



# A generalized inverse eigenvalue problem in structural dynamic model updating

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## ABSTRACT

This paper is concerned with the problem of the best approximation for a given matrix pencil under a given spectral constraint and a submatrix pencil constraint. Such a problem arises in structural dynamic model updating. By using the Moore–Penrose generalized inverse and the singular value decomposition (SVD) matrices, the solvability condition and the expression for the solution of the problem are presented. A numerical algorithm for solving the problem is developed.

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## 1. Introduction

Throughout this paper, we denote the set of real  $m \times n$  matrices by  $\mathbf{R}^{m \times n}$ , the set of all orthogonal matrices in  $\mathbf{R}^{n \times n}$  by  $\mathbf{O}^{n \times n}$ , the transpose and the Moore–Penrose generalized inverse of a real matrix  $A$  by  $A^T$  and  $A^+$ , respectively. For  $A \in \mathbf{R}^{n \times n}$ ,  $\text{tr}(A)$  denotes the trace of the matrix  $A$ . For  $A, B \in \mathbf{R}^{m \times n}$ , an inner product in  $\mathbf{R}^{m \times n}$  is defined by  $(A, B) = \text{tr}(B^T A)$ , then  $\mathbf{R}^{m \times n}$  is a Hilbert space. The matrix norm  $\| \cdot \|$  induced by the inner product is the Frobenius norm.

Using the finite element technique, the dynamic analysis of a mechanical or civil structure is modeled by the generalized eigenvalue problem

$$K_a x = \lambda M_a x, \quad (1)$$

where  $K_a, M_a \in \mathbf{R}^{n \times n}$  are the analytical stiffness and mass matrices, respectively. In some applications, due to the complexity of the structure no reasonable analytical model of the stiffness and mass matrices can be evaluated, a preliminary estimate of the stiffness and mass matrices may be obtained by the finite element technique. Very often natural frequencies and mode shapes (eigenvalues and eigenvectors) of a finite element model described by (1) do not match very well with experimentally measured frequencies and mode shapes obtained from a real-life vibration test.

The information on the dynamic behavior of a structure is available from a vibration test, where the excitation and the response of the structure at many points are measured experimentally. Identification techniques (see, e.g. [7]) extract a part of the eigenpairs of the structure from the measurements. However, one usually obtains an incomplete set of eigenpairs from

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the vibration tests (see, e.g., [5,13]). The finite element model updating problem, roughly speaking, is how to incorporate the measured modal data into the finite element model to produce an adjusted finite element model with modal properties that closely match the experimental modal data. Then the updated model may be considered to be a better dynamic representation of the structure. This model may be used with greater confidence for the analysis of the structure under different boundary conditions.

The finite element model updating in structural dynamics has been addressed for many years. The most common approach is to modify the analytical mass and stiffness matrices to satisfy the dynamic equation based on the measured modal data. Let  $X \in \mathbb{R}^{n \times p}$  be the measured modal matrix,  $\Lambda \in \mathbb{R}^{p \times p}$  the measured natural frequencies matrix, where  $n \geq p$ , and  $\Lambda$  is diagonal. The measured mode shapes and frequencies are assumed correct and have to satisfy the dynamic equation

$$KX = MX\Lambda, \quad (2)$$

where  $M, K \in \mathbb{R}^{n \times n}$  are the mass and stiffness matrices to be corrected, respectively. To date, some methods have been proposed to correct the analytical mass and stiffness matrices from measured modal data (see, e.g., [2,3,6,9–13,15,16,19,20,23–25,27]). However, the analytical mass and stiffness matrices  $M_a$  and  $K_a$  are adjusted globally. From a practical viewpoint, a spatial representation of the structural-element property changes that resulted from the model errors is generally preferred for engineering applications. Model errors can be localized by using sensitivity analysis (see, e.g., [18,26]), residual force approach [17], least-squares approach [21], assigned eigenstructure [8]. Based on the localization of modeling errors, it is usual practice to adjust partial elements of the analytical mass and stiffness matrices  $M_a$  and  $K_a$  using measured mode shapes and natural frequencies. Among current developments for the structural dynamic model updating, one challenge that is of practical importance is to correct the inaccurate elements of the analytical mass and stiffness matrices  $M_a$  and  $K_a$  while maintaining the accurate elements invariant.

The elements to be corrected in the analytical mass and stiffness matrices  $M_a$  and  $K_a$  are determined by the error-localization processes (see, e.g., [8,17,18,21,26]). For index sets  $\alpha = \{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$  and  $\beta = \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$ , we assume that the elements that lie in the rows of the matrices  $M_a$  and  $K_a$  indexed by  $\alpha$  and the columns indexed by  $\beta$  are accurate while the others need to be corrected. In general,  $n > r, k \geq p$ . It is well known that there are permutation matrices  $P$  and  $Q$  such that

$$P^T K_a Q = \begin{bmatrix} K_{11}^{(a)} & K_{12}^{(a)} \\ K_{21}^{(a)} & K_{22}^{(a)} \end{bmatrix} \begin{matrix} r \\ n-r \\ k \quad n-k \end{matrix}, \quad P^T M_a Q = \begin{bmatrix} M_{11}^{(a)} & M_{12}^{(a)} \\ M_{21}^{(a)} & M_{22}^{(a)} \end{bmatrix} \begin{matrix} r \\ n-r \\ k \quad n-k \end{matrix}.$$

The elements of submatrices  $K_{11}^{(a)}, M_{11}^{(a)}$  are precisely accurate while the elements of other blocks are to be updated. The dynamic equation (2) is equivalent to  $(P^T K Q)(Q^T X) = (P^T M Q)(Q^T X)\Lambda$ . Without loss of generality, we shall assume that  $\alpha = \beta$ , the  $r \times r$  leading principal submatrices of the analytical mass and stiffness matrices  $M_a$  and  $K_a$  are accurate, i.e., the  $r \times r$  leading principal submatrices of the mass and stiffness matrices  $M$  and  $K$  to be corrected are known. Thus, the problem can be mathematically formulated as the following problems.

**Problem I.** Given a full column rank matrix  $X \in \mathbb{R}^{n \times p}$ , a diagonal matrix  $\Lambda \in \mathbb{R}^{p \times p}$  and matrices  $K_0 \in \mathbb{R}^{r \times r}, M_0 \in \mathbb{R}^{r \times r}$ , find real  $n \times n$  matrices  $K, M$  such that

$$KX = MX\Lambda, \quad K([1, r]) = K_0, \quad M([1, r]) = M_0, \quad (3)$$

where  $K([1, r])$  is the  $r \times r$  leading principal submatrix of the matrix  $K$ .

**Problem II.** Given  $n \times n$  matrices  $K_a, M_a$  with  $K_a([1, r]) = K_0, M_a([1, r]) = M_0$ , find  $(\hat{K}, \hat{M}) \in \mathbf{S_E}$  such that

$$\|K_a - \hat{K}\|^2 + \|M_a - \hat{M}\|^2 = \inf_{(K, M) \in \mathbf{S_E}} (\|K_a - K\|^2 + \|M_a - M\|^2), \quad (4)$$

where  $\mathbf{S_E}$  is the solution set of Problem I.

Let  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p), X = [x_1, \dots, x_p]$  where  $x_i \in \mathbb{R}^n$ , then the set  $\mathbf{S_E}$ , further the matrices  $\hat{K}$  and  $\hat{M}$ , is determined by the eigenvalues  $\lambda_1, \dots, \lambda_p$  and corresponding eigenvectors  $x_1, \dots, x_p$ , so Problems I and II is a generalized inverse eigenvalue problem.

The paper is organized as follows. In Section 2, using the Moore–Penrose generalized inverse and the singular value decomposition of a matrix, we give necessary and sufficient conditions for the set  $\mathbf{S_E}$  to be nonempty and construct the set  $\mathbf{S_E}$  explicitly when it is nonempty. In Section 3, we show that there exists a unique solution in Problem II if the set  $\mathbf{S_E}$  is nonempty, and present the expression of the unique solution  $(\hat{K}, \hat{M})$  of Problem II. In Section 4, a numerical algorithm for solving Problems I and II is described and a numerical example is provided.

## 2. The solution of Problem I

We first introduce the results about the existence conditions and expression of solutions to matrix equations.

**Lemma 1** ([4]). If  $E \in \mathbf{R}^{m \times n}$ ,  $F \in \mathbf{R}^{p \times q}$ , and  $G \in \mathbf{R}^{m \times q}$ , then the matrix equation

$$EZF = G$$

has a solution  $Z \in \mathbf{R}^{n \times p}$  if and only if

$$EE^+GF^+F = G,$$

in which case the general solution of the equation  $EZF = G$  can be expressed as

$$Z = E^+GF^+ + Y - E^+EYFF^+,$$

where  $Y \in \mathbf{R}^{n \times p}$  is an arbitrary matrix.

Let the partitions of the matrices  $X$ ,  $K$  and  $M$  be

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad X_1 \in \mathbf{R}^{r \times p}, X_2 \in \mathbf{R}^{(n-r) \times p}, \quad (5)$$

and

$$K = \begin{bmatrix} K_0 & K_1 \\ K_2 & K_3 \end{bmatrix} \begin{matrix} r \\ n-r \\ r & n-r \end{matrix}, \quad M = \begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix} \begin{matrix} r \\ n-r \\ r & n-r \end{matrix}, \quad (6)$$

where the matrices  $K_i$ ,  $M_i$  ( $i = 1, 2, 3$ ) are yet to be determined.

We assume that the singular value decomposition (SVD) (see, e.g., [14]) of the matrix  $X_2$  is

$$X_2 = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T, \quad (7)$$

where  $U = [U_1, U_2] \in \mathbf{OR}^{(n-r) \times (n-r)}$ ,  $V = [V_1, V_2] \in \mathbf{OR}^{p \times p}$ ,  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_s)$ ,  $\sigma_i > 0$  ( $i = 1, \dots, s$ ),  $s = \text{rank}(X_2)$ ,  $U_1 \in \mathbf{R}^{(n-r) \times s}$ ,  $V_1 \in \mathbf{R}^{p \times s}$ , and the SVD of the matrix  $X_2 \Lambda V_2$  is

$$X_2 \Lambda V_2 = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} Q^T, \quad (8)$$

where  $P = [P_1, P_2] \in \mathbf{OR}^{(n-r) \times (n-r)}$ ,  $Q = [Q_1, Q_2] \in \mathbf{OR}^{(p-s) \times (p-s)}$ ,  $\Omega = \text{diag}(\omega_1, \dots, \omega_t)$ ,  $\omega_i > 0$  ( $i = 1, \dots, t$ ),  $t = \text{rank}(X_2 \Lambda V_2)$ ,  $P_1 \in \mathbf{R}^{(n-r) \times t}$ ,  $Q_1 \in \mathbf{R}^{(p-s) \times t}$ .

**Theorem 1.** Suppose that  $K_0 \in \mathbf{R}^{r \times r}$ ,  $M_0 \in \mathbf{R}^{r \times r}$ ,  $X \in \mathbf{R}^{n \times p}$  with  $\text{rank}(X) = p$ , and  $\Lambda \in \mathbf{R}^{p \times p}$  is a diagonal matrix. Let the partitions of  $X$ ,  $K$  and  $M$  be (5) and (6), and the SVDs of  $X_2$  and  $X_2 \Lambda V_2$  be given in (7) and (8), respectively. Then Problem I is solvable if and only if

$$(K_0 X_1 - M_0 X_1 \Lambda) V_2 [I - (X_2 \Lambda V_2)^+ X_2 \Lambda V_2] = 0, \quad (9)$$

in which case the solution set  $\mathbf{S}_E$  can be expressed as

$$\mathbf{S}_E = \left\{ (K, M) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n} \mid K = \begin{bmatrix} K_0 & K_1 \\ K_2 & K_3 \end{bmatrix}, M = \begin{bmatrix} M_0 & M_1 \\ M_2 & M_3 \end{bmatrix} \right\}, \quad (10)$$

where

$$\begin{cases} K_1 = K_{10} + LP_2^T X_2 \Lambda X_2^+ + WU_2^T, \\ (K_2, K_3) = (M_2, M_3) X \Lambda X^+ + G(I - XX^+), \end{cases} \quad (11)$$

$$\begin{cases} M_1 = M_{10} + LP_2^T, \\ M_{10} = (K_0 X_1 - M_0 X_1 \Lambda) V_2 (X_2 \Lambda V_2)^+, \end{cases} \quad (12)$$

$$K_{10} = (M_0 X_1 \Lambda - K_0 X_1) [I - V_2 (X_2 \Lambda V_2)^+ X_2 \Lambda V_2] X_2^+, \quad (13)$$

and  $L \in \mathbf{R}^{r \times (n-r-t)}$ ,  $W \in \mathbf{R}^{r \times (n-r-s)}$ ,  $G \in \mathbf{R}^{(n-r) \times n}$ ,  $[M_2, M_3] \in \mathbf{R}^{(n-r) \times n}$  are arbitrary matrices.

**Proof.** From (5) and (6), the Eq. (3) is equivalent to the following two equations

$$K_0X_1 + K_1X_2 = M_0X_1\Lambda + M_1X_2\Lambda, \quad (14)$$

$$[K_2, K_3]X = [M_2, M_3]X\Lambda. \quad (15)$$

It follows from Lemma 1 that the Eq. (14) with respect to unknown matrix  $K_1$  has a solution if and only if

$$(M_0X_1\Lambda - K_0X_1 + M_1X_2\Lambda)X_2^+X_2 = M_0X_1\Lambda - K_0X_1 + M_1X_2\Lambda. \quad (16)$$

From the SVD (7) of  $X_2$ , the Eq. (16) may be reduced to

$$M_1X_2\Lambda V_2 = K_0X_1V_2 - M_0X_1\Lambda V_2. \quad (17)$$

Using Lemma 1, it is easy to verify that the Eq. (17) with respect to unknown matrix  $M_1$  has a solution if and only if the condition (9) holds. In this case, the general solution of the Eq. (17) is

$$M_1 = M_{10} + LP_2^T, \quad (18)$$

where  $M_{10}$  is given by (12) and  $L \in \mathbf{R}^{r \times (n-r-t)}$  is an arbitrary matrix.

Substituting (18) into (14) and applying Lemma 1 again, we get

$$K_1 = K_{10} + LP_2^T X_2 \Lambda X_2^+ + WU_2^T, \quad (19)$$

where  $K_{10}$  is given by (13) and  $W \in \mathbf{R}^{r \times (n-r-s)}$  is an arbitrary matrix.

Since the matrix  $X$  is of full column rank, the Eq. (15) with respect to unknown matrix  $[K_2, K_3]$  is always solvable, and the general solution of the Eq. (15) can be expressed as (11), where  $G \in \mathbf{R}^{(n-r) \times n}$  is an arbitrary matrix and  $[M_2, M_3]$  can be chosen arbitrarily. Substituting (18) and (19) into (6) yields (10).  $\square$

### 3. The solution of Problem II

In order to solve Problem II, we need the following lemma (see [22]).

**Lemma 2.** Suppose that  $A \in \mathbf{R}^{q \times m}$ ,  $\Delta \in \mathbf{R}^{q \times q}$  and  $\Gamma \in \mathbf{R}^{m \times m}$  where  $\Delta^2 = \Delta = \Delta^T$  and  $\Gamma^2 = \Gamma = \Gamma^T$ . Then

$$\|A - \Delta D \Gamma\| = \min_{E \in \mathbf{R}^{q \times m}} \|A - \Delta E \Gamma\|$$

if and only if  $\Delta(A - D)\Gamma = 0$ , in which case,

$$\|A - \Delta D \Gamma\| = \|A - \Delta A \Gamma\|.$$

For the given matrices  $K_a, M_a \in \mathbf{R}^{n \times n}$  with  $K_a([1, r]) = K_0$ ,  $M_a([1, r]) = M_0$ , let

$$K_a = \begin{bmatrix} K_0 & K_1^{(a)} \\ K_2^{(a)} & K_3^{(a)} \end{bmatrix} \begin{matrix} r \\ n-r \end{matrix}, \quad M_a = \begin{bmatrix} M_0 & M_1^{(a)} \\ M_2^{(a)} & M_3^{(a)} \end{bmatrix} \begin{matrix} r \\ n-r \end{matrix}. \quad (20)$$

**Theorem 2.** Let  $K_a, M_a \in \mathbf{R}^{n \times n}$  with  $K_a([1, r]) = K_0$ ,  $M_a([1, r]) = M_0$  and the partition of  $K_a, M_a$  be given by (20). If the condition (9) holds, then Problem II has a unique solution and the solution can be expressed as

$$\hat{K} = \begin{bmatrix} K_0 & K_{10}U_1U_1^T + K_1^{(a)}U_2U_2^T + LP_2^T X_2 \Lambda X_2^+ \\ K_2 & K_3 \end{bmatrix}, \quad (21)$$

$$\hat{M} = \begin{bmatrix} M_0 & M_{10} + LP_2^T \\ M_2 & M_3 \end{bmatrix}, \quad (22)$$

where

$$L = [(K_1^{(a)} - K_{10})(X_2 \Lambda X_2^+)^T + M_1^{(a)} - M_{10}]P_2[I + P_2^T X_2 \Lambda (X_2^T X_2)^+ \Lambda X_2^T P_2]^{-1}, \quad (23)$$

$$[K_2, K_3] = [M_c^{(a)} + K_c^{(a)}(X \Lambda X^+)^T]TX \Lambda X^+ + K_c^{(a)}(I - XX^+), \quad (24)$$

$$[M_2, M_3] = [M_c^{(a)} + K_c^{(a)}(X \Lambda X^+)^T]T, \quad (25)$$

and  $M_c^{(a)} = [M_2^{(a)}, M_3^{(a)}]$ ,  $K_c^{(a)} = [K_2^{(a)}, K_3^{(a)}]$ ,  $T = (I + X \Lambda (X^T X)^{-1} \Lambda X^T)^{-1}$ .

**Proof.** It follows from [Theorem 1](#) that the set  $\mathbf{S_E}$  is nonempty if the condition (9) is satisfied. It is easy to verify that  $\mathbf{S_E}$  is a closed convex subset of  $\mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n}$ . From the best approximation theorem (see, e.g., [1]), it follows that there exists a unique solution  $(\hat{K}, \hat{M})$  in  $\mathbf{S_E}$  such that (4) holds.

We now focus our attention on seeking the unique solution  $(\hat{K}, \hat{M})$  in  $\mathbf{S_E}$ . For any pair of matrices  $(K, M) \in \mathbf{S_E}$  given in (10), let  $M_c = [M_2, M_3]$ . Then

$$\begin{aligned} \|K_a - K\|^2 + \|M_a - M\|^2 &= \|K_1^{(a)} - K_{10} - LP_2^T X_2 \Lambda X_2^+ - WU_2^T\|^2 + \|M_c^{(a)} - M_c\|^2 \\ &\quad + \|M_1^{(a)} - M_{10} - LP_2^T\|^2 + \|K_c^{(a)} - M_c X \Lambda X^+ - G(I - XX^+)\|^2 \\ &= \left\| K_1^{(a)} - K_{10} - LP_2^T X_2 \Lambda X_2^+ - [0, W] \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \right\|^2 + \|M_c^{(a)} - M_c\|^2 \\ &\quad + \|M_1^{(a)} - M_{10} - LP_2^T\|^2 + \|K_c^{(a)} - M_c X \Lambda X^+ - G(I - XX^+)\|^2 \\ &= \|(K_1^{(a)} - K_{10} - LP_2^T X_2 \Lambda X_2^+)U_1\|^2 + \|M_c^{(a)} - M_c\|^2 \\ &\quad + \|M_1^{(a)} - M_{10} - LP_2^T\|^2 + \|K_c^{(a)} - M_c X \Lambda X^+ - G(I - XX^+)\|^2 \\ &\quad + \|(K_1^{(a)} - K_{10} - LP_2^T X_2 \Lambda X_2^+)U_2 - W\|^2. \end{aligned}$$

Observe that  $X_2^+ U_2 = 0$ , therefore,  $\|K_a - K\|^2 + \|M_a - M\|^2 = \min$  if and only if

$$W = (K_1^{(a)} - K_{10})U_2, \quad (26)$$

$$f(L) = \|(K_1^{(a)} - K_{10})U_1 - LP_2^T X_2 \Lambda X_2^+ U_1\|^2 + \|(M_1^{(a)} - M_{10}) - LP_2^T\|^2 = \min, \quad (27)$$

$$\|K_c^{(a)} - M_c X \Lambda X^+ - G(I - XX^+)\|^2 + \|M_c^{(a)} - M_c\|^2 = \min. \quad (28)$$

From (27), we have

$$\begin{aligned} f(L) &= \text{tr}(U_1^T (K_1^{(a)} - K_{10})^T (K_1^{(a)} - K_{10}) U_1) + \text{tr}(P_2 L^T L P_2^T) + \text{tr}((P_2^T X_2 \Lambda X_2^+ U_1)^T L^T L P_2^T X_2 \Lambda X_2^+ U_1) \\ &\quad - 2\text{tr}((M_1^{(a)} - M_{10}))^T L P_2^T + \text{tr}((M_1^{(a)} - M_{10})^T (M_1^{(a)} - M_{10})) - 2\text{tr}(U_1^T (K_1^{(a)} - K_{10})^T L P_2^T X_2 \Lambda X_2^+ U_1). \end{aligned}$$

Consequently,

$$\frac{\partial f(L)}{\partial L} = -2(K_1^{(a)} - K_{10})U_1 U_1^T (X_2 \Lambda X_2^+)^T P_2 + 2LP_2^T X_2 \Lambda X_2^+ U_1 U_1^T (X_2 \Lambda X_2^+)^T P_2 - 2(M_1^{(a)} - M_{10})P_2 + 2L.$$

Setting  $\frac{\partial f(L)}{\partial L} = 0$ , we obtain

$$L = [(K_1^{(a)} - K_{10})U_1 U_1^T (X_2 \Lambda X_2^+)^T + M_1^{(a)} - M_{10}]P_2[I + P_2^T X_2 \Lambda X_2^+ U_1 U_1^T (X_2 \Lambda X_2^+)^T P_2]^{-1}. \quad (29)$$

Using the SVD (7) of  $X_2$ , it is easy to verify that (29) may be reduced to (23).

Since  $(I - XX^+)^2 = I - XX^+ = (I - XX^+)^T$ , [Lemma 2](#) implies that (28) attains a minimum with respect to the matrix  $G$  if and only if

$$(K_c^{(a)} - M_c X \Lambda X^+)(I - XX^+) = G(I - XX^+),$$

namely,

$$K_c^{(a)}(I - XX^+) = G(I - XX^+), \quad (30)$$

in which case, the minimization problem (28) is equivalent to

$$\|K_c^{(a)} XX^+ - M_c X \Lambda X^+\|^2 + \|M_c^{(a)} - M_c\|^2 = \min. \quad (31)$$

By the similar way of finding the minimum of the function  $f(L)$ , we can solve the minimization problem (31), and then obtain (25).

Substituting (25) and (30) into (11) yields (24). Finally, substituting (23)–(26) into (10), we get the expressions (21) and (22). Thus the proof is complete.  $\square$

#### 4. Numerical algorithm and example

Based on [Theorems 1](#) and [2](#) we can describe an algorithm for solving [Problem II](#) as follows.

**Algorithm 1.** (1) Input matrices  $X, \Lambda, K_0, M_0, K_a$  and  $M_a$ ;  
(2) Form the matrix  $X_1, X_2$  according to (5);

- (3) Compute the SVD (7) of the matrix  $X_2$ ;
- (4) Compute the SVD (8) of the matrix  $X_2 \Lambda V_2$ ;
- (5) If  $\|(K_0 X_1 - M_0 X_1 \Lambda) V_2 [I - (X_2 \Lambda V_2)^+ X_2 \Lambda V_2]\| < \text{tol}$ , i.e., the condition (9) holds computationally, go to (6); otherwise, Problem I has no solution, and stop;
- (6) Compute the matrices  $M_{10}$  and  $K_{10}$  by (12) and (13), respectively;
- (7) Compute the matrix  $L$  by (23);
- (8) Compute the matrices  $[M_2, M_3]$  and  $[K_2, K_3]$  by (25) and (24), respectively;
- (9) Compute the unique solution  $(\hat{K}, \hat{M})$  of Problem II by (21) and (22).

We now give a numerical example to show the application of the above-obtained solvability theory and numerical algorithm for solving Problem II. All codes are run in MATLAB (version 6.5) with machine precision  $10^{-16}$  on a Pentium IV personal computer.

**Example 1.** This is an example for updating the mass and stiffness matrices of a vibrating system described in (1). Let  $n = 10$ ,  $r = 6$ ,  $p = 4$ , and the matrices  $X$ ,  $\Lambda$ ,  $K_0$ ,  $M_0$ ,  $K_a$  and  $M_a$  be given by

$$X = \begin{bmatrix} 0.4364 & 0.4369 & -0.4394 & -0.4447 \\ -0.0401 & -0.1392 & 0.2301 & 0.3281 \\ 0.3166 & 0.0306 & 0.1738 & 0.2881 \\ -0.0396 & -0.1249 & 0.1386 & 0.0522 \\ 0.2013 & -0.2575 & 0.2081 & -0.1481 \\ -0.0367 & -0.0588 & -0.1112 & -0.2024 \\ 0.1004 & -0.2985 & -0.2314 & -0.1400 \\ -0.0297 & 0.0295 & -0.1189 & 0.2081 \\ 0.0279 & -0.1315 & -0.2657 & 0.3375 \\ -0.0173 & 0.0679 & 0.0913 & -0.0182 \end{bmatrix},$$

$$\Lambda = \text{diag}(0.0047, 0.1851, 1.4605, 5.6999),$$

$$K_0 = \begin{bmatrix} 12 & 18 & -12 & 18 & 0 & 0 \\ 18 & 36 & -18 & 18 & 0 & 0 \\ -12 & -18 & 24 & 0 & -12 & 18 \\ 18 & 18 & 0 & 72 & -18 & 18 \\ 0 & 0 & -12 & -18 & 24 & 0 \\ 0 & 0 & 18 & 18 & 0 & 72 \end{bmatrix},$$

$$M_0 = \begin{bmatrix} 1.56 & 0.66 & 0.54 & -0.39 & 0 & 0 \\ 0.66 & 0.36 & 0.39 & -0.27 & 0 & 0 \\ 0.54 & 0.39 & 3.12 & 0 & 0.54 & -0.39 \\ -0.39 & -0.27 & 0 & 0.72 & 0.39 & -0.27 \\ 0 & 0 & 0.54 & 0.39 & 3.12 & 0 \\ 0 & 0 & -0.39 & -0.27 & 0 & 0.72 \end{bmatrix},$$

$$K_a = \begin{bmatrix} 12 & 18 & -12 & 18 & 0 & 0 & 0 & 0 & 0 & 0 \\ 18 & 36 & -18 & 18 & 0 & 0 & 18 & -18 & 18 & 0 \\ -12 & -18 & 24 & 0 & -12 & 18 & -18 & -12 & 18 & 0 \\ 18 & 18 & 0 & 72 & -18 & 18 & 18 & 0 & 0 & 0 \\ 0 & 0 & -12 & -18 & 24 & 0 & 18 & 0 & 0 & 0 \\ 0 & 0 & 18 & 18 & 0 & 72 & 0 & -12 & -18 & 0 \\ 0 & 18 & -18 & 18 & 18 & 0 & 72 & 0 & 0 & 0 \\ 0 & -18 & -12 & 0 & 0 & -12 & 0 & 24 & 0 & 18 \\ 0 & 18 & 18 & 0 & 0 & -18 & 0 & 0 & 72 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 & 18 & 72 \end{bmatrix},$$

and

$$M_a = \begin{bmatrix} 1.56 & 0.66 & 0.54 & -0.39 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.66 & 0.36 & 0.39 & -0.27 & 0 & 0 & -0.27 & 0.39 & -0.27 & 0 \\ 0.54 & 0.39 & 3.12 & 0 & 0.54 & -0.39 & 0.39 & 0.54 & -0.39 & 0 \\ -0.39 & -0.27 & 0 & 0.72 & 0.39 & -0.27 & -0.27 & 0 & 0 & 0 \\ 0 & 0 & 0.54 & 0.39 & 3.12 & 0 & -0.39 & 0 & 0 & 0 \\ 0 & 0 & -0.39 & -0.27 & 0 & 0.72 & 0 & 0.54 & 0.39 & 0 \\ 0 & -0.27 & 0.39 & -0.27 & -0.39 & 0 & 0.72 & 0 & 0 & 0 \\ 0 & 0.39 & 0.54 & 0 & 0 & 0.54 & 0 & 3.12 & 0 & -0.39 \\ 0 & -0.27 & -0.39 & 0 & 0 & 0.39 & 0 & 0 & 0.72 & -0.27 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.39 & -0.27 & 0.72 \end{bmatrix}.$$

From Algorithm 1, we obtain  $\text{rank}(X_2) = 3$ ,  $X_2 \Lambda V_2 \in \mathbf{R}^{4 \times 1}$  and  $\text{rank}(X_2 \Lambda V_2) = 1$ , i.e.,  $X_2 \Lambda V_2$  is of full column rank. The condition (9) holds. Thus, it follows from Theorem 1 that  $\mathbf{S}_E$  is nonempty. Using Algorithm 1, we obtain the unique solution of Problem II as follows.

$$\hat{K} = \begin{bmatrix} 12.0000 & 18.0000 & -12.0000 & 18.0000 & 0 & 0 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ 18.0000 & 36.0000 & -18.0000 & 18.0000 & 0 & 0 & -5.6206 & -9.1400 & 12.7274 & 3.5436 \\ -12.0000 & -18.0000 & 24.0000 & 0 & -12.0000 & 18.0000 & -10.0406 & -15.3140 & 19.1992 & -1.1078 \\ 18.0000 & 18.0000 & 0 & 72.0000 & -18.0000 & 18.0000 & 2.9034 & 4.1637 & -4.3621 & 2.6872 \\ 0 & 0 & -12.0000 & -18.0000 & 24.0000 & 0 & -11.3345 & 18.3744 & -0.5873 & 2.3425 \\ 0 & 0 & 18.0000 & 18.0000 & 0 & 72.0000 & -17.7012 & 17.8572 & -3.6664 & -3.8362 \\ 17.4213 & 10.0810 & -24.5463 & 9.9114 & -10.8951 & -5.9247 & 36.2994 & 2.1481 & -17.5161 & 8.4875 \\ -0.3684 & -15.9315 & -6.5375 & 1.6101 & 7.9335 & -12.0828 & 5.4120 & 22.6464 & 1.5228 & 16.9951 \\ -1.9737 & 16.1176 & 10.5726 & -1.6848 & -12.2382 & -19.5662 & -15.3423 & 0.0836 & 62.9409 & 21.8670 \\ -1.0513 & 0.9639 & 1.8509 & 0.9552 & 4.0868 & 0.3733 & 3.5353 & 17.3161 & 19.1068 & 71.2868 \end{bmatrix},$$

$$\hat{M} = \begin{bmatrix} 1.5600 & 0.6600 & 0.5400 & -0.3900 & 0 & 0 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ 0.6600 & 0.3600 & 0.3900 & -0.2700 & 0 & 0 & 0.1885 & 1.5378 & 0.7257 & -0.4403 \\ 0.5400 & 0.3900 & 3.1200 & 0 & 0.5400 & -0.3900 & 0.3602 & 1.5358 & 1.6417 & -0.2488 \\ -0.3900 & -0.2700 & 0 & 0.7200 & 0.3900 & -0.2700 & -0.0832 & -0.3575 & -0.3630 & 0.0492 \\ 0 & 0 & 0.5400 & 0.3900 & 3.1200 & 0 & 0.4214 & 1.9007 & -1.6382 & -0.8004 \\ 0 & 0 & -0.3900 & -0.2700 & 0 & 0.7200 & 0.1342 & 5.2487 & -4.2343 & -1.4975 \\ 3.2124 & -3.6540 & -4.5483 & -1.3344 & -0.3103 & 3.9070 & 11.4594 & 1.8054 & 7.7319 & -3.2294 \\ 2.8526 & -0.7954 & 0.0219 & -0.9448 & -1.8174 & 0.7355 & 0.1411 & 3.9306 & 1.1579 & -0.6448 \\ -2.2846 & 1.7866 & 0.9487 & -0.6402 & -3.8046 & -0.4615 & 3.4906 & 4.7395 & 10.6966 & -2.4076 \\ -0.4978 & 0.7537 & 0.9425 & -0.1384 & -1.1501 & -0.7427 & -0.8582 & 0.4973 & 0.9035 & 0.7067 \end{bmatrix}.$$

It is easy to calculate  $\|\hat{K}X - \hat{M}X\Lambda\| = 4.8921\text{e}-14$ . Therefore, the prescribed eigenvalues (the diagonal elements of the matrix  $\Lambda$ ) and eigenvectors (the column vectors of the matrix  $X$ ) are embedded in the new model  $\hat{K}x = \lambda\hat{M}x$ , and  $K_0, M_0$  are the  $6 \times 6$  leading principal submatrices of the matrices  $\hat{K}, \hat{M}$ , respectively.

## 5. Concluding remarks

Structural dynamic model updating with eigeninformation and a submatrix pencil constraint or the connectivity of the original finite element model has been a longstanding open problem. Many efforts in both theoretic and computational aspects have been made in response to the demand of engineering applications. The results are limited and hardly satisfactory so far. One of the most fundamental challenges is to characterize when the model updating problem with eigeninformation and the submatrix pencil constraint is solvable.

In this paper, the model updating problem with eigeninformation and a submatrix pencil constraint is formulated as the problem of approximating a given matrix pencil under a given spectral constraint and a submatrix pencil restriction. By using the Moore–Penrose generalized inverse and the singular value decomposition of matrices, we develop a necessary and sufficient condition under which the corresponding problem is solvable, and we present a numerical method for solving the problem. The approach is demonstrated using a numerical example and reasonable results are produced.

If the elements to be updated in analytical mass and stiffness matrices  $M_a$  and  $K_a$  lie in  $(i_k, j_k)$ -positions ( $k = 1, \dots, m$ ) or the positions of the elements to be corrected in both  $M_a$  and  $K_a$  are different, then by comparing the corresponding elements of both sides of the dynamic equation (2), Problem I may be reduced to a linear system  $Ay = b$ , where the components of  $y$  are the elements to be updated,  $A$  and  $b$  are the known matrix and vector resulted from the eigeninformation and accurate elements in  $M_a$  and  $K_a$ , and Problem II can be transformed into finding a nearest vector to a given vector  $\tilde{y}$ , an estimation of the elements to be updated in  $M_a$  and  $K_a$ , under linear restriction  $Ay = b$ . It is verified that the nearest vector is  $A^+b + (I - A^+A)\tilde{y}$ .

In practice the matrices  $M$  and  $K$  are often structured, for example,  $M$  and  $K$  are symmetric. However, Algorithm 4.1 does not guarantee that the updated matrices will be symmetric. Can such a structured matrix pencil be updated with eigeninformation and a submatrix pencil constraint? Can the physical feasibility of the updated matrices be maintained? These problems are the subject of further investigation.

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## References

- [1] J.P. Aubin, Applied Functional Analysis, John Wiley & Sons, Inc., 1979.
- [2] M. Baruch, Optimization procedure to correct stiffness and flexibility matrices using vibration tests, AIAA J. 16 (1978) 1208–1210.
- [3] M. Baruch, Optimal correction of mass and stiffness matrices using measured modes, AIAA J. 22 (1982) 1623–1626.
- [4] A. Ben-Israel, T.N.E. Greville, Generalized Inverse: Theory and Applications, 2nd edition, Springer, New York, 2003.
- [5] A. Berman, W.G. Flannely, Theory of incomplete models of dynamic structures, AIAA J. 9 (1971) 1481–1487.
- [6] A. Berman, E.J. Nagy, Improvement of a large analytical model using test data, AIAA J. 21 (1983) 1168–1173.
- [7] H.H.F. Chu, Modal Testing and Modal Refinement, American Society of Mechanical Engineers, New York, 1983.
- [8] R.G. Cobb, B. Liebst, Structural damage identification using assigned partial eigenstructure, AIAA J. 35 (1997) 152–158.
- [9] H. Dai, Optimal approximation of matrix pencil under linear restriction and its application, Numer. Math. Sinica 11 (1989) 29–37 (in Chinese).

- [10] H. Dai, Optimal approximation of real symmetric matrix pencil under spectral restriction, *Numer. Math. J. Chinese Univ.* 12 (1990) 177–187 (in Chinese).
- [11] H. Dai, Stiffness matrix correction using test data, *Acta Aeronaut. Astronaut. Sinica* 15 (1994) 1091–1094 (in Chinese).
- [12] H. Dai, About an inverse eigenvalue problem arising in vibration analysis, *RAIRO Math. Model. Numer. Anal.* 29 (1995) 421–434.
- [13] M.I. Friswell, J.E. Mottershead, *Finite Element Model Updating in Structural Dynamics*, Kluwer Academic Publishers, Dordrecht, 1995.
- [14] G.H. Golub, C.F. Van Loan, *Matrix Computations*, 3rd edition, The Johns Hopkins University Press, Baltimore, 1996.
- [15] J. He, Structural modification, *Philos. Trans. R. Soc. Lond. A* 359 (2001) 187–204.
- [16] L.W. Li, A new method for structural model updating and stiffness identification, *Mech. Syst. Signal Process.* 16 (2002) 155–167.
- [17] M. Link, Identification and correction of errors in analytical models using test data-theoretical and practical bounds, in: *Proc. of the 8th International Modal Analysis Conference*, 1990, pp. 570–578.
- [18] W. Lubier, A. Lotze, Application of sensitivity methods for error localization in finite element systems, in: *Proc. of the 8th International Modal Analysis Conference*, 1990, pp. 598–604.
- [19] C. Minas, D.J. Inman, Matching finite element models to modal data, *J. Vibration Acoustics* 112 (1990) 84–92.
- [20] J.E. Mottershead, M.I. Friswell, Model updating in structural dynamics: A survey, *J. Sound Vibration* 167 (1993) 347–375.
- [21] J.C. O'Callahan, C.-M. Chou, Localization of model errors in optimized mass and stiffness matrices using modal test data, *Internat. J. Anal. Experimental Anal.* 4 (1989) 8–14.
- [22] W.F. Trench, Inverse eigenproblems and associated approximation problems for matrices with generalized symmetry or skew symmetry, *Linear Algebra Appl.* 380 (2004) 199–211.
- [23] F.-S. Wei, D.-W. Zhang, Mass matrix modification using element correction method, *AIAA J.* 27 (1989) 119–121.
- [24] F.-S. Wei, Structural dynamic model improvement using vibration test data, *AIAA J.* 28 (1990) 175–177.
- [25] F.-S. Wei, Mass and stiffness interaction effects in analytical model modification, *AIAA J.* 28 (1990) 1686–1688.
- [26] H.-Q. Xie, Sensitivity analysis of eigenvalue problem, Ph.D. Dissertation, Nanjing University of Aeronautics and Astronautics, 2003.
- [27] O. Zhang, A. Zerva, D.-W. Zhang, Stiffness matrix adjustment using incomplete measured modes, *AIAA J.* 35 (1997) 917–919.