



Product integration methods based on discrete spline quasi-interpolants and application to weakly singular integral equations[☆]

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ARTICLE INFO

Article history:

Received 29 August 2009

Received in revised form 17 November 2009

Keywords:

Integral equations
Quadrature formulae
Quasi-interpolation
Singular kernels

ABSTRACT

Quadrature formulae are established for product integration rules based on discrete spline quasi-interpolants on a bounded interval. The integrand considered may have algebraic or logarithmic singularities. These formulae are then applied to the numerical solution of integral equations with weakly singular kernels.

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1. Introduction

A classical and efficient technique for the numerical evaluation of integrals of the form

$$I_g(f) = \int_0^1 g(t)f(t)dt, \quad (1.1)$$

where f is a smooth function and g is absolutely integrable in $I = [0, 1]$, is to approximate $I_g(f)$ by $I_g(\tilde{f})$, where \tilde{f} is an approximant of f , for which $I_g(\tilde{f})$ can be easily calculated. This method is called the *product integration rule*. We may write

$$I_g(f) \simeq \sum_{i=0}^n g_i f(t_i), \quad (1.2)$$

where the abscissae t_i depend on I , and the weights g_i depend on I and also on the weight function g . Formulae of the form (1.2) which are exact on polynomials of degree $2n + 1$ are called Gauss-type rules and they have been widely studied in the literature [1,2,14]. In [3], the authors have considered the case where \tilde{f} is the $(n - 1)$ th degree continuous piecewise Lagrangian interpolant of f and they have studied the convergence order when g is smooth or has algebraic or logarithmic singularities.

The product integration methods can also be used for approximating solutions of some integral equations with weakly singular kernels like integrals of the form $\int_0^1 H(x, t)|x - y|^{\alpha-1}u(t)dt$, $0 < \alpha < 1$, see for example [4,5,3,6]. In this case, if we fix x and we write $f(t) = H(x, t)u(t)$, then we obtain an integral of the form (1.1).

[☆] Research supported in part by AI MA/182/08.

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In this paper, we propose to approximate f by a discrete spline quasi-interpolant (abbr. dQI) Q_m of order m (degree $m - 1$) which can be defined as

$$Q_m f := \sum_{j \in J} f(t_j) L_j,$$

where $\{t_j, j \in J\}$ is a sequence of data points in I and $\{L_j, j \in J\}$ is a sequence of *quasi-Lagrange splines* that are (finite) linear combinations of B-splines generating some space \mathbb{S} of splines of order m . Let Π_{m-1} be the space of polynomials of degree $m - 1$. It is clear that $\Pi_{m-1} \subset \mathbb{S}$, then in order to provide an optimal approximation, the quasi-interpolant Q_m is constructed to be exact on Π_{m-1} , i.e. $Q_m e_r = e_r$ for all monomials $e_r(x) := x^r$, $0 \leq r \leq m - 1$. This implies, as Q_m is local and bounded, that $f - Q_m f = \mathcal{O}(h^m)$ on a partition with meshlength h , for f smooth.

By replacing f by $Q_m f$ in (1.1), we obtain the following approximation:

$$I_g(\bar{f}) = \sum_{j \in J} f(t_j) \int_0^1 g(x) L_j(x) dx \quad (1.3)$$

which of type (1.2) with $g_j = \int_0^1 g L_j$. The paper is organized as follows. In Section 2, we briefly introduce the spline quasi-interpolants on an interval together with results on the approximation error for smooth functions. In Section 3, we present quadrature formulae with weights, based on these quasi-interpolants, and we give the rates of convergence for both smooth and singular cases. In Section 4, we use these formulae for solving integral equations with singular kernels. Finally, we give in Section 5 several numerical examples in order to illustrate the theoretical results.

2. Discrete spline quasi-interpolants on an interval

We denote by $\mathcal{S}_m(I, \mathcal{X}_n)$ the space of splines of order m (degree $m - 1$) and of class C^{m-2} on the uniform partition $\mathcal{X}_n := \{x_i = ih, i = 0, \dots, n\}$ of the interval I with meshlength $h = \frac{1}{n}$. A B-spline basis of this space is $\{B_j, j \in J\}$, with $J = \{1, 2, \dots, n + m - 1\}$. According to these notations, $\text{supp}(B_j) = [x_{j-m}, x_j]$ and $\mathcal{N}_j = \{x_{j-m+1}, \dots, x_{j-1}\}$ is the set of the $m - 1$ interior knots in the support of B_j . As usual, we add multiple knots at the endpoints: $0 = x_0 = x_1 = \dots = x_{-m+1}$ and $1 = x_n = x_{n+1} = \dots = x_{n+m-1}$. The Schoenberg–Marsden operator is the simplest dQI exact on Π_1 defined by

$$Q_2 f := \sum_{j \in J} f(\theta_j) B_j,$$

where $\theta_j, j \in J$, are the Greville points

$$\theta_j = \frac{1}{m-1} \sum_{\ell=1}^{m-1} x_{j-\ell}.$$

The dQIs of order $m \geq 3$ used here are the spline operators introduced in [7] and defined by:

$$Q_m f := \sum_{j \in J} \mu_j(f) B_j.$$

The coefficients $\mu_j(f)$ are linear combinations of values of f on the set \mathcal{X}_n for m even and on the set \mathcal{T}_n for m odd, where

$$\mathcal{T}_n = \{t_j, 1 \leq j \leq n + 2\}, \quad t_1 = 0, \quad t_{n+2} = 1, \quad t_j = \frac{1}{2}(x_{j-2} + x_{j-1}).$$

Therefore, for m odd, we set $f(\mathcal{T}_n) = \{f_j = f(t_j), 1 \leq j \leq n + 2\}$, and for m even, we set $f(\mathcal{X}_n) = \{f_j = f(x_j), 0 \leq j \leq n\}$. Moreover, Q_m is constructed to be exact on Π_{m-1} , i.e.

$$Q_m p = p, \quad \forall p \in \Pi_{m-1}.$$

Then, it is well known (see e.g. [8], chapter 5) that for any subinterval $I_k = [x_{k-1}, x_k]$, $1 \leq k \leq n$, and for any function f we have

$$\|f - Q_m f\|_{\infty, I_k} \leq (1 + \|Q_m\|_{\infty}) d_{\infty, J_k}(f, \Pi_{m-1}),$$

where $J_k = [x_{k-m}, x_{k+m-1}]$ and the distance of f to polynomials is defined by

$$d_{\infty, J_k}(f, \Pi_{m-1}) = \inf\{\|f - p\|_{\infty, J_k}, p \in \Pi_{m-1}\}.$$

Here, as usual, $\|f - p\|_{\infty, J_k} = \max_{x \in J_k} |f(x) - p(x)|$. Therefore, as Q_m is bounded for all h , then for f smooth enough, e.g. $f \in C^m(I)$, we get

$$\|f - Q_m f\|_{\infty} = \mathcal{O}(h^m). \quad (2.1)$$

3. Product integration rules: Error estimates

We can write the spline dQI Q_m under the following quasi-Lagrange form

$$Q_m f := \sum_{j \in J'} f_j L_j, \quad (3.1)$$

where $J' = \{0, 1, \dots, n\}$ for m even, $J' = \{1, 2, \dots, n+2\}$ for m odd, and the *quasi-Lagrange functions* L_j are linear combinations of a finite number of B-splines B_j . Then if we approximate $f(x)$ by $\tilde{f}(x) = Q_m f(x)$, we obtain the following product integration rule

$$I_g(f) \simeq \int_0^1 g(x) Q_m f(x) dx = \sum_{i \in J'} A_i f(t_i),$$

with $A_i = \int_0^1 g(x) L_i(x) dx$. The error estimate is given by

$$E_g(f) = \int_0^1 g(x) \{f(x) - Q_m f(x)\} dx.$$

Using (2.1), we can easily prove that for all m and for all $g \in L_1(I)$ we have

$$E_g(f) = O(h^m).$$

Moreover, in the case where m is odd, superconvergence results are obtained for some weight functions g .

3.1. Smooth weight g

Theorem 1. Assume that m is odd and let $g \in W_1^1(I)$, where $W_1^1(I)$ denote the Sobolev space of functions with integrable derivatives. Then, for $f \in C^{m+1}(I)$, we have

$$E_g(f) = O(h^{m+1}). \quad (3.2)$$

Proof. We have

$$E_g(f) = \int_0^1 g(x) \{f(x) - Q_m f(x)\} dx.$$

We suppose, without loss of generality, that n is even. Let ϕ be the piecewise constant function defined by

$$\begin{cases} \phi(x) = \phi_j = \frac{1}{h} \int_{t_{j-1}}^{t_j} g(t) dt, & x \in [t_{j-1}, t_j], 2 \leq j \leq \bar{n} + 1, \\ \phi(x) = \phi(1-x), & x \in [t_{\bar{n}+1}, t_{n+2}], \end{cases} \quad (3.3)$$

where $\bar{n} = \frac{n+2}{2}$. Then we have

$$\begin{aligned} |E_g(f)| &= |E_{g-\phi}(f) + E_\phi(f)| \\ &\leq \|g - \phi\|_{1,I} \|f - Q_m f\|_{\infty,I} + \left| \int_0^1 \phi(x) \{f(x) - Q_m f(x)\} dx \right|. \end{aligned} \quad (3.4)$$

Let us study the first term in (3.4). We have

$$\begin{aligned} \|g - \phi\|_{1,I} &= \int_0^1 |g(x) - \phi(x)| dx = \sum_{i=2}^{n+2} \int_{t_{i-1}}^{t_i} |g(x) - \phi(x)| dx \\ &= \sum_{j=2}^{\bar{n}+1} \int_{t_{j-1}}^{t_j} |g(x) - \phi_j| dx + \sum_{j=\bar{n}+2}^{n+2} \int_{t_{j-1}}^{t_j} |g(x) - \phi_{n+4-j}| dx. \end{aligned}$$

For $x, t \in [t_{j-1}, t_j]$, $g(x) - g(t) = \int_t^x g'(s) ds$, therefore

$$\begin{aligned} |g(x) - \phi_j| &\leq \frac{1}{h} \int_{t_{j-1}}^{t_j} |g(x) - g(t)| dt \leq \frac{1}{h} \int_{t_{j-1}}^{t_j} \int_t^x |g'(s)| ds dt \\ &\leq \int_{t_{j-1}}^{t_j} |g'(s)| ds, \end{aligned}$$

then

$$\begin{aligned} \sum_{j=2}^{\bar{n}+1} \int_{t_{j-1}}^{t_j} |g(x) - \phi_j| dx &\leq \sum_{j=2}^{\bar{n}+1} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} |g'(s)| ds dx \\ &\leq h \sum_{j=2}^{\bar{n}+1} \int_{t_{j-1}}^{t_j} |g'(s)| ds. \end{aligned}$$

Similarly we have

$$\sum_{j=\bar{n}+2}^{n+2} \int_{t_{j-1}}^{t_j} |g(x) - \phi_{n+4-j}| dx \leq h \sum_{j=\bar{n}+2}^{n+2} \int_{t_{j-1}}^{t_j} |g'(s)| ds,$$

then we deduce that

$$\|g - \phi\|_{1,I} \leq h \|g'\|_{1,I}.$$

Hence, as $\|f - Q_m f\|_{\infty,I} = O(h^m)$, we obtain

$$\|g - \phi\|_{1,I} \|f - Q_m f\|_{\infty,I} = O(h^{m+1}). \quad (3.5)$$

On the other hand, let $v(t)$ be the error $t^m - Q_m(t^m)$, $t \in I$, i.e.

$$v(t) = t^m - \sum_{j=1}^{n+2} t_j^m L_j(t).$$

Using the symmetries of the knots t_j and the exactness of Q_m on \mathbb{P}_{m-1} , we easily prove that

$$v(1-t) = -v(t), \quad t \in \left[0, \frac{1}{2}\right]. \quad (3.6)$$

Indeed, consider the polynomial $r = t^m + (1-t)^m \in \mathbb{P}_{m-1}$, then we have $v(t) + v(1-t) = r - Q_m r$. Thus, as Q_m is exact on \mathbb{P}_{m-1} , $Q_m r = r$, we obtain (3.6).

Now, by using (3.6) and the symmetries of the function ϕ , we get, by setting $s = 1-t$ in the second integral

$$\begin{aligned} 0 &= \int_0^1 \phi(t) v(t) dt + \int_0^1 \phi(1-t) v(1-t) dt \\ &= 2 \int_0^1 \phi(t) v(t) dt. \end{aligned} \quad (3.7)$$

Then we deduce

$$\left| \int_0^1 \phi(x) \{f(x) - Q_m f(x)\} dx \right| = O(h^{m+1}). \quad (3.8)$$

Consequently, from (3.5) and (3.8) we deduce the estimate (3.2). ■

3.2. Singular weight: $g_1(x) = |t-x|^{\alpha-1}$, $0 < \alpha < 1$

We first need to remind the definition of the Nikol'skii space $N_1^\nu[0, 1]$, $\nu > 0$, introduced in [5].

Definition 2. Let I be a compact subinterval of \mathbb{R} . For any integer $k > 0$, let $W_1^k(I)$ be the Sobolev space of functions with integrable k th derivatives. The Nikol'skii space $N_1^\nu(I)$, $\nu > 0$, is defined as the set of all functions $u \in W_1^{[\nu]}(I)$ such that $\|u\|_{\nu,1,I} < \infty$, where

$$\begin{aligned} \|u\|_{\nu,1,I} &= \|u\|_{1,I} + |u|_{\nu,1,I} \quad \text{and} \quad |u|_{\nu,1,I} = \sup_{\varepsilon > 0} \{\varepsilon^{-\nu_0} \|\Delta_\varepsilon u^{[\nu]}\|_{1,I_\varepsilon}\}, \\ I_\varepsilon &= \{t \in I : t + \varepsilon \in I\}, \quad \varepsilon \in \mathbb{R}, \\ \nu &= [\nu] + \nu_0, \quad [\nu] \in \mathbb{N}, \quad 0 < \nu_0 < 1, \quad \text{and} \quad \Delta_\varepsilon u(x) = u(x + \varepsilon) - u(x). \end{aligned}$$

Theorem 3. Suppose that m is odd, and $g_1(x) = |t-x|^{\alpha-1}$, $0 < \alpha < 1$. Then, for all $f \in C^{m+1}(I)$ we have

$$E_{g_1}(f) = O(h^{m+\alpha}). \quad (3.9)$$

Proof. Let ϕ be the function defined in (3.3) and based on g_1 . Then we have

$$\begin{aligned} |E_{g_1}(f)| &= |E_{g_1-\phi}(f) + E_\phi(f)| \\ &\leq \|g_1 - \phi\|_{1,I} \|f - Q_m f\|_{\infty,I} + \left| \int_0^1 \phi(x) \{f(x) - Q_m f(x)\} dx \right|. \end{aligned}$$

According to [4], the function g_1 is an element of the Nikol'skii space $N_1^\alpha[0, 1]$.

Then by using Lemma A1, given in [5], we deduce that

$$\|g_1 - \phi\|_{1,I} \leq C_1 h^\alpha, \quad (3.10)$$

$$|\phi| \leq C_2 h^{\alpha-1}, \quad (3.11)$$

where C_1 and C_2 are constants independent of n and t .

On the other hand, we prove, as in Theorem 1, that there exist two constants C_3 and C_4 such that

$$\|f - Q_m f\|_{\infty,I} \leq C_3 h^m,$$

and

$$\left| \int_0^1 (f - Q_m f)(x) dx \right| \leq C_4 h^{m+1}.$$

Hence, by combining the above inequalities, we obtain the estimate (3.9). ■

3.3. Singular weight: $g_2(x) = \log(|t - x|)$

Theorem 4. Suppose that m is odd, and $g_2(x) = \log(|t - x|)$. Then, for all $f \in C^{m+1}(I)$ we have

$$E_{g_2}(f) = O(h^{m+1} \log(h)). \quad (3.12)$$

Proof. In this case, the Nikol'skii space can be modified by redefining the semi-norm to be

$$|u| = \sup_{\varepsilon > 0} \{(\varepsilon \log(\varepsilon))^{-1} \|\Delta_\varepsilon u\|_{1,I_\varepsilon}\}.$$

Then, by using the same arguments as in the proof of Theorem 3, and by replacing (3.10) and (3.11) by

$$\|g_2 - \phi\|_{1,I} \leq C'_2 h \log(h), \quad (3.13)$$

and

$$|\phi| \leq C'_2 \log(h), \quad (3.14)$$

where C'_1 and C'_2 are constants independent of n and t , we get the superconvergence result (3.12). ■

4. Application to integral equations

We consider the linear integral equation of the second kind

$$u(x) - \int_0^1 k(x, t) u(t) dt = f(x), \quad x \in I,$$

where the kernel function $k(x, t)$ is not necessarily continuous, but its associated integral operator \mathcal{K} is still compact on $C(I)$. In this setting, discontinuous kernel functions $k(x, t)$ may have an infinite singularity, the most important examples being $g_1(x, t) = |t - x|^{\alpha-1}$, for some $0 < \alpha < 1$, and $g_2(x, t) = \log |t - x|$.

In each case, we write the equation in the form

$$u(x) - \int_0^1 H(x, t) g(x, t) u(t) dt = f(x), \quad x \in I. \quad (4.1)$$

Continuity and differentiability properties of the exact solution $u(x)$ of (4.1) have been studied by many authors (see [9, 10, 13]). In our case, we assume that $f \in C[a, b]$, $H(x, t)$ and $u(x)$ are several times continuously differentiable. For the numerical solution of the Eq. (4.1), we consider the product integration method based on the dQI given in Section 2.

4.1. Description of the method

For $u \in C(I)$ we set

$$[H(x, t)u(t)]_m = Q_m H(x, \cdot)u(\cdot) = \sum_{j \in J'} H(x, \tau_j)u(\tau_j)L_j(t), \quad x \in I. \quad (4.2)$$

Then, we define a numerical approximation to the integral operator in (4.1) by

$$\mathcal{K}_n u(x) = \int_0^1 [H(x, t)u(t)]_m g(x, t) dt, \quad x \in I. \quad (4.3)$$

This can also be written as

$$\begin{aligned} \mathcal{K}_n u(x) &= \sum_{j \in J'} \left(\int_0^1 L_j(t)g(x, t) dt \right) H(x, \tau_j)u(\tau_j) \\ &= \sum_{j \in J'} w_j(x)H(x, \tau_j)u(\tau_j), \end{aligned}$$

where $\tau_j, j \in J'$, are the evaluation points of the spline dQI Q_m . Thus, we approximate the integral equation (4.1) by

$$u_n(x) - \sum_{j \in J'} w_j(x)H(x, \tau_j)u_n(\tau_j) = f(x), \quad x \in I. \quad (4.4)$$

This is equivalent to first solve the system of linear equations

$$u_n(\tau_i) - \sum_{j \in J'} w_j(\tau_i)H(\tau_i, \tau_j)u_n(\tau_j) = f(\tau_i), \quad i \in J',$$

then, using the Nyström interpolation formula, we obtain the following approximating solution

$$u_n(x) = f(x) + \sum_{j \in J'} w_j(x)H(x, \tau_j)u_n(\tau_j), \quad x \in I. \quad (4.5)$$

Before proving the existence of the approximate solution (4.5), we give a technique which we utilise for the evaluation of the weight functions $w_j(x)$ appearing in (4.5).

4.2. Evaluation of the weight functions $w_j(x)$

For the computation of the approximate solution $u_n(x)$, given in (4.5), we need to calculate the weight functions $w_j(x)$ defined by

$$w_j(x) = \int_0^1 L_j(t)g(x, t) dt, \quad j \in J'.$$

As each Lagrange function L_j is a linear combination of a finite number of B-splines B_i , it suffices to compute the integrals

$$\int_0^1 B_j(t)g(x, t) dt, \quad j \in J'. \quad (4.6)$$

By expressing each B-spline in the Bernstein basis $\{\beta_r^{m-1}(t) = \binom{m-1}{r} (1-t)^r t^{m-1-r}, r = 0, 1, \dots, m-1\}$ of Π_{m-1} , the integrals in (4.6) are obtained as linear combinations of

$$\int_0^1 \beta_r^{m-1}(t)g(x, t) dt, \quad j \in J'.$$

The latter are symbolically calculated with the aid of a computer algebra system. As examples, we give in the two following subsections the explicit values of the integrals (4.6) in the case of the C^1 quadratic B-splines and the weight functions $g(x, t) = |x - t|^{\alpha-1}$, $0 < \alpha < 1$, and $g(x, t) = \log|x - t|$, $x \in I$.

4.2.1. Case of $g(x, t) = \log(|x - t|)$

Let

$$b_j(x) = \int_0^1 B_j(t) \log|t - x| dt,$$

and

$$\phi_r(x) = \int_0^1 \log |t - x| \beta_r^2(t) dt, \quad r = 0, 1, 2.$$

These last expressions are symbolically calculated with the aid of a computer algebra system. For $3 \leq j \leq n$, we have

$$\begin{aligned} \int_0^1 B_j(t) \log |t - x| dt &= \sum_{r=1}^3 \int_{x_{j-r}}^{x_{j-r+1}} B_j(t) \log |t - x| dt \\ &= h \sum_{r=1}^3 \int_0^1 \log |th + x_{j-r} - x| B_j(th + x_{j-r}) dt \\ &= h \sum_{r=1}^3 \int_0^1 \log \left| t + j - r - \frac{x}{h} \right| B_j(th + x_{j-r}) dt + h \log h. \end{aligned}$$

By writing $B_j(th + x_{j-r})$, $r = 1, 2, 3$, in terms of the Bernstein basis functions we obtain

$$B_j(th + x_{j-3}) = \frac{1}{2} \beta_0^2, \quad B_j(th + x_{j-2}) = \frac{1}{2} \beta_0^2 + \beta_1^2 + \frac{1}{2} \beta_2^2, \quad B_j(th + x_{j-1}) = \frac{1}{2} \beta_2^2.$$

Then, for $3 \leq j \leq n$, we get

$$\begin{aligned} b_j(x) &= \int_0^1 B_j(t) \log |t - x| dt = h \log h + \frac{1}{2} \left[\phi_0 \left(\frac{x}{h} + 3 - j \right) + \phi_0 \left(\frac{x}{h} + 2 - j \right) \right. \\ &\quad \left. + 2\phi_1 \left(\frac{x}{h} + 2 - j \right) + \phi_2 \left(\frac{x}{h} + 2 - j \right) + \phi_0 \left(\frac{x}{h} + 1 - j \right) \right]. \end{aligned}$$

For the extremal B-splines we have

$$\begin{aligned} b_1(x) &= \frac{1}{3} h \log h + h \phi_2 \left(\frac{x}{h} \right), \\ b_2(x) &= \frac{2}{3} h \log h + \frac{h}{2} \left[\phi_0 \left(\frac{x}{h} \right) + 2\phi_1 \left(\frac{x}{h} \right) + \phi_2 \left(\frac{x}{h} - 1 \right) \right], \\ b_{n+1}(x) &= \frac{2}{3} h \log h + \frac{h}{2} \left[\phi_0 \left(\frac{x}{h} + 2 - n \right) + 2\phi_1 \left(\frac{x}{h} + 1 - n \right) + \phi_2 \left(\frac{x}{h} + 1 - n \right) \right], \\ b_{n+2}(x) &= \frac{1}{3} h \log h + h \phi_0 \left(\frac{x}{h} + 1 - n \right). \end{aligned}$$

4.2.2. Case of $g(t) = |x - t|^{\alpha-1}$, $0 < \alpha < 1$, $x \in I$.

Let

$$\psi_r(x) = \int_0^1 \beta_r^2(t) |x - t|^{\alpha-1} dt, \quad r = 0, 1, 2,$$

where their explicit expressions are calculated with the aid of a computer algebra system. For the computation of the integrals

$$b_j(x) = \int_0^1 B_j(t) |t - x|^{\alpha-1} dt,$$

we proceed as in the above case and we obtain

$$\begin{aligned} b_1(x) &= h^\alpha \psi_2 \left(\frac{x}{h} \right), \\ b_2(x) &= \frac{1}{2} h^\alpha \left[\psi_0 \left(\frac{x}{h} \right) + 2\psi_1 \left(\frac{x}{h} \right) + \psi_2 \left(\frac{x}{h} - 1 \right) \right], \\ b_{n+1}(x) &= \frac{1}{2} h^\alpha \left[\psi_0 \left(\frac{x}{h} + 2 - n \right) + 2\psi_1 \left(\frac{x}{h} + 1 - n \right) + \psi_2 \left(\frac{x}{h} + 1 - n \right) \right], \\ b_{n+2}(x) &= h^\alpha \psi_0 \left(\frac{x}{h} + 1 - n \right), \end{aligned}$$

and for $3 \leq j \leq n$,

$$\begin{aligned} b_j(x) &= \frac{1}{2} h^\alpha \left[\psi_0 \left(\frac{x}{h} + 3 - j \right) + \psi_0 \left(\frac{x}{h} + 2 - j \right) + 2\psi_1 \left(\frac{x}{h} + 2 - j \right) \right. \\ &\quad \left. + \psi_2 \left(\frac{x}{h} + 2 - j \right) + \psi_0 \left(\frac{x}{h} + 1 - j \right) \right], \quad x \in [0, 1]. \end{aligned}$$

5. Error estimate for the approximate solution

We consider the Eq. (4.1) with $H(x, t)$ assumed to be continuous. Furthermore, we assume that $g(x, t)$ satisfies

$$c_g \equiv \sup_{x \in I} \int_0^1 |g(x, t)| dt < \infty, \quad (5.1)$$

and

$$\lim_{h \rightarrow 0^+} \omega_g(h) = 0, \quad (5.2)$$

where

$$\omega_g(h) \equiv \sup_{|x-y| < h} \int_0^1 |g(x, t) - g(y, t)| dt.$$

These properties are satisfied by both $\log |t - x|$ and $|t - x|^{\alpha-1}$, $0 < \alpha < 1$.

Theorem 5. Assume the function $g(x, t)$ satisfies (5.1) and (5.2), and $H(x, t)$ is continuous for $a \leq x, t \leq b$. For a given function $f \in C[a, b]$, assume the integral equation

$$u(x) - \int_a^b H(x, t)g(x, t)u(t)dt = f(x), \quad a \leq x \leq b,$$

is uniquely solvable. Consider the numerical approximation (4.3), with $[H(x, t)u(t)]_m$ defined by (4.2). Then for all sufficiently large n , say $n > N$, the Eq. (4.4) is uniquely solvable, and the operators $(I - \mathcal{K}_n)^{-1}$ are uniformly bounded. Moreover,

$$\|u - u_n\|_\infty \leq c \|\mathcal{K}u - \mathcal{K}_n u\|_\infty, \quad \forall n > N, \quad (5.3)$$

for a suitable $c > 0$.

Proof. From the proof of Theorem 4.1.1 given in [11], it suffices to show that the operators $\{\mathcal{K}_n\}$ form a collectively compact and pointwise convergent family to the operator \mathcal{K} on $C(I)$, which is equivalent to prove the following properties:

A1 \mathcal{K} and \mathcal{K}_n , $n \geq 1$, are linear operators.

A2 $\mathcal{K}_n u \rightarrow \mathcal{K}u$ when $n \rightarrow \infty$, for all $u \in C(I)$.

A3 The set $\{\mathcal{K}_n, n \geq 1\}$ is collectively compact.

From the definitions of \mathcal{K} and \mathcal{K}_n , we easily prove the property **A1**. On the other hand, according to [12], we have

$$|z(x) - Q_m z(x)| \leq c \omega(z, h), \quad \text{for all } z \in C(I),$$

where c is a constant independent of h . Then

$$\begin{aligned} |\mathcal{K}u(x) - \mathcal{K}_n u(x)| &\leq c c_g \omega(H(x, \cdot)u(\cdot), h) \\ \|\mathcal{K}u - \mathcal{K}_n u\|_\infty &\leq c c_g \max_{x \in I} \omega(H(x, \cdot)u(\cdot), h) \end{aligned}$$

and the latter converges to zero by the uniform continuity of $H(x, t)u(t)$ over the square $0 \leq x, t \leq 1$. This proves that $\mathcal{K}_n u \rightarrow \mathcal{K}u$ when $n \rightarrow \infty$, for all $u \in C(I)$; and **A2** is proved. To show **A3**, we prove that the set

$$\mathcal{S} = \{\mathcal{K}_n u | n \geq 1 \text{ and } \|u\|_\infty \leq 1\},$$

has a compact closure in $C(I)$. For this, we need to show that \mathcal{S} is equicontinuous and $\|\mathcal{K}_n u\|_\infty$ is bounded. On other hand, as

$$|[H(x, t)u(t)]_m| = |Q_m H(x, \cdot)u(\cdot)| \leq \|Q_m\|_\infty \|H(x, \cdot)u(\cdot)\|_\infty,$$

we deduce that

$$\|\mathcal{K}_n u\|_\infty = \max_{x \in I} \left| \int_a^b [H(x, t)u(t)]_m g(x, t) dt \right| \leq c_H c_g \|Q_m\|_\infty, \quad u \in C(I) \text{ and } \|u\|_\infty \leq 1,$$

with

$$c_H \equiv \max_{0 \leq t, s \leq 1} |H(t, s)|.$$

This shows the uniform boundedness of $\{\mathcal{K}_n\}$, with

$$\|\mathcal{K}_n\|_\infty \leq c_H c_g \|Q_m\|_\infty, \quad n \geq 1.$$

For the equicontinuity of \mathcal{J} , write

$$\begin{aligned}\mathcal{K}_n u(x) - \mathcal{K}_n u(\tau) &= \int_0^1 [H(x, t)u(t)]_m g(x, t) dt - \int_0^1 [H(\tau, t)u(t)]_m g(\tau, t) dt \\ &= \int_0^1 [H(x, t) - H(\tau, t)]u(t)]_m g(x, t) dt + \int_0^1 [H(\tau, t)u(t)]_m (g(x, t) - g(\tau, t)) dt.\end{aligned}$$

Now, the assumptions on $g(x, t)$ and $H(x, t)$, together with $\|u\|_\infty \leq 1$, imply

$$\left| \int_0^1 [H(x, t) - H(\tau, t)]u(t)]_m g(x, t) dt \right| \leq c_g \|Q_m\|_\infty \|u\|_\infty \max_{t \in I} |H(x, t) - H(\tau, t)|,$$

and

$$\left| \int_0^1 [H(\tau, t)u(t)]_m (g(x, t) - g(\tau, t)) dt \right| \leq c_H \|Q_m\|_\infty \|u\|_\infty \omega_g(|x - \tau|).$$

The combination of these results shows the desired equicontinuity of \mathcal{J} , and completes the proof of the property **A3**. ■

Theorem 6. Suppose that m is odd. For $H(x, \cdot)u(\cdot) \in C^{m+1}(I)$, we have the following error estimates

$$\|u - u_n\|_\infty \leq \begin{cases} Ch^{m+\alpha}, & \text{for } g(x, t) = |x - t|^{\alpha-1}, \\ C'h^{m+1} \log h, & \text{for } g(x, t) = \log |x - t|. \end{cases}$$

Proof. It follows from the fact that $\|\mathcal{K}u - \mathcal{K}_n u\|_\infty = \max_{x \in I} E_g(H(x, \cdot)u(\cdot))$, and from Theorem 5 and the results given in Section 3. ■

6. Numerical examples

In this section, we present the numerical solutions of four Fredholm integral equations using the product integration method presented above based on the following dQIs:

- The C^1 quadratic spline dQI Q_3 defined by

$$Q_3 f := \sum_{j \in J'} f_j L_j,$$

where $f_j = f(t_j)$, $t_j \in T_n$, $j \in J' = \{1, 2, \dots, n+2\}$ and the *quasi-Lagrange functions* L_j are linear combinations of at most three C^1 quadratic B-splines, for example

$$L_j := \frac{1}{8}(-B_{j-1} + 10B_j - B_{j+1}), \quad \text{for } 4 \leq j \leq n-1.$$

At the endpoints, the quasi-Lagrange functions have specific forms (see [7]).

- The C^2 cubic spline dQI defined by

$$Q_4 f := \sum_{j=0}^n f_j L_j,$$

where the *quasi-Lagrange functions* L_j are linear combinations of a finite number of C^2 cubic B-splines, e.g. for $4 \leq j \leq n$

$$L_j := -\frac{1}{6}(-B_{j+3} + 8B_{j+2} - B_{j+1}),$$

where $f_j = f(x_j)$, $x_j \in X_n$. At the end points, the quasi-Lagrange functions have specific forms (see [7]). We compare the obtained numerical results with those obtained by using the classical continuous piecewise quadratic and cubic Lagrangian interpolants (see [3,6]), denoted respectively by P_3 and P_4 .

Example 1. Consider the following integral equation

$$u(x) - \int_0^1 |x - t|^{-\frac{1}{2}} u(t) dt = f(x), \quad x \in [0, 1],$$

where f is chosen so that $u(x) = e^x$.

In Tables 1 and 2, we give the max errors, obtained by using the product integration method based on the above two dQIs and the Lagrangian interpolant P_3 and P_4 , in terms of the number n of subintervals. We also give the numerical approximation orders denoted by NAO.

Table 1

n	$\ u - u_n\ _\infty$			
	Q_3	NAO	P_3	NAO
4	2.42 (−3)		2.68 (−3)	
8	1.82 (−4)	3.7	2.15 (−4)	3.6
16	1.15 (−5)	4.0	2.12 (−5)	3.3
32	9.39 (−7)	3.6	1.94 (−6)	3.5

Table 2

n	$\ u - u_n\ _\infty$			
	Q_4	NAO	P_4	NAO
9	6.41 (−5)		9.13 (−5)	
18	2.83 (−6)	4.5	5.21 (−6)	4.1
36	1.13 (−7)	4.6	2.75 (−7)	4.2
72	6.52 (−9)	4.1	1.72 (−8)	4.0

Table 3

n	$\ u - u_n\ _\infty$			
	Q_3	NAO	P_3	NAO
4	5.24 (−2)		1.49 (−1)	
8	5.07 (−3)	3.4	1.49 (−2)	3.3
16	4.42 (−4)	3.5	1.44 (−3)	3.4
32	4.25 (−5)	3.4	1.33 (−4)	3.4

Table 4

n	$\ u - u_n\ _\infty$			
	Q_4	NAO	P_4	NAO
9	6.22 (−3)		1.02 (−2)	
18	4.23 (−4)	3.9	7.13 (−4)	3.8
36	2.66 (−5)	4.0	4.62 (−5)	3.9
72	1.65 (−6)	4.0	2.89 (−6)	4.0

Table 5

n	$\ u - u_n\ _\infty$			
	Q_3	NAO	P_3	NAO
4	1.76 (−4)		7.23 (−4)	
8	1.73 (−5)	3.4	5.81 (−5)	3.6
16	1.53 (−6)	3.5	4.38 (−6)	3.7
32	8.02 (−8)	3.6	3.18 (−7)	3.8

Example 2. For the equation

$$u(x) - \int_0^\pi \sin(x-t)|x-t|^{-\frac{1}{2}} u(t) dt = f(x), \quad x \in [0, \pi],$$

with exact solution $u(x) = \cos(x)$, we obtain the following results (see [Tables 3 and 4](#)).

Example 3. We consider the integral equation

$$u(x) - \int_0^1 \log|x-t| u(t) dt = f(x), \quad x \in [0, 1],$$

where the function $f(x)$ is chosen so that $u(x) = e^x$. We obtain the following results (see [Tables 5 and 6](#)).

Example 4. We consider the integral equation [[11](#), chap.4]

$$u(x) - \int_0^\pi \log|\cos x - \cos t| u(t) dt = 1, \quad 0 \leq x \leq \pi \quad (6.1)$$

Table 6

n	$\ u - u_n\ _\infty$			
	Q_4	NAO	P_4	NAO
9	4.30 (−6)		8.29 (−6)	
18	3.06 (−7)	3.8	6.15 (−7)	3.7
36	2.05 (−8)	3.9	4.36 (−8)	3.8
72	1.27 (−9)	4.0	2.88 (−9)	3.9

Table 7

n	$\ u - u_n\ _\infty$			
	Q_3	NAO	P_3	NAO
4	7.32 (−6)		1.65 (−5)	
8	6.27 (−7)	3.6	1.13 (−6)	3.9
16	4.54 (−8)	3.8	7.25 (−8)	3.9
32	3.05 (−9)	3.9	4.56 (−9)	4.0

Table 8

n	$\ u - u_n\ _\infty$			
	Q_4	NAO	P_4	NAO
9	3.52 (−7)		5.93 (−7)	
18	2.58 (−8)	3.8	4.25 (−8)	3.8
36	1.71 (−9)	3.9	2.81 (−9)	3.9
72	1.08 (−10)	4.0	1.77 (−10)	4.0

with exact solution $u(x) = \frac{1}{1+\pi \log 2} \simeq 0.31470429802$. The kernel $k(x, t) = \log |\cos x - \cos t|$ can be written in the form

$$\begin{aligned}
 k(x, t) &= \log \left| 2 \sin \frac{1}{2}(x-t) \sin \frac{1}{2}(x+t) \right| \\
 &= \log \left\{ \frac{2 \sin \frac{1}{2}(x-t) \sin \frac{1}{2}(x+t)}{(x-t)(x+t)(2\pi - t - x)} \right\} + \log |x-t| + \log(x+t) + \log(2\pi - x - t) \\
 &= \sum_{j=1}^4 H_j(x, t) g_j(x, t),
 \end{aligned} \tag{6.2}$$

with $g_1 = H_2 = H_3 = H_4 \equiv 1$ and

$$\begin{aligned}
 H_1(x, t) &= \log \left\{ \frac{2 \sin \frac{1}{2}(x-t) \sin \frac{1}{2}(x+t)}{(x-t)(x+t)(2\pi - t - x)} \right\} \\
 g_2(x, t) &= \log |x-t|, \quad g_3(x, t) = \log(x+t), \\
 g_4(x, t) &= \log(2\pi - x - t).
 \end{aligned}$$

Then, the integral operator in (6.1) is given by

$$\mathcal{K}u(x) = \sum_{j=1}^4 \mathcal{K}_j u(x) = \sum_{j=1}^4 \int_0^\pi H_j(x, t) g_j(x, t) u(t) dt, \quad u \in C[0, \pi].$$

The function $H_1(x, t)$ is infinitely differentiable on $[0, \pi]$ and the functions g_2, g_3 and g_4 are singular functions for which the weight functions in (4.5) are calculated as in Section 4.2. For the case of the operator K_1 with kernel $H_1(x, t)g_1(x, t)$, we use the regular Nyström method based on the dQI. For this example we obtain the following results (see Tables 7 and 8).

7. Final remarks

The methods developed in this paper can be extended to the class of kernels $k(t, s) = g(|t-s|)$ considered in [6], where $g \in C^{m-1}([0, 1])$ and satisfies the following algebraic estimates

$$|g(t)| \leq c(|\log(t)| + 1), \quad |g^{(k)}(t)| \leq ct^{-k}, \quad k = 1, \dots, m-1,$$

or

$$|g^{(j)}(t)| \leq \gamma_j t^{\alpha-1-j}, \quad 0 < \alpha < 1, \quad 0 < t \leq 1, \quad j = 0, \dots, m-1.$$

In this paper we have supposed that the exact solution of the integral equation (4.1) is sufficiently smooth. The case where the solution has some singularities will be treated in a further paper.

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