



Super-efficiency in stochastic data envelopment analysis: An input relaxation approach

M. Khodabakhshi*

Department of Mathematics, Faculty of Science, Lorestan University, Khorram Abad, Iran

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ABSTRACT

This paper addresses super-efficiency issue based on input relaxation model in stochastic data envelopment analysis. The proposed model is not limited to using the input amounts of evaluating DMU, and one can obtain a total ordering of units by using this method. The input relaxation super-efficiency model is developed in stochastic data envelopment analysis, and its deterministic equivalent, also, is derived which is a nonlinear program. Moreover, it is shown that the deterministic equivalent of the stochastic super-efficiency model can be converted to a quadratic program. As an empirical example, the proposed method is applied to the data of textile industry of China to rank efficient units. Finally, when allowable limits of data variations for evaluating DMU are permitted, the sensitivity analysis of the proposed model is discussed.

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1. Introduction

The basic DEA results group the DMUs into two sets, efficient and inefficient DMUs. Often decision makers are interested in a complete ranking in order to refine the evaluation of the units. Several authors have proposed methods for ranking the efficient DMUs. Adler et al. [1] provided the detailed discussion regarding the differences among the research works and the choices of the models in the ranking areas. Some DEA researchers have initiated an area called super-efficiency to rank the DEA efficient DMUs and developed various models. In this direction, researchers focused on ranking only DEA efficient DMUs based on the results obtained either from CCR [2] or BCC [3] models. The research in super-efficiency area was first developed by Andersen and Petersen [4]. In their research, they ranked DEA efficient DMUs in such a way that superior DEA efficient DMUs may have efficiency scores greater than unity. Their approach became very popular and many research works extended their idea by addressing new issues such as outlier detection, sensitivity analysis and scale classification. Interesting research works in this area can be found in [5–17].

On the other hand, Thrall [7] pointed out that the model developed by Andersen and Petersen [4] (called AP model) may result in infeasibility and instability when some inputs are close to zero. Similarly, Zhu [8] showed that when the constant-return-to-scale DEA models are used, the infeasibility could occur in the super-efficiency evaluation if and only if there is a zero in the data. Mehrabian et al. [11] developed a super-efficiency model (called MAJ model) that does not have the drawbacks of infeasibility and instability as the AP model. However, the MAJ and the AP model used different ways to evaluate DEA efficiency scores. The DEA efficient DMUs in the AP model are obtained from the CCR model and that of the MAJ model are obtained through its own model. A deficiency of the MAJ model is that at the optimal, all the inputs of the DMU being evaluated need to increase by the same variable. It is very difficult to explain the meaning of the variable. Xue and Harker [12] showed the necessary and sufficient conditions of infeasibility in super-efficiency evaluation when the

* Tel.: +98 9122586521; fax: +98 661 2201333.

E-mail address: mkhbakhshi@yahoo.com.

BCC model with variable returns to scale is used. In ranking DEA efficiency DMUs, they identified four classes: (i) super-efficient, (ii) strongly efficient, (iii) efficient; and (iv) weakly efficient. Although their research results are rather insightful, Xue and Harker [12] didn't provide new models to compute super-efficiency scores and rank DEA efficient DMUs. Tone [13] proposed a new super-efficiency model (called SuperSBM(I)) based on the measurement of slacks. Although Tone's model did not have similar deficiencies as the AP and MAJ models but it can be difficult in ranking if any inputs of DMUs are zero. Overall, the research areas in super-efficiency are widely cited and have been applied in a wide range of settings such as financial institutions, industries, public regulations, education as well as health care.

Stochastic formulation of the original models were introduced to incorporate possible uncertainty in the inputs and/or outputs (e.g. [18–27]). Morita and Seiford [28] studied robustness of the efficiency results when input and output data are subject to stochastic measurement error, while Jess et al. [29] introduced a semi-infinite programming model in DEA to study an interesting chemical engineering problem. Cooper et al. [19] have provided chance constrained programming models that are directed to determining where efficient and inefficient behaviour will occur with associated probabilities. Cooper et al. [18] incorporated congestion models in corresponding chance constrained programming models. They showed how the task of identifying congestion may be accomplished with deterministic models rather than their chance constrained (stochastic) counterparts under suitable assumptions. Although one must then deal with a nonlinear programming problem, the task can be reduced to solving a quadratic programming problem.

In this paper, we discuss “super-efficiency” issue based on input relaxation model in stochastic data envelopment analysis. Our model is always feasible, and we can obtain a total ordering of units by applying this approach. In addition, it allows zero inputs or outputs and doesn't need extra procedures to process zero inputs or outputs. We extend, in this paper, the model introduced in [14], allowing deterministic inputs and outputs to be stochastic. We then obtain deterministic equivalent to our stochastic super-efficiency input relaxation model. We show that the deterministic equivalent can be transformed to quadratic programming model. The quadratic model is applied to the data of textile industry of China.

The rest of the paper is organized as follows: The input relaxation model is described in Section 2. In Section 3, we provide a super-efficiency model based on the model that introduced in Section 2, and its characterizations will be described theoretically. In addition, the input relaxation super-efficiency model is illustrated by means of a numerical example. In Section 4, stochastic version of the proposed input relaxation model super-efficiency is developed, and its deterministic equivalent is also obtained. Furthermore, it is shown that the deterministic equivalent of the stochastic super-efficiency model can be converted to a quadratic program. As an empirical example, we apply the model to data of textile industry of China. Section 5 discusses sensitivity analysis. Section 6 concludes the paper.

2. Preliminaries

Suppose that all inputs and outputs are non negative deterministic elements. Let DMU_j , ($j = 1, 2, \dots, n$) be n decision making units (DMU) that convert m inputs x_{ij} ($i = 1, \dots, m$) into s outputs y_{rj} ($r = 1, \dots, s$). The model for improving output called input relaxation model, recently introduced in [30,31,14,20,32,33], is

$$\begin{aligned} \text{Maximize} \quad & \phi_o + \varepsilon \left(\sum_{i=1}^m s_{i1}^- + \sum_{r=1}^s s_r^+ - \sum_{i=1}^m s_{i2}^+ \right) \\ \text{Subject to} \quad & x_{io} = \sum_{j=1}^n \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+, \quad i = 1, \dots, m \\ & 0 = \sum_{j=1}^n \lambda_j y_{rj} - \phi_o y_{ro} - s_r^+, \quad r = 1, \dots, s \\ & 1 = \sum_{j=1}^n \lambda_j \\ & s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0, \end{aligned} \quad (1)$$

where $o \in \{1, \dots, n\}$, ϕ_o is maximum possible proportional outputs amount that DMU_o can produce, and the first and second slacks in the input constraints are slacks for decrement and increment of the i th input. The model allows evaluating DMU to overuse the available sources. It often happens in real application that some DMUs can produce far more outputs, should we allow some input relaxation by loosening some of the existing constraints on inputs. While loosening one or more constraints may not always be possible, when it is, model (1) can result in DMU's with considerable increase in the output that is mostly due to some slight changes in one or more inputs (e.g. [30,20]).

The columns correspond to s_{i1}^- and s_{i2}^+ are linearly dependent, so that at the basic optimal solution at most one of these variables is positive. It is obvious that s_{i1}^- and s_{i2}^+ are, respectively, maximized and minimized at the optimal solution. The conditions of efficiency for evaluating DMU_o can therefore be stated as follows.

Definition 1. DMU_o is efficient for the input relaxation model if the following two conditions are satisfied:

- (i) $\phi_o^* = 1$
- (ii) $s_{i1}^{-*} = s_{i2}^{+*} = s_r^{+*} = 0 \quad \forall i \text{ \& } \forall r$.

Table 1
Data of 5 stores.

Store	A	B	C	D	E
Employee	2	3	4	5	6
Sale	1	3	3	2	3

Example. Suppose there are 5 branch stores which we labeled A to E at the head of each column in Table 1. The number of employees and sales (measured in 100,000 dollars) are recorded in each column as input and output, respectively. Note that DMU_C has 1 unit excess input in comparison to DMU_B, while they produce the same output. Evaluating DMU_A by the input relaxation model, one obtains $\phi_A^* = 3$, $s_{12}^{+*} = 1$, $\lambda_B^* = 1$, and all other variables equal zero. The results of solving DMU_A by this model shows that if input of DMU_A is increased by $s_{12}^+ = 1$, its output is increased triple. The input of DMU_C is first increased by $s_{12}^{+*} = 1$. This new input then increases the previous output up to three times. Such information is certainly useful in decision making process.

3. Input relaxation super-efficiency model

Excluding the column vector correspond to DMU_o from the LP coefficients matrix of model (1), input relaxation super-efficiency model introduced in [14] is defined as follows:

$$\begin{aligned}
 &\text{Maximize} \quad \phi_o^s \\
 &\text{Subject to} \quad x_{io} = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+, \quad i = 1, \dots, m \\
 &\quad \quad \quad 0 = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} - \phi_o^s y_{ro} - s_r^+, \quad r = 1, \dots, s \\
 &\quad \quad \quad 1 = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \\
 &\quad \quad \quad s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0.
 \end{aligned} \tag{2}$$

This is similar to an output-oriented version of Andersen and Petersen's model under variable returns to scale for model (1). Inefficient DMUs are assigned an index of efficiency greater than 1 that could be interpreted as the minimum increase in output vector that is required to make a DMU efficient. Efficient DMUs have an index equal to or less than 1. It represents the maximum possible proportional decrease in an output vector retaining DMU efficiency. While the most of the super-efficiency models have the difficulty of the infeasibility, we show that our super-efficiency model is always feasible.

It is worth emphasizing that we can solve the input relaxation super-efficiency model to find efficient DMUs as well as ranking them. This derives from this fact that efficient DMUs have efficiency score less than or equal to 1 as will be shown in the Proposition 3. Therefore, DMUs with super-efficiency score not greater than 1 are efficient with input relaxation model. For inefficient ones super-efficiency scores which are equal to efficiency scores in the input relaxation model are greater than 1.

Proposition 1. *The super-efficiency model is feasible and bounded.*

Proof. Assuming without loss of generality that DMU_o ≠ DMU₁, then $\phi_o^s = 0$, $\lambda_1 = 1$, $\lambda_j = 0, j \notin \{1, o\}$, $s_{i1}^- = x_{io}$, $s_{i2}^+ = x_{i1}$, $\forall i$, $s_r^+ = y_{r1}$, $\forall r$, is a solution of (2). In addition, as $\lambda_j \leq 1, j \neq o$, then $\phi_o^s \leq \max_j \frac{y_{1j}}{y_{1o}}$, assuming that $y_{1o} > 0$. Therefore, problem (2) is feasible and bounded, and $\phi_o^s \geq 0$. □

Proposition 2. *The value of super-efficiency score is always greater than or equal to zero.*

Proof. It is obvious by the proof of the Proposition 1. □

We now give two results without proofs since the proofs can be extracted from [14] easily.

Proposition 3. *Let DMU_o be an efficient DMU under the input relaxation model, then the super-efficiency score will be less than or equal to 1.*

Proposition 4. *If (X_o, Y_o) and (X_o, Y'_o) represent two input-output combinations of DMU_o such that $Y'_o \geq Y_o$, then super-efficiency score correspond to (X_o, Y'_o) is less than or equal to that of (X_o, Y_o) . Therefore, the (X_o, Y'_o) won't be ranked below (X_o, Y_o) .*

Table 2

Data for the numerical example.

DMU	I_1	I_2	O_1	O_2
1	2	2.5	2.5	0.5
2	2.5	2	2	0.75
3	1.5	1	1	2.5
4	1.5	2	3	1
5	2	1	1.5	3
6	1.2	1.5	2.5	5.5
7	3	0	4	1
8	0.5	3	3	5.5
9	2	3	4.8	4
10	3	2.5	3.5	3
11	3	4	4	4
12	4	1	5	5

We, now, compare our input relaxation super-efficiency model to some earlier super-efficiency models in the literature. Output-oriented version of Andersen and Petersen's super-efficiency model under variable returns to scale can be defined as below.

$$\begin{aligned}
 &\text{Maximize} \quad \phi_o^s \\
 &\text{Subject to} \quad x_{io} = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} + s_i^-, \quad i = 1, \dots, m \\
 &\quad \quad \quad 0 = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} - \phi_o^s y_{ro} - s_r^+, \quad r = 1, \dots, s \\
 &\quad \quad \quad 1 = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \\
 &\quad \quad \quad s_i^-, \lambda_j, s_r^+ \geq 0.
 \end{aligned} \tag{3}$$

The following theorem indicates the relationship between the input relaxation super-efficiency model and the Andersen and Peterson's model (3). The proof of the theorem we give below is obvious.

Theorem. Suppose that DMU_o is efficient under the input relaxation model. In this case if $\phi_o^s \leq 1$ in the input relaxation super-efficiency model (2), then $\phi_o^s \leq 1$ in the Andersen and Petersen's model (3).

Tone [13] introduced two models called SuperSBM (I) and (O) that rank the DEA efficient DMUs obtained from the SBM model or equivalently the CCR model. The letters (I) and (O) refer to input and output, respectively. SuperSBM (O), which is output oriented is defined as below:

$$\begin{aligned}
 &\delta_o^* = \min \quad \frac{1}{1/s \sum_{r=1}^s \frac{\bar{y}_r}{y_{ro}}} \\
 &\text{Subject to} \quad \bar{X} \geq \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j X_j \\
 &\quad \quad \quad \bar{Y} \leq \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j Y_j \\
 &\quad \quad \quad \mathbf{0} \leq \bar{Y} \leq Y_o \text{ \& } \bar{X} = X_o \\
 &\quad \quad \quad \lambda \geq \mathbf{0}.
 \end{aligned} \tag{4}$$

Note that in the SuperSBM (O), if any of the outputs is zero, the objective function of the model can't be defined since the denominator of the correspond fraction in the summation in the objective function will become zero. Although Tone [13], extended his model to consider the DMUs with zero outputs or inputs, the input relaxation super-efficiency model is more direct and easier to use than the SuperSBM (O) model which requires additional work for zero outputs.

A numerical example: We use the data of Table 2 in that each DMU uses two inputs to produce two outputs. Table 3 shows the computational results for efficient units. The two middle columns of the Table 3 represent the optimal values of the objective functions related to the model (1) and the model (2), respectively. The rank of DMUs, also, are shown in the last column.

Table 3

Computational results of the super-efficiency input relaxation model.

DMU	ϕ^*	ϕ^{s*}	Rank
6	1.0000	1.0000	3
8	1.0000	0.9836	2
12	1.0000	0.8727	1

The results of the input relaxation model in Table 3 show DMU 6, DMU 8 and DMU 12 are efficient. The value of the ϕ^* for these DMUs is unity. Therefore, we use the super-efficiency input relaxation model to rank them. The values of the ϕ^{s*} for these DMUs are 1, 0.9836 and 0.8727, respectively. It means that DMU 6 can remain efficient only with the current output, while DMU 8 could reduce its output proportionally to 98% of the current outputs to remain efficient. In addition, even if DMU 12 decrease its outputs, proportionally, to 87% of the current outputs it can remain efficient. For these DMUs the lower the efficiency index, ϕ^{s*} , the better the DMU. Hence, DMU 12 ranks the first, DMU 8 ranks the second, and DMU 6 ranks the third.

4. Stochastic input relaxation super-efficiency model

In this section, we develop stochastic input relaxation super-efficiency model which permits the possible presence of stochastic variability in the data.

As we know, DEA doesn't allow stochastic variations in input and output, therefore, DEA efficiency measurement may be sensitive to such variations. For example, a DMU which is measured as efficient relative to other DMUs, may turn inefficient if such random variations are considered. To remove this weakness in the conventional DEA models, some authors incorporated stochastic input and output variations into the DEA. See, for example, Huang, Li [34], Cooper et al. [9], and Khodabakhshi and Asgharian [20] among others. In what follows, we introduce stochastic version of the proposed super-efficiency input relaxation model which allows for the possibility of stochastic variations in input–output data.

The development in this section is similar to the papers provided by Cooper et al. [18], and Khodabakhshi, Asgharian [20].

Following Cooper et al. [18], let $\tilde{x}_j = (\tilde{x}_{1j}, \dots, \tilde{x}_{mj})^t$, $\tilde{y}_j = (\tilde{y}_{1j}, \dots, \tilde{y}_{sj})^t$ be random input and output related to DMU_j ($j = 1, \dots, n$). Let also $x_j = (x_{1j}, \dots, x_{mj})^t$, $y_j = (y_{1j}, \dots, y_{sj})^t$ show the corresponding vectors of expected values of inputs and outputs for DMU_j.

Suppose that all input and output components are jointly Normally distributed in the following chance constrained version of stochastic input relaxation model with inequality constraints, where slack variables are all excluded from the objective function.

$$\begin{aligned}
 &\text{Maximize } \phi_o \\
 &\text{Subject to } P \left\{ \sum_{j=1}^n \lambda_j \tilde{x}_{ij} - s_{i2}^+ \leq \tilde{x}_{io} \right\} \geq 1 - \alpha, \quad i = 1, \dots, m \\
 &\quad P \left\{ \sum_{j=1}^n \lambda_j \tilde{y}_{rj} \geq \phi_o \tilde{y}_{ro} \right\} \geq 1 - \alpha, \quad r = 1, \dots, s \\
 &\quad 1 = \sum_{j=1}^n \lambda_j \\
 &\quad s_{i2}^+, \lambda_j \geq 0,
 \end{aligned} \tag{5}$$

where α is a predetermined value between 0 and 1 which specifies the significance level. Since a solution with $\phi_o = 1$, $\lambda_o = 1$, $\lambda_j = 0$, ($j \neq o$) always exists, the optimal value of objective function is greater than or equal to 1. Stochastic efficiency with the input relaxation model can therefore be defined as below.

Definition 2 (Stochastic Efficiency for the Input Relaxation Model). DMU_o is stochastically efficient if and only if the following conditions are satisfied:

- (i) $\phi_o^* = 1$.
- (ii) Slack variables are zero in all alternative optimal solutions.

DMU_o is called stochastically inefficient if it doesn't fulfill the conditions of Definition 2. In other words, if for an optimal solution $\phi_o^* > 1$, or some of slacks are non zero, then DMU_o is stochastically inefficient. In fact, if $\phi_o^* > 1$, then all outputs for evaluating DMU_o can be increased to $\phi_o^* y_{ro}$, ($r = 1, \dots, s$) by using a convex combination of the other DMUs at the significance level α .

The corresponding stochastic version of the input relaxation model (1) is therefore

$$\begin{aligned}
 &\text{Maximize} \quad \phi_o + \varepsilon \left(\sum_{i=1}^m s_{i1}^- + \sum_{r=1}^s s_r^+ - \sum_{i=1}^m s_{i2}^+ \right) \\
 &\text{Subject to} \quad P \left\{ \sum_{j=1}^n \lambda_j \tilde{x}_{ij} + s_{i1}^- \leq \tilde{x}_{io} + s_{i2}^+ \right\} = 1 - \alpha, \quad i = 1, \dots, m \\
 &\quad \quad \quad P \left\{ \sum_{j=1}^n \lambda_j \tilde{y}_{rj} - \phi_o \tilde{y}_{ro} \geq s_r^+ \right\} = 1 - \alpha, \quad r = 1, \dots, s \\
 &\quad \quad \quad 1 = \sum_{j=1}^n \lambda_j \\
 &\quad \quad \quad s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0.
 \end{aligned} \tag{6}$$

Model (6), which includes a non-Archimedean ε , suggests the following restatement of the definition of stochastic efficiency.

Definition 3. DMU_o is called stochastically efficient for the input relaxation model at significance level α if the following conditions are fulfilled.

- (i) $\phi_o^* = 1$
- (ii) $s_{i1}^{-*} = s_{i2}^{+*} = s_r^{+*} = 0 \quad \forall i \text{ \& } \forall r$.

It is worth emphasizing that optimal values of slack variables may not be attainable in an ε -free model. A non-Archimedean ε is therefore needed in model (6) to avoid such undesirable feature.

4.1. Stochastic input relaxation super-efficiency model

Based on the previous assumptions the stochastic version of the proposed super-efficiency model can be defined as below:

$$\begin{aligned}
 &\text{Maximize} \quad \phi_o^s \\
 &\text{Subject to} \quad P \left\{ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \tilde{x}_{ij} + s_{i1}^- \leq \tilde{x}_{io} + s_{i2}^+ \right\} = 1 - \alpha, \quad i = 1, \dots, m \\
 &\quad \quad \quad P \left\{ \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \tilde{y}_{rj} - \phi_o^s \tilde{y}_{ro} \geq s_r^+ \right\} = 1 - \alpha, \quad r = 1, \dots, s \\
 &\quad \quad \quad 1 = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \\
 &\quad \quad \quad s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0,
 \end{aligned} \tag{7}$$

where α is a predetermined value between 0 and 1 which specifies the significance level, and P represents the probability. DMU_o is stochastically super-efficient at significance α if the optimal value of the objective function is less than 1. Therefore, if $\phi_o^{s*} < 1$ it means that DMU_o can reduce its output to ϕ_o^{s*} percent of its current output and still remain efficient, hence the lower the ϕ_o^{s*} , the better the DMU. In the next subsection, the deterministic equivalent of the above stochastic super-efficiency model is obtained.

4.2. Deterministic equivalent for the stochastic super-efficiency model

In what follows, we exploit Normality assumption to introduce a deterministic equivalent to model (7). We first need to recall a well-known fact about normally distributed random vectors that is used below. Suppose that $\vec{X}_k \sim N(\vec{\mu}_{k \times 1}, \Sigma_{k \times k})$, where $\vec{\mu}_{k \times 1}$ and $\Sigma_{k \times k}$ are, respectively, the mean value vector and the variance–covariance matrix. Then for any matrix $A_{m \times k}$ we have $A\vec{X} \sim N(A\vec{\mu}, A\Sigma_{k \times k}A^T)$, where A^T is the transpose of A . Using this result, one can obtain the following deterministic

equivalent to the stochastic input relaxation model, model (6).

$$\begin{aligned}
 &\text{Maximize} \quad \phi_o + \varepsilon \left(\sum_{i=1}^m s_{i1}^- + \sum_{r=1}^s s_r^+ - \sum_{i=1}^m s_{i2}^+ \right) \\
 &\text{Subject to} \quad \sum_{j=1}^n \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+ - \Phi^{-1}(\alpha) \sigma_i^l(\lambda) = x_{i0}, \quad i = 1, \dots, m \\
 &\quad \phi_o y_{r0} - \sum_{j=1}^n \lambda_j y_{rj} + s_r^+ - \Phi^{-1}(\alpha) \sigma_r^o(\phi_o, \lambda) = 0, \quad r = 1, \dots, s \\
 &\quad \sum_{j=1}^n \lambda_j = 1 \\
 &\quad s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0,
 \end{aligned} \tag{8}$$

where Φ is the cumulative distribution function (cdf) of a standard Normal random variable and Φ^{-1} is its inverse. It is assumed that x_{ij} and y_{rj} are the means of the input and output variables.

It is obvious, from the forms of $\sigma_i^l(\lambda)$ and $\sigma_r^o(\phi_o, \lambda)$, that model (8) is a nonlinear program. Following Cooper et al. [18], we show that this nonlinear program can be transformed to a quadratic programming problem. Suppose that w_i^l and w_r^o are nonnegative variables. Replacing w_i^l and w_r^o , respectively, by $\sigma_i^l(\lambda)$ and $\sigma_r^o(\phi_o, \lambda)$ and adding the following quadratic equality constraints

$$\begin{aligned}
 (w_i^l)^2 &= (\sigma_i^l(\lambda))^2 \\
 (w_r^o)^2 &= (\sigma_r^o(\phi_o, \lambda))^2
 \end{aligned}$$

model (8) is transformed to a quadratic programming problem. One can therefore obtain the optimal values $\phi_o^*, s_{i1}^{+*}, s_{i2}^{+*}$ and s_r^{+*} by solving the quadratic program.

One of the following three cases must naturally occur for the i th input of evaluating DMU_o:

- (i) increase, which corresponds to $s_{i2}^{+*} > 0$
- (ii) decrease, which corresponds to $s_{i1}^{-*} > 0$
- (iii) no change, which corresponds to $s_{i1}^{-*} = s_{i2}^{+*}$.

To make sure that at most one of the s_{i1}^- and s_{i2}^+ is positive in our model, we add the following constraint on s_{i1}^- and s_{i2}^+

$$s_{i1}^- \cdot s_{i2}^+ = 0.$$

It is worth noting that in the deterministic input relaxation model (2) at most one of the three aforementioned cases could occur in the basic optimal solution. For, we use the simplex method and the corresponding columns of s_{i1}^{-*}, s_{i2}^{+*} in model (1) are linearly dependent. We, finally, have the following deterministic equivalent to our stochastic model, model (6).

$$\begin{aligned}
 &\text{Maximize} \quad \phi_o + \varepsilon \left(\sum_{i=1}^m s_{i1}^- + \sum_{r=1}^s s_r^+ - \sum_{i=1}^m s_{i2}^+ \right) \\
 &\text{Subject to} \quad \sum_{j=1}^n \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+ - \Phi^{-1}(\alpha) w_i^l = x_{i0}, \quad i = 1, \dots, m \\
 &\quad \phi_o y_{r0} - \sum_{j=1}^n \lambda_j y_{rj} + s_r^+ - \Phi^{-1}(\alpha) w_r^o = 0, \quad r = 1, \dots, s \\
 &\quad \sum_{j=1}^n \lambda_j = 1 \\
 &\quad (w_i^l)^2 = \sum_{j \neq o} \sum_{k \neq o} \lambda_j \lambda_k \text{Cov}(\tilde{x}_{ij}, \tilde{x}_{ik}) + 2(\lambda_o - 1) \sum_{j \neq o} \lambda_j \text{Cov}(\tilde{x}_{ij}, \tilde{x}_{io}) + (\lambda_o - 1)^2 \text{Var}(\tilde{x}_{io}) \\
 &\quad (w_r^o)^2 = \sum_{k \neq o} \sum_{j \neq o} \lambda_k \lambda_j \text{Cov}(\tilde{y}_{rk}, \tilde{y}_{rj}) + 2(\lambda_o - \phi_o) \sum_{k \neq o} \lambda_k \text{Cov}(\tilde{y}_{rk}, \tilde{y}_{ro}) + (\lambda_o - \phi_o)^2 \text{Var}(\tilde{y}_{ro}) \\
 &\quad s_{i1}^- \cdot s_{i2}^+ = 0 \\
 &\quad s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+, w_i^l, w_r^o \geq 0.
 \end{aligned} \tag{9}$$

Table 4
Results of the model (8).

DMU	ϕ^*	s_{11}^{-*}	s_{21}^{-*}	s_{12}^{+*}	s_{22}^{+*}	s^{+*}
1981	4.0124428	0	0.	334.22	6.83	0
1982	3.97752466	0	0.	310.92	5.53	0
1983	3.71152426	0	0.	299.72	9.61	0
1984	3.3890515	0	0.	305.92	8.59	0
1985	2.99845835	0	0.	153.22	14.08	0
1986	3.07765436	0	0.	122.72	13.24	0
1987	2.76797602	0	0.	82.12	10.78	0
1988	2.35063409	0	0.	7.92	2.97	0
1989	1.99016563	12.78	0.	0	0.72	0
1990	2.16052255	21.78	0.	0	8.45	0
1991	1.70048387	32.78	0.	0	12.29	0
1992	1.51060567	19.78	0.	0	9.19	0
1993	1.26974514	0	0.	39.22	1.61	0
1994	1	0	0.	0	0	0
1995	1	0	0.	0	0	0
1996	1	0	0.	0	0	0
1997	1	0	0.	0	0	0

Similarly, one can obtain the following deterministic equivalent to the stochastic super-efficiency model, model (7).

$$\begin{aligned}
 & \text{Maximize} \quad \phi_o^s \\
 & \text{Subject to} \quad \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+ - \Phi^{-1}(\alpha) \sigma_i^l(\lambda) = x_{io}, \quad i = 1, \dots, m \\
 & \quad \phi_o^s y_{ro} - \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} + s_r^+ - \Phi^{-1}(\alpha) \sigma_r^o(\phi_o^s, \lambda) = 0, \quad r = 1, \dots, s \\
 & \quad \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j = 1 \\
 & \quad s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0.
 \end{aligned} \tag{10}$$

We, finally, have the following deterministic equivalent to our stochastic input relaxation super-efficiency model.

$$\begin{aligned}
 & \text{Maximize} \quad \phi_o^s \\
 & \text{Subject to} \quad \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+ - \Phi^{-1}(\alpha) w_i^l = x_{io}, \quad i = 1, \dots, m \\
 & \quad \phi_o^s y_{ro} - \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj} + s_r^+ - \Phi^{-1}(\alpha) w_r^o = 0, \quad r = 1, \dots, s \\
 & \quad \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j = 1 \\
 & \quad (w_i^l)^2 = \sum_{j \neq o} \sum_{k \neq o} \lambda_j \lambda_k \text{Cov}(\tilde{x}_{ij}, \tilde{x}_{ik}) - 2 \sum_{j \neq o} \lambda_j \text{Cov}(\tilde{x}_{ij}, \tilde{x}_{io}) + \text{Var}(\tilde{x}_{io}) \\
 & \quad (w_r^o)^2 = \sum_{k \neq o} \sum_{j \neq o} \lambda_k \lambda_j \text{Cov}(\tilde{y}_{rk}, \tilde{y}_{rj}) - 2 \phi_o^s \sum_{k \neq o} \lambda_k \text{Cov}(\tilde{y}_{rk}, \tilde{y}_{ro}) + (\phi_o^s)^2 \text{Var}(\tilde{y}_{ro}) \\
 & \quad s_{i1}^- * s_{i2}^+ = 0 \\
 & \quad s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+, w_i^l, w_r^o \geq 0.
 \end{aligned} \tag{11}$$

4.3. An empirical example

In this subsection, data of Chinese textile industry is used for illustration. These data are based on two inputs, labor and capital, and a single output. Both capital and output values are stated in units of 1 million Ren Min Bi (Chinese monetary

Table 5

Numerical results of the stochastic super-efficiency, model (10).

DMU	Stochastic super-efficiency score	Rank
1994	0.8935206	1
1995	0.9936376	4
1996	0.9705907	3
1997	0.9634141	2

unit) and labor is expressed in units of 1000 persons. Following [35], we consider every year in this industry as a DMU. The input and output data are presented in Appendix.

The computational results of the model (1) show that just DMU 1994 is efficient. To see a detailed discussion on the computational results of the model (1) and compare to the results of the output oriented BCC model refer to [30]. However, for DMU 1994 $\phi^{s*} = 0.96$ in the model (2) which means if this DMU reduces its output to $\phi^{s*} = 0.96\%$ of the current output it remains efficient.

Models (9) and (11) are applied to the data represented in Appendix to obtain and rank efficient units.

Let $\alpha = 0.4$ for which $\Phi^{-1}(\alpha) \approx -0.25$. This rather large value of α is deliberately chosen to illustrate differences between the results based on model (11) and model (2). It is worth noting that model (11) and model (2) produce similar results when α is small.

The computational results of the equivalent deterministic problems are presented in Tables 4 and 5. We assume that all DMUs have the same variance, but they can have different means. The variances for the outputs and the inputs can therefore be estimated by:

$$\text{Var}(\tilde{y}_r) = \frac{1}{16} \sum_{j=1}^{17} (y_{rj} - \bar{y}_r)^2 \quad \& \quad \text{Var}(\tilde{x}_i) = \frac{1}{16} \sum_{j=1}^{17} (x_{ij} - \bar{x}_i)^2$$

where

$$\bar{y}_r = \frac{1}{17} \sum_{j=1}^{17} y_{rj} \quad \& \quad \bar{x}_i = \frac{1}{17} \sum_{j=1}^{17} x_{ij}$$

and x_{ij} and y_{rj} are the observed values of inputs and outputs for DMU_j which we used as an estimate for the expected values of the stochastic inputs and outputs. We will also assume that outputs and inputs for different DMUs are independent. This independence assumption then implies that $\text{Cov}(\tilde{y}_{rk}, \tilde{y}_{rj}) = 0$ and also $\text{Cov}(\tilde{x}_{ij}, \tilde{x}_{ik}) = 0$. Numerical results of Models (9) and (11) obtained by GAMS software are represented in Tables 4 and 5, respectively. To see a detailed discussion of the stochastic input relaxation model on the textile industry one may refer to [20]. Based on the results in the Table 4, if DMU 1987 uses 82 more labor, and 10.78 more capital than the previous amounts, its output could be more than 2.76 times the current output. Moreover, if DMU 1982 uses 5.53 more capital than the previous amount, its output would be 3.98 times the previous amount, while about more than 310 unit labor could be employed. Besides producing high output, attracting labor is a merit in a country like China which have to employ 14–16 million persons each year.

Here, we want to rank efficient DMUs identified by stochastic input relaxation model. In Table 4, DMUs 1994, 1995, 1996 and 1997 have $\phi^* = 1$ and the optimal values of slacks for them are zero. Therefore, these DMUs are efficient in the model (5), while just one of them, 1994, is efficient in model (1).

The computational results of stochastic super-efficiency input relaxation model are presented in Table 5. This results show that four DMUs, 1994–97, have stochastic super-efficiency score less than 1 in the model (11). Therefore, these DMUs are stochastic super-efficient in the model (11). In other words, even if these DMUs produce less output than their current outputs they can still remain efficient in model (5). For example, DMU 1994 has super-efficiency score $\tilde{\phi}^{s*} = 0.89$ that means if this DMU reduces its output to 0.89 of its current output it can remain efficient. This DMU has the lowest stochastic super-efficiency score, so it ranks the first. Note that DMU 1994 was the only efficient DMU with the model (1). The next top DMU is 1997 with super-efficiency score $\tilde{\phi}^{s*} = 0.96$ which ranks the second. Finally, DMUs 1996 and 1995 with stochastic super-efficiency scores 0.97 and 0.99 rank the third and the fourth, respectively.

5. Sensitivity analysis

In our sensitivity analysis discussion, following [18], we permit allowable limits of data variations for only one DMU at a time. These sensitivity analysis are to be found in [36,37] that contrast with other approaches to sensitivity analysis in DEA that allow all data for all DMUs to be varied simultaneously until at least one DMU changes its status from efficient to inefficient, or vice versa. We are going to simplify matter in the previous part by assuming that only DMU_o has random variations in its input and outputs, i.e. $\sigma_{io}^I \neq 0$, $\sigma_{ro}^O \neq 0$, $\sigma_{ij}^I = 0$, and $\sigma_{rj}^O = 0$ ($j \neq o$) for all i and r . In this case, the model

(10) can be written:

$$\begin{aligned}
 & \text{Maximize} \quad \phi_o^s \\
 & \text{Subject to} \quad x'_{io} = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x'_{ij} + s_{i1}^- - s_{i2}^+, \quad i = 1, \dots, m \\
 & \quad 0 = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y'_{rj} - \phi_o^s y'_{ro} - s_r^+, \quad r = 1, \dots, s \\
 & \quad 1 = \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j \\
 & \quad s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0,
 \end{aligned} \tag{12}$$

where

$$\begin{aligned}
 y'_{ro} &= y_{ro} - \sigma_{ro}^o \Phi^{-1}(\alpha), \quad r = 1, \dots, s \\
 y'_{rj} &= y_{rj}, \quad j \neq o, r = 1, \dots, s
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 x'_{io} &= x_{io} + \sigma_{io}^i \Phi^{-1}(\alpha), \quad i = 1, \dots, m \\
 x'_{ij} &= x_{ij}, \quad j \neq o, i = 1, \dots, m.
 \end{aligned} \tag{14}$$

Therefore, the model (12) is the deterministic equivalent of stochastic model (7) under the above assumptions. This model is, also, the input relaxation super-efficiency model (2) for DMU_o with adjusted input and output values x'_{ij} , $i = 1, \dots, m$ and y'_{rj} , $r = 1, \dots, s$ as defined in (13) and (14).

Proposition 5. If $\alpha = 0.5$. Then results of the super-efficiency model (2) and model (12) are the same.

Proof. Since $\Phi^{-1}(0.5) = 0$, it is obvious. \square

Proposition 6. For $0 < \alpha < 0.5$,

- (i) Suppose that for DMU_o $\phi_o^{s*} \leq 1$ in model (2), then $\tilde{\phi}_o^{s*} \leq 1$ in model (12).
- (ii) Suppose that for DMU_o $\tilde{\phi}_o^{s*} > 1$ in model (12), then $\phi_o^{s*} > 1$ in the model (2).

Proof. (i) Note that since $0 < \alpha < 0.5$, $\Phi^{-1}(\alpha) < 0$. Therefore, $y'_{ro} \geq y_{ro}$ and $x'_{io} \leq x_{io}$. Thus, if $\tilde{\phi}_o^{s*} > 1$, then there exists a solution with $\phi_o^s = \tilde{\phi}_o^{s*} > 1$ for the super-efficiency model (2), when evaluating DMU_o, which is in contrast with $\phi_o^{s*} \leq 1$. Therefore, this contradiction shows that we must have $\tilde{\phi}_o^{s*} \leq 1$.

(ii) This follows directly from (i). \square

Proposition 7. For $0.5 < \alpha < 1$.

- (i) Suppose that for DMU_o $\tilde{\phi}_o^{s*} \leq 1$ in model (12), then $\phi_o^{s*} \leq 1$ in the model (2).
- (ii) Suppose that for DMU_o $\phi_o^{s*} > 1$ in model (2), then $\tilde{\phi}_o^{s*} > 1$ in model (12).

Proof. (i) Note that since $0.5 < \alpha < 1$, $\Phi^{-1}(\alpha) > 0$. Therefore, $y_{ro} \geq y'_{ro}$ and $x_{io} \leq x'_{io}$. Thus, if $\phi_o^{s*} > 1$, then there exists a solution with $\tilde{\phi}_o^s = \phi_o^{s*} > 1$ for the model (12), when evaluating DMU_o, which is in contrast with $\tilde{\phi}_o^{s*} \leq 1$. Therefore, this contradiction shows that we must have $\phi_o^{s*} \leq 1$.

(ii) This follows directly from (i). \square

Proposition 8. For $0.5 < \alpha < 1$.

Suppose that for DMU_o $\phi_o^{s*} \leq 1$ in model (2), then $\tilde{\phi}_o^{s*} \leq 1$ in model (12), if

$$\sum_{r=1}^s \sigma_{ro}^o < \sum_{r=1}^s \theta_r^{+*} / \Phi^{-1}(\alpha),$$

where $\sum_{r=1}^s \theta_r^{+*}$ is the optimal value of

$$\begin{aligned}
 &\text{Minimize} && \sum_{r=1}^s \theta_r^+ \\
 &\text{Subject to} && \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij} \leq x_{i0} + \theta_i^-, \quad i = 1, \dots, m \\
 &&& \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj} \geq y_{r0} - \theta_r^+, \quad r = 1, \dots, s \\
 &&& 1 = \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j \\
 &&& \theta_i^-, \theta_r^+, \lambda_j \geq 0.
 \end{aligned} \tag{15}$$

Proof. Suppose on the contrary that $\phi^{\tilde{s}*} > 1$ in the model (12). This implies that there is a $\bar{\lambda}$ with $\bar{\lambda}_0 = 0$, $\bar{\lambda}_j \geq 0$ ($j \neq 0$) and $1 = \sum_{j=1}^n \bar{\lambda}_j$ such that

$$\begin{aligned}
 \sum_{j \neq 0} \bar{\lambda}_j y'_{rj} &> y'_{r0}, \quad r = 1, \dots, s \\
 \sum_{j \neq 0} \bar{\lambda}_j x'_{ij} &\leq x'_{i0} + s_{i2}^+, \quad i = 1, \dots, m
 \end{aligned}$$

in which $s_{i2}^+ \geq 0$, by the definition of y' and x'

$$\begin{aligned}
 \sum_{j \neq 0} \bar{\lambda}_j y'_{rj} &> y_{r0} - \Phi^{-1}(\alpha) \sigma_{r0}^0 \\
 \sum_{j \neq 0} \bar{\lambda}_j x'_{ij} &\leq x_{i0} + s_{i2}^+ + \Phi^{-1}(\alpha) \sigma_{i0}^I
 \end{aligned}$$

letting

$$\bar{\theta}_r^+ = \Phi^{-1}(\alpha) \sigma_{r0}^0, \quad r = 1, \dots, s$$

and

$$\bar{\theta}_i^- = \Phi^{-1}(\alpha) \sigma_{i0}^I + s_{i2}^+, \quad i = 1, \dots, m$$

we find that $(\bar{\theta}^+, \bar{\theta}^-, \bar{\lambda})$ satisfies relation (15) for which $\sum_{r=1}^s \bar{\theta}_r^+ < \sum_{r=1}^s \theta_r^{+*}$, a contradiction to assumption that $\theta_r^{+*}, \bar{\theta}_i^*$ are optimal for model (15). Hence, $\phi^{\tilde{s}*} \leq 1$ in model (12). \square

Proposition 9. For $0 < \alpha < 0.5$.

Suppose that for DMU_0 $\phi^{\tilde{s}*} \leq 1$ in model (12), then $\phi^{s*} \leq 1$ in model (2), if $\sum_{r=1}^s \sigma_{r0}^0 < \sum_{r=1}^s \theta_r^{+*} / (-\bar{\Phi}^{-1}(\alpha))$ where $\sum_{r=1}^s \theta_r^{+*}$ is the optimal value of

$$\begin{aligned}
 &\text{Minimize} && \sum_{r=1}^s \theta_r^+ \\
 &\text{Subject to} && \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij} \leq x'_{i0} + \theta_i^-, \quad i = 1, \dots, m \\
 &&& \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj} \geq y'_{r0} - \theta_r^+, \quad r = 1, \dots, s \\
 &&& 1 = \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j \\
 &&& \theta_i^-, \theta_r^+, \lambda_j \geq 0.
 \end{aligned} \tag{16}$$

Table 6

Data of textile industry.

Year	Labor	Capital	Output
1981	389.00	19.86	856.02
1982	412.30	21.16	866.85
1983	423.50	17.08	956.04
1984	417.30	18.10	1082.94
1985	570.00	12.61	1273.20
1986	600.50	13.45	1230.72
1987	641.10	15.91	1410.66
1988	715.30	23.72	1728.16
1989	736.00	25.97	2109.57
1990	745.00	18.24	2291.08
1991	756.00	14.40	2533.27
1992	743.00	17.50	2899.16
1993	684.00	25.08	3520.74
1994	691.00	25.45	4949.93
1995	673.00	29.35	4604.00
1996	634.00	23.05	4722.29
1997	595.00	25.02	4760.28

Proof. Suppose on the contrary that $\phi^{s*} > 1$ in model (2). Therefore, there exist $\bar{\lambda}$ with $\lambda_o = 0$, $\bar{\lambda}_j \geq 0$ ($j \neq 0$) and $1 = \sum_{j=1}^n \bar{\lambda}_j$ such that

$$\sum_{j \neq 0} \bar{\lambda}_j y_{rj} = \sum_{j \neq 0} \bar{\lambda}_j y'_{rj} > y_{ro}$$

$$\sum_{j \neq 0} \bar{\lambda}_j x_{ij} = \sum_{j \neq 0} \bar{\lambda}_j x'_{ij} \leq x_{io} + s_{i2}^+$$

in which s_{i2}^+ is non-negative, by definition of x'_o and y'_o we then have

$$\sum_{j \neq 0} \bar{\lambda}_j y_{rj} = \sum_{j \neq 0} \bar{\lambda}_j y'_{rj} > y_{ro} = y'_{ro} + \Phi^{-1}(\alpha) \sigma_{ro}^o$$

$$\sum_{j \neq 0} \bar{\lambda}_j x_{ij} = \sum_{j \neq 0} \bar{\lambda}_j x'_{ij} \leq x'_{io} + s_{i2}^+ - \Phi^{-1}(\alpha) \sigma_{io}^l$$

letting $\bar{\theta}_r^+ = -\Phi^{-1}(\alpha) \sigma_{ro}^o$, $r = 1, \dots, s$ and $\bar{\theta}_i^- = -\Phi^{-1}(\alpha) \sigma_{io}^l + s_{i2}^+$, we find that $(\bar{\theta}_1^+, \dots, \bar{\theta}_s^+, \bar{\theta}_1^-, \dots, \bar{\theta}_m^-)$ is a solution of (16) for which $\sum_{r=1}^s \bar{\theta}_r^+ < \sum_{r=1}^s \theta_r^{+*}$, a contradiction to assumption that $\theta_r^{+*}, \bar{\theta}_i^*$ are optimal for model (16). Therefore, we must have $\phi^{s*} \leq 1$ in the model (2). \square

6. Conclusion

In this paper, we proposed a super-efficiency measure based on input relaxation model in stochastic data envelopment analysis, and we described its characterizations theoretically and empirically. The proposed method will be practically useful to rank efficient units obtained by the input relaxation model.

In addition to developing stochastic version of the proposed super-efficiency model, we obtained the deterministic equivalent of the stochastic version which can be converted to a quadratic problem. As an empirical example, the proposed method is applied to data of textile industry of China to rank efficient units. Sensitivity analysis of the proposed super-efficiency model, when allowable limits of data variations for evaluating DMU are permitted, on parameter α , which specifies significance level, has discussed. Finally, developing the proposed super-efficiency measure in fuzzy DEA can be suggested for further research.

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Appendix

The data of textile industry was collected from the Statistical Year Book of China, published by the Chinese Bureau of Statistics. All values have been adjusted to a 1991 base period to eliminate the impact of price variations. The adjusted input and output data are presented in Table 6.

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