$

is obtained. Therefore we have

$$\begin{aligned}\|P_N\Delta^{-2}\xi_1\|_X &= R\|\psi_N\|_X\\ &\le RC_{17}\|\Delta\psi_N\|_{L^2(\mathbf{T}_\alpha)}\\ &\le RC_{20}\beta,\end{aligned}$$

where

$$C_{20} := C_{17}C_1C_5(\tau_2C_{18}+\tau_1C_{19}).$$

Note that $C_{17}=O(N)$, $C_1=O(1/N^3)$, $C_5=O(1/N)$ and $C_{18}=C_{19}=O(N^2)$, then $C_{20}=O(1/N)$.

5.3.2. Estimation of $\|P_N\Delta^{-2}\xi_2\|_X$

Setting $\xi_2=\sum_{(m,n)\in Q}A_{mn}\cos(m\alpha x+ny)$, it can be shown that

$$\begin{aligned}\|P_N\Delta^{-2}\xi_2\|_X^2 &\le \|\Delta^{-2}\xi_2\|_X^2\\ &= \frac{2\pi^2}{\alpha}\sum_{(m,n)\in Q}(\alpha^2m^2+n^2)^{-1}A_{mn}^2\\ &\le \max_{(m,n)\in Q}\frac{1}{\alpha^2m^2+n^2}\|\xi\|_{L^2(\mathbf{T}_\alpha)}^2\\ &= \alpha^{-2}\|\xi\|_{L^2(\mathbf{T}_\alpha)}^2,\end{aligned}$$

then

$$\|P_N\Delta^{-2}\xi_2\|_X \le \alpha^{-1}\|\xi_2\|_{L^2(\mathbf{T}_\alpha)}.$$

Hence $\|\xi_2\|_{L^2(\mathbf{T}_\alpha)}$ should be estimated. Since

$$\begin{aligned}\|\xi_2\|_{L^2(\mathbf{T}_\alpha)} \le R\Big(&\|J(\psi_*^{(2)},\Delta\phi_N)\|_{L^2(\mathbf{T}_\alpha)}+\|J(\psi_N^{(2)}+\psi_*^{(2)},\Delta\psi_N^{(1)}+\Delta\psi_*^{(1)})\|_{L^2(\mathbf{T}_\alpha)}\\ &+\|J(\psi_N^{(1)}+\psi_*^{(1)},\Delta\psi_N^{(2)}+\Delta\psi_*^{(2)})\|_{L^2(\mathbf{T}_\alpha)}\Big),\end{aligned}$$

for each L^2-norm can be bounded as follows:

$$\begin{aligned}\|J(\psi_*^{(2)},\Delta\phi_N)\|_{L^2(\mathbf{T}_\alpha)} &= \|(\psi_*^{(2)})_x\Delta(\phi_N)_y-(\psi_*^{(2)})_y\Delta(\phi_N)_x\|_{L^2(\mathbf{T}_\alpha)}\\ &\le \|\Delta(\phi_N)_y\|_{L^\infty(\mathbf{T}_\alpha)}\|(\psi_*^{(2)})_x\|_{L^2(\mathbf{T}_\alpha)}+\|\Delta(\phi_N)_x\|_{L^\infty(\mathbf{T}_\alpha)}\|(\psi_*^{(2)})_y\|_{L^2(\mathbf{T}_\alpha)}\\ &\le (\tau_4C_2+\tau_3C_4)\|\psi_*^{(2)}\|_X\\ &\le (\tau_4C_2+\tau_3C_4)\beta,\end{aligned}$$

$$\begin{aligned}&\|J(\psi_N^{(2)}+\psi_*^{(2)},\Delta\psi_N^{(1)}+\Delta\psi_*^{(1)})\|_{L^2(\mathbf{T}_\alpha)}\\ &\quad\le \|(\psi_N^{(2)})_x\Delta(\psi_N^{(1)})_y-(\psi_N^{(2)})_y\Delta(\psi_N^{(1)})_x\|_{L^2(\mathbf{T}_\alpha)}+\|(\psi_N^{(2)})_x\Delta(\psi_*^{(1)})_y-(\psi_N^{(2)})_y\Delta(\psi_*^{(1)})_x\|_{L^2(\mathbf{T}_\alpha)}\\ &\qquad+\|(\psi_*^{(2)})_x\Delta(\psi_N^{(1)})_y-(\psi_*^{(2)})_y\Delta(\psi_N^{(1)})_x\|_{L^2(\mathbf{T}_\alpha)}+\|(\psi_*^{(2)})_x\Delta(\psi_*^{(1)})_y-(\psi_*^{(2)})_y\Delta(\psi_*^{(1)})_x\|_{L^2(\mathbf{T}_\alpha)}\\ &\quad\le \|(\psi_N^{(2)})_x\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_N^{(1)})_y\|_{L^2(\mathbf{T}_\alpha)}+\|(\psi_N^{(2)})_y\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_N^{(1)})_x\|_{L^2(\mathbf{T}_\alpha)}\\ &\qquad+\|(\psi_N^{(2)})_x\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_*^{(1)})_y\|_{L^2(\mathbf{T}_\alpha)}+\|(\psi_N^{(2)})_y\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_*^{(1)})_x\|_{L^2(\mathbf{T}_\alpha)}\\ &\qquad+\|(\psi_*^{(2)})_x\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_N^{(1)})_y\|_{L^2(\mathbf{T}_\alpha)}+\|(\psi_*^{(2)})_y\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_N^{(1)})_x\|_{L^2(\mathbf{T}_\alpha)}\\ &\qquad+\|(\psi_*^{(2)})_x\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_*^{(1)})_y\|_{L^2(\mathbf{T}_\alpha)}+\|(\psi_*^{(2)})_y\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_*^{(1)})_x\|_{L^2(\mathbf{T}_\alpha)}\\ &\quad\le (C_{13}+C_{15})\gamma^2+(C_{13}+C_{14}+C_{15}+C_{16})\beta\gamma+(C_{14}+C_{16})\beta^2,\end{aligned}$$

and similarly

$$\begin{aligned}&\|J(\psi_N^{(1)}+\psi_*^{(1)},\Delta\psi_N^{(2)}+\Delta\psi_*^{(2)})\|_{L^2(\mathbf{T}_\alpha)}\\ &\quad\le (C_{13}+C_{15})\gamma^2+(C_{13}+C_{14}+C_{15}+C_{16})\beta\gamma+(C_{14}+C_{16})\beta^2.\end{aligned}$$

Then

$$\|\xi_2\|_{L^2(\mathbf{T}_\alpha)} \le R((\tau_4C_2+\tau_3C_4)\beta+2(C_{13}+C_{15})\gamma^2+2(C_{13}+C_{14}+C_{15}+C_{16})\beta\gamma+2(C_{14}+C_{16})\beta^2).$$

5.3.3. Conclusion

Considering the circumstances mentioned above, $Z_1(U)>0$ can be determined satisfying

$$\begin{aligned}&\rho R\left(C_{20}\beta+\alpha^{-1}\left\{(\tau_4C_2+\tau_3C_4)\beta+2(C_{13}+C_{15})\gamma^2+2(C_{13}+C_{14}+C_{15}+C_{16})\beta\gamma+2(C_{14}+C_{16})\beta^2\right\}\right)\\ &\quad\le Z_1(U).\end{aligned}$$

5.4. Estimation of Y_2

From Lemma 4.1 and the definition of F, it holds that

$$\|(I-P_N)F(0)\|_X \le C_5\|f(0)\|_{L^2(\mathbf{T}_\alpha)}.$$

Then $Y_2 > 0$ can be determined satisfying

$$C_5\| - \Delta^2\phi_N - RJ(\phi_N, \Delta\phi_N) - \cos(y)\|_{L^2(\mathbf{T}_\alpha)} \le Y_2$$

using approximate solution $\phi_N \in X_N$.

5.5. Estimation of $Z_2(U)$

From Lemma 4.1 and the definition of F, for all $\psi_1, \psi_2 \in U$

$$\|(I-P_N)F'[\psi_1](\psi_2)\|_X \le C_5\|f'[\psi_1](\psi_2)\|_{L^2(\mathbf{T}_\alpha)}$$

holds. Since

$$f'[\psi_1](\psi_2) = -R\left(J(\phi_N, \Delta\psi_2) + J(\psi_2, \Delta\phi_N) + J(\psi_2, \Delta\psi_1) + J(\psi_1, \Delta\psi_2)\right),$$

the latter two L^2-norm estimates have been obtained before. And we also get

$$\begin{aligned}
\|J(\phi_N, \Delta\psi_2)\|_{L^2(\mathbf{T}_\alpha)} &= \|(\phi_N)_x\Delta(\psi_2)_y - (\phi_N)_y\Delta(\psi_2)_x\|_{L^2(\mathbf{T}_\alpha)} \\
&\le \|(\phi_N)_x\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_2)_y\|_{L^2(\mathbf{T}_\alpha)} + \|(\phi_N)_y\|_{L^\infty(\mathbf{T}_\alpha)}\|\Delta(\psi_2)_x\|_{L^2(\mathbf{T}_\alpha)} \\
&\le (\tau_1+\tau_2)(\beta+\gamma), \\
\|J(\psi_2, \Delta\phi_N)\|_{L^2(\mathbf{T}_\alpha)} &= \|(\psi_2)_x\Delta(\phi_N)_y - (\psi_2)_y\Delta(\phi_N)_x\|_{L^2(\mathbf{T}_\alpha)} \\
&\le \|\Delta(\phi_N)_y\|_{L^\infty(\mathbf{T}_\alpha)}\|(\psi_2)_x\|_{L^2(\mathbf{T}_\alpha)} + \|\Delta(\phi_N)_x\|_{L^\infty(\mathbf{T}_\alpha)}\|(\psi_2)_y\|_{L^2(\mathbf{T}_\alpha)} \\
&\le \tau_4\|(\psi_N^{(2)} + \psi_*^{(2)})_x\|_{L^2(\mathbf{T}_\alpha)} + \tau_3\|(\psi_N^{(2)} + \psi_*^{(2)})_y\|_{L^2(\mathbf{T}_\alpha)} \\
&\le (\tau_4C_2 + \tau_3C_4)\beta + (\tau_4\alpha^{-2} + \tau_3C_3)\gamma.
\end{aligned}$$

Then

$$\begin{aligned}
\|f'(\psi_1)\psi_2\|_{L^2(\mathbf{T}_\alpha)} \le R\Big(&(\tau_1+\tau_2+\tau_4C_2+\tau_3C_4)\beta + (\tau_1+\tau_2+\tau_4\alpha^{-2}+\tau_3C_3)\gamma \\
&+ 2(C_{13}+C_{15})\gamma^2 + 2(C_{13}+C_{14}+C_{15}+C_{16})\beta\gamma + 2(C_{14}+C_{16})\beta^2\Big).
\end{aligned} \tag{55}$$

Therefore $Z_2(U) > 0$ can be determined satisfying

$$\begin{aligned}
C_5R\Big(&(\tau_1+\tau_2+\tau_4C_2+\tau_3C_4)\beta + (\tau_1+\tau_2+\tau_4\alpha^{-2}+\tau_3C_3)\gamma + 2(C_{13}+C_{15})\gamma^2 \\
&+ 2(C_{13}+C_{14}+C_{15}+C_{16})\beta\gamma + 2(C_{14}+C_{16})\beta^2\Big) \le Z_2(U).
\end{aligned}$$

5.6. Verification algorithm

For the results stated above, we formulate the following verification algorithm. Here, assume the computation of ϕ_N, Y_1, Y_2 and ρ have been done.

- $k = 0$
  Set the initial values $\gamma^{(0)} > 0$ and $\beta^{(0)} > 0$.
- $k \ge 1$
  (1) For a fixed small constant $\varepsilon > 0$, set

  $$\hat{\gamma}^{(k)} := (1+\varepsilon)\gamma^{(k-1)}, \qquad \hat{\beta}^{(k)} := (1+\varepsilon)\beta^{(k-1)}.$$

  (2) The kth candidate set $U^{(k)}$ is defined by

  $$\begin{aligned}
  U_N^{(k)} &:= \{v_N \in X_N \mid \|v_N\|_X \le \hat{\gamma}^{(k)}\}, \\
  U_*^{(k)} &:= \{v_* \in X_* \mid \|v_*\|_X \le \hat{\beta}^{(k)}\}, \\
  U^{(k)} &:= U_N^{(k)} + U_*^{(k)}.
  \end{aligned}$$

  (3) Compute values $Z_1(U^{(k)})$ and $Z_2(U^{(k)})$ for kth iteration satisfying

  $$\begin{aligned}
  &\rho R\Big(C_{20}\hat{\beta}^{(k)} + \alpha^{-1}\Big\{(\tau_4C_2+\tau_3C_4)\hat{\beta}^{(k)} + 2(C_{13}+C_{15})(\hat{\gamma}^{(k)})^2 + 2(C_{13}+C_{14}+C_{15}+C_{16})\hat{\beta}^{(k)}\hat{\gamma}^{(k)} \\
  &\quad + 2(C_{14}+C_{16})(\hat{\beta}^{(k)})^2\Big\}\Big) \le Z_1(U^{(k)}), \\
  &C_5R\Big((\tau_1+\tau_2+\tau_4C_2+\tau_3C_4)\hat{\beta}^{(k)} + (\tau_1+\tau_2+\tau_4\alpha^{-2}+\tau_3C_3)\hat{\gamma}^{(k)} + 2(C_{13}+C_{15})(\hat{\gamma}^{(k)})^2 \\
  &\quad + 2(C_{13}+C_{14}+C_{15}+C_{16})\hat{\beta}^{(k)}\hat{\gamma}^{(k)} + 2(C_{14}+C_{16})(\hat{\beta}^{(k)})^2\Big) \le Z_2(U^{(k)}).
  \end{aligned}$$

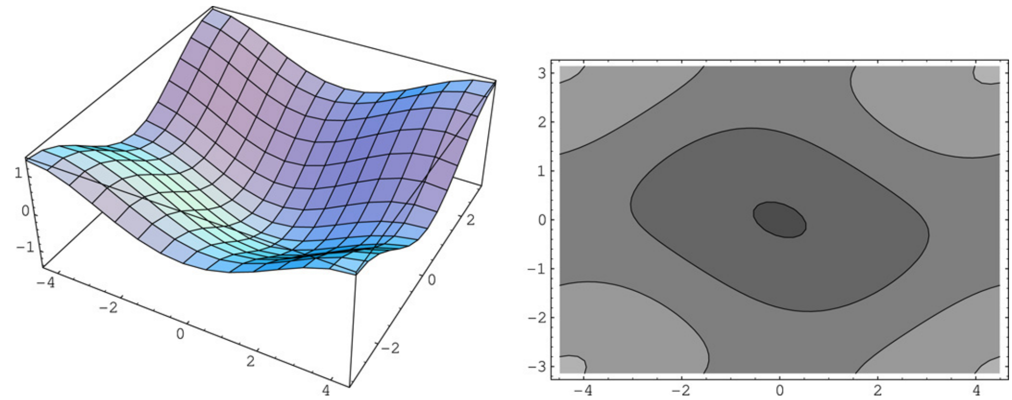

Fig. 1. Shape of approximate solution.

(4) If $Y_1 + Z_1(U^{(k)}) < \hat{\gamma}^{(k)}$ and $Y_2 + Z_2(U^{(k)}) < \hat{\beta}^{(k)}$ hold then stop, and there exists a desired solution in $U^{(k)} \subset X$ uniquely.
(5) Setting $k := k + 1$, $\gamma^{(k)} := Y_1 + Z_1(U^{(k)})$ and $\beta^{(k)} := Y_2 + Z_2(U^{(k)})$ and return to step 1. If k reaches a maximum iteration number or if $\gamma^{(k)}$ and $\beta^{(k)}$ exceed a criterion then stop, and the verification fails.

In actual computation, almost all cost of verification procedures center on the estimation of ρ in (48). In our algorithm, when the Reynolds number tends to be large, a larger truncation number N should be needed because each $Z_1(U^{(k)})$ and $Z_2(U^{(k)})$ is in proportional to R.

6. Some verification results

We now show some verification results. The interval arithmetic in each verification step was implemented using Sun ONE Studio 7, Compiler Collection Fortran 95 on FUJITSU PRIMEPOWER850 (CPU: SPARC64-GP 1.3 GHz, OS: Solaris8). The approximate solutions were obtained by Newton–Raphson method using usual floating point arithmetic by double precision.

6.1. Result 1

The Reynold number is $R = 4$ and aspect ratio is $\alpha = 0.7$. In order to show *concrete* approximate solution, the obtained approximate solution was translated to decimal digit. We adopt ϕ_N as each interval coefficient encloses the decimal value rigorously. Fig. 1 shows the shape of approximate solution ϕ_N.

Fig. 2 shows the obtained parameters as the result of verified computation for $N = 45$. The computation (elapsed) time is 38,196 s (10.61 h).

The verification algorithm executed successfully under the following values:

$$Y_1 + Z_1(U^{(k)}) = 0.2104483239393 \times 10^{-9},$$
$$Y_2 + Z_2(U^{(k)}) = 0.1195514641468 \times 10^{-9},$$

and we can assure that there exists a non-trivial solution ϕ around the approximate solution ϕ_N bounded

$$\|\phi - \phi_N\|_X \le 0.32999978808611105 \times 10^{-9}$$

with local uniqueness. Moreover the solution is unique for the bound of

$$\|\phi - \phi_N\|_X \le 0.9603388265712344 \times 10^{-4}.$$

6.2. Result 2

Fig. 3 shows obtained parameters as the result of verified computation for $R = 13$, $\alpha = 0.7$ and $N = 80$. The computation (elapsed) time is 2,328,377 s (646.77 h).

The verification algorithm executed successfully under the following values:

$$Y_1 + Z_1(U^{(k)}) = 0.2984666091441 \times 10^{-9},$$
$$Y_2 + Z_2(U^{(k)}) = 0.2550977301831 \times 10^{-9},$$

and we can assure that there exists a non-trivial solution ϕ around the approximate solution ϕ_N bounded

$$\|\phi - \phi_N\|_X \le 0.5535643393271721 \times 10^{-9}$$

$$
\begin{array}{ll}
\tau_1 = 0.679988962444568 & \tau_2 = 0.760572136105463 \\
\tau_3 = 0.688531888675846 & \tau_4 = 0.762504737664696 \\
C_1 = 0.299524522791540 \times 10^{-4} & C_2 = 0.964468963388759 \times 10^{-3} \\
C_3 = 1 & C_4 = 0.472589792060492 \times 10^{-3} \\
C_5 = 0.310559006211180 \times 10^{-1} & C_6 = 1 \\
C_7 = 0.217391304347826 \times 10^{-1} & C_8 = 0.133158578910294 \\
C_9 = 0.972725460991748 & C_{10} = 0.143318650381104 \times 10^{1} \\
C_{11} = 0.442078072440795 \times 10^{1} & C_{12} = 0.454510495768881 \times 10^{-1} \\
C_{13} = 0.309454650708556 \times 10^{1} & C_{14} = 0.146352379637580 \times 10^{1} \\
C_{15} = 0.253907054371308 \times 10^{1} & C_{16} = 0.145439563886989 \times 10^{1} \\
C_{17} = 0.549295002708017 \times 10^{2} & C_{18} = 0.173027925853025 \times 10^{4} \\
C_{19} = 0.247182751218608 \times 10^{4} & C_{20} = 0.153123721033578 \\
Y_1 = 0.486767083804449 \times 10^{-11} & Y_2 = 0.177855233274369 \times 10^{-11} \\
\rho = 0.277961085501260 \times 10^{1} &
\end{array}
$$

Fig. 2. Obtained parameters.

$$
\begin{array}{ll}
\tau_1 = 0.804392084576680 & \tau_2 = 0.280788924200297 \\
\tau_3 = 1.08299013162018 & \tau_4 = 0.282159189340473 \\
C_1 = 0.548593709375779 \times 10^{-5} & C_2 = 0.311052633216067 \times 10^{-3} \\
C_3 = 1 & C_4 = 0.152415790275873 \times 10^{-3} \\
C_5 = 0.176366843033510 \times 10^{-1} & C_6 = 1 \\
C_7 = 0.123456790123457 \times 10^{-1} & C_8 = 0.133158578910294 \\
C_9 = 0.972725460991748 & C_{10} = 0.143318650381104 \times 10^{1} \\
C_{11} = 0.442078072440795 \times 10^{1} & C_{12} = 0.255799572311638 \times 10^{-1} \\
C_{13} = 0.309454650708556 \times 10^{1} & C_{14} = 0.145038357500699 \times 10^{1} \\
C_{15} = 0.253907054371308 \times 10^{1} & C_{16} = 0.144521575558961 \times 10^{1} \\
C_{17} = 0.976524449258696 \times 10^{2} & C_{18} = 0.546853691584870 \times 10^{4} \\
C_{19} = 0.781219559406957 \times 10^{4} & C_{20} = 0.738811961720739 \times 10^{-1} \\
Y_1 = 0.130027857641770 \times 10^{-10} & Y_2 = 0.376053834380867 \times 10^{-11} \\
\rho = 0.115923517738789 \times 10^{1} &
\end{array}
$$

Fig. 3. Obtained parameters.

with local uniqueness. Moreover the solution is unique for the bound of

$$\|\phi - \phi_N\|_X \le 0.12104328219862762 \times 10^{-3}.$$

For some fixed Reynolds number R, we can prove the existence of steady-state solutions for the Kolmogorov flows by computer-assisted proof. We cannot say for certain the continuity of the verified solutions with respect to the Reynolds number. These questions must be solved in our future works.

Acknowledgements

We would like to thank the referees for many helpful insights and comments. This work was supported by a Grant-in-Aid from the Ministry of Education, Culture, Sports, Science and Technology of Japan (Nos. 18540127, 15204007, 16104001).

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