Journal of Computational and Applied Mathematics 236 (2012) 3084–3100

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Analysis of a FEM–BEM model posed on the conducting domain for the time-dependent eddy current problem

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ARTICLE INFO

Article history:
Received 7 June 2011
Received in revised form 23 January 2012

Keywords:
Time-dependent electromagnetics
Eddy current problem
Boundary elements
Finite elements

ABSTRACT

The three-dimensional eddy current time-dependent problem is considered. We formulate it in terms of two variables, one lying only on the conducting domain and the other on its boundary. We combine finite elements (FEM) and boundary elements (BEM) to obtain a FEM–BEM coupled variational formulation. We establish the existence and uniqueness of the solution in the continuous and the fully discrete case. Finally, we investigate the convergence order of the fully discrete scheme.

1. Introduction

The eddy current model is commonly used in many problems in sciences and industry, for example, in induction heating, electromagnetic braking, electric generation, etc. An overview of the mathematical analysis of the eddy current model and its numerical solution in harmonic regime can be found in the recent book [1], which provides a large list of references on this subject.

In this paper, we deal with the numerical solution of the time-dependent eddy current problem, which is naturally formulated in the whole space, with adequate decay conditions at infinity. The literature on the numerical analysis of time-dependent problems of this kind is more scarce. Among the few papers devoted to this subject, both in bounded and unbounded domains, by using finite element (FEM), boundary element (BEM) or coupled FEM–BEM methods, we can mention [2–8]. These articles differ from each other by the physical quantities chosen for the formulation (magnetic field, electric field or different kind of potentials) and by the way of treating the decay condition to reduce the problem to a bounded domain.

We consider a FEM–BEM method to compute the eddy currents generated in a three-dimensional conductor Ω_C by a time-dependent source current. The problem is reformulated by expressing the magnetic and the electric fields in terms of convenient new variables. We use FEM only on the conducting domain Ω_C, the integral conditions being imposed on its boundary $\partial\Omega_C$. Therefore, the domain where FEM is used results as small as possible, leading to a more efficient method as compared, for instance, with [2,3], where similar formulations but involving FEM in part of the dielectric domain are considered. Another important feature of this approach is that it preserves the coercivity of the original problem. The purpose of this paper is to analyze the convergence of a fully discrete FEM–BEM scheme for this formulation and to investigate the convergence order.

The paper is organized as follows. In Section 2 we give some basic definitions. In Section 3 we introduce the model problem and the assumptions over the data. Then, we introduce a new variable, the time-primitive of the electric field, which plays the role of a vector potential for the magnetic field. In Section 4 we introduce the integral operators and recall

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2012.01.030

their properties. Then, we derive the FEM–BEM formulation and show the existence and uniqueness of the solution to the problem. In Section 5, we introduce a space-discretization of the problem based on Nédélec edge elements in Ω_C and piecewise linear continuous elements for the variable on $\partial\Omega_C$ arising from the integral equations. Then, a backward Euler method is employed for the time discretization. Finally, the results presented in Section 6 prove that the proposed fully discrete scheme is convergent with optimal order.

2. Preliminaries

In the sequel we deal with real valued functions. Boldface letters will denote vectors (in $\mathbb{R}^n$) or vector-valued functions, as well as matrices. The symbol $|\cdot|$ will represent the Euclidean norm for n-dimensional vectors:

$$|\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v} := \sum_{i=1}^{n} v_i^2.$$

In all the paper the conductor $\Omega_C \subset \mathbb{R}^3$ is a bounded connected polyhedron, with a Lipschitz-continuous connected boundary $\Gamma := \partial\Omega_C$, so that the insulator $\Omega_I := \mathbb{R}^3 \setminus \overline{\Omega_C}$ is also connected.

We remark that, under the above conditions, Ω_C and Ω_I have the same number of non-bounding cycles L; namely, there exist L disjoint connected open "cutting" surfaces $\Sigma_j^{\text{int}} \subset \Omega_C$ (respectively $\Sigma_j^{\text{ext}} \subset \Omega_I$), $j = 1, \dots, L$, such that $\widetilde{\Omega}_C := \Omega_C \setminus \bigcup_{j=1}^{L} \Sigma_j^{\text{int}}$ (respectively $\widetilde{\Omega}_I := \Omega_I \setminus \bigcup_{j=1}^{L} \Sigma_j^{\text{ext}}$) is simply connected. The boundary curves $\partial\Sigma_j^{\text{int}}$ and $\partial\Sigma_j^{\text{ext}}$ lie on Γ.

We denote by

$$(f, g)_{0,\Omega_*} := \int_{\Omega_*} fg \, \mathrm{d}\boldsymbol{x}$$

the inner product in $L^2(\Omega_*)$ and $\|\cdot\|_{0,\Omega_*}$ the corresponding norm with $* \in \{C, I\}$. As usual, $\|\cdot\|_{s,\Omega_C}$ stands for the norm of the Hilbertian Sobolev spaces $H^s(\Omega_C)$ for all $s \in \mathbb{R}$. We recall that, for $s \in (0, 1)$, the space $H^s(\Gamma)$ has an intrinsic definition (by localization) on the Lipschitz surface Γ due to their invariance under Lipschitz coordinate transformations. We denote by $\|\cdot\|_{s,\Gamma}$ the norm in $H^s(\Gamma)$. Moreover, $H^{-s}(\Gamma)$ denotes the corresponding dual space.

In this paper, the spaces that are product of function spaces are endowed with the natural product norms and duality pairings without changing the notations; it will be clear from the context when scalar or vector functions are used.

Finally, we introduce the functional spaces

$$\boldsymbol{H}(\mathbf{curl}\,;\Omega_C) := \left\{\boldsymbol{v} \in (L^2(\Omega_C))^3 : \mathbf{curl}\,\boldsymbol{v} \in (L^2(\Omega_C))^3\right\},$$
$$\boldsymbol{H}(\mathrm{div};\Omega_C) := \left\{\boldsymbol{v} \in (L^2(\Omega_C))^3 : \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega_C)\right\},$$

endowed with their natural norms $\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)} := \|\boldsymbol{v}\|^2_{0,\Omega_C} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega_C}$ and $\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div};\Omega_C)} := \|\boldsymbol{v}\|^2_{0,\Omega_C} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{0,\Omega_C}$, respectively.

2.1. Basic spaces for time dependent problems

Since we will deal with a time-dependent problem, we will use spaces of functions defined on a bounded interval $[0, T]$ and with values in a separable Hilbert space V whose norm is denoted here by $\|\cdot\|_V$. We use the notation $\mathcal{C}^0([0, T]; V)$ for the Banach space consisting of all continuous functions $f : [0, T] \to V$. More generally, for any $k \in \mathbb{N}$, $\mathcal{C}^k([0, T]; V)$ denotes the subspace of $\mathcal{C}^0([0, T]; V)$ of all functions f with (strong) derivatives $\mathrm{d}^j f/\mathrm{d}t^j$ in $\mathcal{C}^0([0, T]; V)$ for all $j = 1, \dots, k$. In the sequel, we will use indistinctly the notations $\partial_t f = \mathrm{d}f/\mathrm{d}t$ to express the derivative with respect to t.

We also consider the space $L^2(0, T; V)$ of classes of functions $f : (0, T) \to V$ that are Böchner-measurable and such that

$$\|f\|^2_{L^2(0,T;V)} := \int_0^T \|f(t)\|_V^2 \, \mathrm{d}t < +\infty.$$

Furthermore, we will use

$$H^1(0, T; V) := \left\{f \in L^2(0, T; V) : \partial_t f \in L^2(0, T; V)\right\}.$$

Analogously, we define $H^k(0, T; V)$ for all $k \in \mathbb{N}$.

3. The model problem

The unit normal vector on Γ that points from Ω_C to Ω_I (respectively from Ω_I to Ω_C) is denoted by $\boldsymbol{n}_C$ (respectively $\boldsymbol{n}_I = -\boldsymbol{n}_C$).

Let $\boldsymbol{E}(\boldsymbol{x}, t)$ be the electric field and $\boldsymbol{H}(\boldsymbol{x}, t)$ the magnetic field. Given a time-dependent compactly supported current density $\boldsymbol{J}$, our aim is to furnish an approximate solution to the problem below:

$$\begin{aligned}
&\partial_t(\boldsymbol{\mu}\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } \mathbb{R}^3 \times (0, T),\\
&\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{\sigma}\boldsymbol{E} = \boldsymbol{J} \quad \text{in } \mathbb{R}^3 \times [0, T],\\
&\operatorname{div}(\boldsymbol{\varepsilon}\boldsymbol{E}) = 0 \quad \text{in } \Omega_I \times [0, T],\\
&\boldsymbol{H}(\boldsymbol{x}, t), \boldsymbol{E}(\boldsymbol{x}, t) = \mathcal{O}(|\boldsymbol{x}|^{-1}) \quad \text{as } |\boldsymbol{x}| \to \infty,\\
&\boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3,
\end{aligned} \tag{1}$$

where the asymptotic behavior $(1)_4$ holds uniformly in $[0, T]$.

The initial data $\boldsymbol{H}_0 \in (L^2(\mathbb{R}^3))^3$ has to satisfy $\operatorname{div}(\boldsymbol{\mu}\boldsymbol{H}_0) = 0$ in $\mathbb{R}^3$. Coefficients $\boldsymbol{\sigma}$, $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$ are assumed to be symmetric matrices with bounded entries. The electric conductivity $\boldsymbol{\sigma}$ is positive definite in Ω_C and vanishes in Ω_I. The magnetic permeability $\boldsymbol{\mu}$ is positive definite in all $\mathbb{R}^3$ and satisfies $\boldsymbol{\mu} = \mu_0\mathbf{I}$ in Ω_I ($\mathbf{I}$ being the identity matrix). The electric permittivity $\boldsymbol{\varepsilon}$ is only needed in the dielectric domain in this formulation and we assume it satisfies $\boldsymbol{\varepsilon} = \varepsilon_0\mathbf{I}$ in Ω_I; μ_0 and ε_0 being the corresponding coefficients in vacuum. Finally, we assume that the source current is supported in Ω_C. Moreover, we consider $\boldsymbol{J} \in L^2(0, T; (L^2(\Omega_C))^3)$.

We define $\boldsymbol{H}_C := \boldsymbol{H}|_{\Omega_C}$ and $\boldsymbol{H}_I := \boldsymbol{H}|_{\Omega_I}$; analogously, $\boldsymbol{H}_{C,0} := \boldsymbol{H}_0|_{\Omega_C}$, $\boldsymbol{H}_{I,0} := \boldsymbol{H}_0|_{\Omega_I}$, $\boldsymbol{E}_C := \boldsymbol{E}|_{\Omega_C}$, $\boldsymbol{E}_I := \boldsymbol{E}|_{\Omega_I}$, etc.

We consider the space $\mathbb{H}(\Omega_C)$, defined as

$$\mathbb{H}(\Omega_C) := \left\{\boldsymbol{v} \in (L^2(\Omega_C))^3 : \mathbf{curl}\,\boldsymbol{v} = \mathbf{0},\ \operatorname{div}(\boldsymbol{\sigma}\boldsymbol{v}) = 0,\ \boldsymbol{\sigma}\boldsymbol{v} \cdot \boldsymbol{n}_C = 0 \text{ on } \Gamma\right\}.$$

We recall that each cutting surface Σ_j^{int}, $j = 1, \dots, L$, "cuts" an independent non-bounding cycle in Ω_C. They are connected orientable Lipschitz surfaces with $\partial\Sigma_j^{\text{int}} \subset \Gamma$, such that every curl-free vector field in Ω_C has a global potential in $\widetilde{\Omega}_C$. A basis of $\mathbb{H}(\Omega_C)$ is given by the functions $\boldsymbol{\omega}_j$ which are the $(L^2(\Omega_C))^3$-extension of ∇p_j, where $p_j \in H^1(\Omega_C \setminus \Sigma_j^{\text{int}})$ is the solution of the problem

$$\begin{aligned}
&\operatorname{div}(\boldsymbol{\sigma}\nabla p_j) = 0 \quad \text{in } \Omega_C \setminus \Sigma_j^{\text{int}},\\
&\boldsymbol{\sigma}\nabla p_j \cdot \boldsymbol{n}_C = 0 \quad \text{on } \Gamma \setminus \partial\Sigma_j^{\text{int}},\\
&[\![\boldsymbol{\sigma}\nabla p_j \cdot \boldsymbol{n}_j^{\text{int}}]\!]_{\Sigma_j^{\text{int}}} = 0, \quad j = 1, \dots, L,\\
&[\![p_j]\!]_{\Sigma_j^{\text{int}}} = 1, \quad j = 1, \dots, L,
\end{aligned}$$

having denoted by $[\![\cdot]\!]_{\Sigma_j^{\text{int}}}$ the jump across the surface Σ_j^{int} and by $\boldsymbol{n}_j^{\text{int}}$ a unit normal vector on Σ_j^{int}.

In order to obtain a suitable formulation for problem (1), we introduce the variable

$$\boldsymbol{A}_C(\boldsymbol{x}, t) := -\int_0^t \boldsymbol{E}_C(\boldsymbol{x}, s)\,\mathrm{d}s + \boldsymbol{A}_{C,0}(\boldsymbol{x}) \tag{2}$$

where $\boldsymbol{A}_{C,0}$ is a vector potential of $\boldsymbol{\mu}_C\boldsymbol{H}_{C,0}$; namely, a vector field such that

$$\mathbf{curl}\,\boldsymbol{A}_{C,0} = \boldsymbol{\mu}_C\boldsymbol{H}_{C,0} \quad \text{in } \Omega_C, \tag{3}$$

which is well known to exist because $\operatorname{div}(\boldsymbol{\mu}_C\boldsymbol{H}_{C,0}) = 0$ in Ω_C (see, for instance, [9, Lemma 3.5]). In practice, $\boldsymbol{A}_{C,0}$ can be found, for instance, by solving the following problem:

$$\begin{aligned}
&\mathbf{curl}\,\boldsymbol{A}_{C,0} = \boldsymbol{\mu}_C\boldsymbol{H}_{C,0} \quad \text{in } \Omega_C,\\
&\operatorname{div}(\boldsymbol{\sigma}\boldsymbol{A}_{C,0}) = 0 \quad \text{in } \Omega_C,\\
&\boldsymbol{\sigma}\boldsymbol{A}_{C,0} \cdot \boldsymbol{n}_C = 0 \quad \text{on } \Gamma,\\
&\int_{\Omega_C} \boldsymbol{\sigma}\boldsymbol{A}_{C,0} \cdot \boldsymbol{\omega}_j\,\mathrm{d}\boldsymbol{x} = 0, \quad j = 1, \dots, L.
\end{aligned}$$

We obtain directly from (2) that $\boldsymbol{E}_C = -\partial_t\boldsymbol{A}_C$ in $\Omega_C \times (0, T)$. Moreover, if we apply **curl** to (2) and use $(1)_1$ and (3), we also deduce that $\boldsymbol{\mu}_C\boldsymbol{H}_C = \mathbf{curl}\,\boldsymbol{A}_C$ in $\Omega_C \times [0, T]$ and, replacing the new equalities in $(1)_2$, we have

$$\boldsymbol{\sigma}\partial_t\boldsymbol{A}_C + \mathbf{curl}\,(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\boldsymbol{A}_C) = \boldsymbol{J} \quad \text{in } \Omega_C \times (0, T).$$

We introduce the Beppo Levi space

$$W^1(\Omega_I) := \left\{\varphi \in L^2_{\text{loc}}(\Omega_I) : \frac{\varphi}{\sqrt{1 + |\boldsymbol{x}|^2}} \in L^2(\Omega_I),\ \nabla\varphi \in (L^2(\Omega_I))^3\right\}$$

and recall that the seminorm $\|\nabla(\cdot)\|_{0,\Omega_I}$ is a norm in $W^1(\Omega_I)$ equivalent to the natural norm; i.e., there exists a constant $C > 0$ such that (see, e.g., [10]):

$$\left\| \frac{\varphi}{\sqrt{1+|\boldsymbol{x}|^2}} \right\|^2_{0,\Omega_I} \leq C\, \|\nabla\varphi\|^2_{0,\Omega_I} \quad \forall \varphi \in W^1(\Omega_I).$$

Moreover we define the harmonic Neumann vector-fields in Ω_I by

$$\mathbb{H}(\Omega_I) := \left\{ \boldsymbol{v} \in (L^2(\Omega_I))^3 : \ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0},\ \operatorname{div}\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}_I = 0 \text{ on } \Gamma \right\}.$$

We will also need a basis of the finite dimensional space $\mathbb{H}(\Omega_I)$. To this end, let $\Sigma_j^{\text{ext}},\ j = 1, \ldots, L$, be the orientable cutting surfaces in Ω_I introduced above. We fix a unit normal $\boldsymbol{n}_j^{\text{ext}}$ on each Σ_j^{ext}. Then, for each $j = 1, \ldots, L$, consider the following problem, which admits a unique solution: Find $z_j \in W^1(\Omega_I \setminus \Sigma_j^{\text{ext}})$ such that

$$\begin{aligned}
&\Delta z_j = 0 \quad \text{in } \Omega_I \setminus \Sigma_j^{\text{ext}},\\
&\nabla z_j \cdot \boldsymbol{n}_I = 0 \quad \text{on } \Gamma \setminus \partial\Sigma_j^{\text{ext}},\\
&[\![\nabla z_j \cdot \boldsymbol{n}_j^{\text{ext}}]\!]_{\Sigma_j^{\text{ext}}} = 0,\\
&[\![z_j]\!]_{\Sigma_j^{\text{ext}}} = 1.
\end{aligned} \tag{4}$$

The set $\{\widetilde{\nabla} z_j : j = 1, \ldots, L\}$, where $\widetilde{\nabla} z_j$ are the $(L^2(\Omega_I))^3$-extension of ∇z_j, is a basis of $\mathbb{H}(\Omega_I)$ (see, for instance, [7]).

We have the following representation of curl-free vector-fields in Ω_I (see, e.g., [11, Remark 7]).

Lemma 3.1. *There holds*

$$\left\{ \boldsymbol{u} \in (L^2(\Omega_I))^3 : \ \mathbf{curl}\,\boldsymbol{u} = \mathbf{0} \textit{ in } \Omega_I \right\} = \nabla(W^1(\Omega_I)) \oplus \mathbb{H}(\Omega_I).$$

Moreover, this is an $L^2(\Omega_I)$-orthogonal decomposition.

We know from $(1)_2$ that $\mathbf{curl}\,\boldsymbol{H}_I = \mathbf{0}$ in Ω_I at all time $t \in [0, T]$. Then, the previous lemma ensures the existence, at each time $t \in [0, T]$, of a function $\psi_I(t)$ in $W^1(\Omega_I)$ and real constants $\{\alpha_j(t)\}_{j=1}^L$ such that

$$\boldsymbol{H}_I(\boldsymbol{x}, t) = \nabla\psi_I(\boldsymbol{x}, t) + \sum_{j=1}^{L} \alpha_j(t)\widetilde{\nabla} z_j(\boldsymbol{x}) \quad \text{in } \Omega_I \times [0, T]. \tag{5}$$

Moreover, taking divergence in the Eq. $(1)_1$ and using that $\boldsymbol{\mu} = \mu_0\mathbf{I}$ in Ω_I, we obtain that $\partial_t(\operatorname{div}\boldsymbol{H}_I) = 0$ in $\Omega_I \times (0, T)$. Hence, since we know that $\operatorname{div}\boldsymbol{H}_I(\boldsymbol{x}, 0) = \operatorname{div}\boldsymbol{H}_{I,0} = 0$ in Ω_I, we conclude that $\operatorname{div}\boldsymbol{H}_I = 0$ in $\Omega_I \times [0, T]$. Then, using (5) and $(4)_1$, we obtain that

$$\Delta\psi_I = 0 \quad \text{in } \Omega_I \times [0, T].$$

On the other hand, multiplying $(1)_1$ by $\widetilde{\nabla} z_i$, using a Green's formula and the fact that $\boldsymbol{E}_I \times \boldsymbol{n}_I = -\boldsymbol{E}_C \times \boldsymbol{n}_C$, we obtain

$$\int_{\Omega_I} \partial_t(\mu_0\boldsymbol{H}_I) \cdot \widetilde{\nabla} z_i \,\mathrm{d}\boldsymbol{x} = -\int_\Gamma \boldsymbol{E}_C \times \boldsymbol{n}_C \cdot \widetilde{\nabla} z_i \,\mathrm{d}\zeta, \quad i = 1, \ldots, L.$$

Replacing $\boldsymbol{H}_I$ by $\nabla\psi_I + \sum_{j=1}^L \alpha_j \widetilde{\nabla} z_j$ and $\boldsymbol{E}_C$ by $-\partial_t\boldsymbol{A}_C$, using the orthogonality between $\nabla W^1(\Omega_I)$ and $\mathbb{H}(\Omega_I)$ and integrating by parts in Ω_I, we obtain

$$\mu_0 \sum_{j=1}^{L} \alpha_j'(t) \int_{\Omega_I} \widetilde{\nabla} z_j \cdot \widetilde{\nabla} z_i \,\mathrm{d}\boldsymbol{x} = \int_\Gamma \partial_t\boldsymbol{A}_C(t) \times \boldsymbol{n}_C \cdot \widetilde{\nabla} z_i \,\mathrm{d}\zeta, \quad i = 1, \ldots, L.$$

Next, integrating in time between 0 and s $(0 < s < T)$ and recalling that $\boldsymbol{A}_C(\boldsymbol{x}, 0) = \boldsymbol{A}_{C,0}(\boldsymbol{x})$, we obtain

$$\mu_0 \sum_{j=1}^{L} \alpha_j(s) \int_{\Omega_I} \widetilde{\nabla} z_j \cdot \widetilde{\nabla} z_i \,\mathrm{d}\boldsymbol{x} - \int_\Gamma \boldsymbol{A}_C(s) \times \boldsymbol{n}_C \cdot \widetilde{\nabla} z_i \,\mathrm{d}\zeta = \mu_0 \sum_{j=1}^{L} \alpha_j(0) \int_{\Omega_I} \widetilde{\nabla} z_j \cdot \widetilde{\nabla} z_i \,\mathrm{d}\boldsymbol{x} - \int_\Gamma \boldsymbol{A}_{C,0} \times \boldsymbol{n}_C \cdot \widetilde{\nabla} z_i \,\mathrm{d}\zeta, \tag{6}$$

with $i = 1, \ldots, L$. From (4), Green's formula yields

$$\int_{\Omega_I} \widetilde{\nabla} z_j \cdot \widetilde{\nabla} z_i \,\mathrm{d}\boldsymbol{x} = \int_{\Sigma_j^{\text{ext}}} \frac{\partial z_i}{\partial \boldsymbol{n}_j} \,\mathrm{d}\zeta,$$

for all $i, j = 1, \dots, L$. Then, we introduce the matrix

$$\mathbf{N} := \left(\int_{\Sigma_j^{\text{ext}}} \frac{\partial z_i}{\partial \boldsymbol{n}_j} \, \mathrm{d}\zeta \right)_{1 \le i,j \le L}. \tag{7}$$

It is clear that $\mathbf{N}$ is symmetric and positive definite. We also define the matrix $\mathbf{Z}$ and the vector $\boldsymbol{\alpha}$ by

$$\mathbf{Z} := \begin{bmatrix} \widetilde{\nabla} z_1 & \cdots & \widetilde{\nabla} z_L \end{bmatrix}^{\mathrm{t}} \quad \text{and} \quad \boldsymbol{\alpha} := \begin{bmatrix} \alpha_1 & \cdots & \alpha_L \end{bmatrix}^{\mathrm{t}}. \tag{8}$$

Thus, we can write Eq. (6) as follows:

$$\mu_0 \mathbf{N} \boldsymbol{\alpha} - \int_\Gamma \mathbf{Z} \left(\boldsymbol{A}_C \times \boldsymbol{n}_C \right) \mathrm{d}\zeta = \mu_0 \mathbf{N} \boldsymbol{\alpha}_0 - \int_\Gamma \mathbf{Z} \left(\boldsymbol{A}_{C,0} \times \boldsymbol{n}_C \right) \mathrm{d}\zeta,$$

where $\boldsymbol{\alpha}_0 := \boldsymbol{\alpha}(0)$ is known.

In conclusion, we are led to the following problem:

Find $\boldsymbol{A}_C \in L^2(0, T; \boldsymbol{H}(\mathbf{curl}\,; \Omega_C)) \cap H^1(0, T; (L^2(\Omega_C))^3)$, $\psi_I \in L^2(0, T; W^1(\Omega_I))$ and $\boldsymbol{\alpha} \in L^2(0, T; \mathbb{R}^L)$ such that

$$\begin{aligned}
&\boldsymbol{\sigma} \partial_t \boldsymbol{A}_C + \mathbf{curl}\,(\boldsymbol{\mu}_C^{-1} \mathbf{curl}\, \boldsymbol{A}_C) = \boldsymbol{J} \quad \text{in } \Omega_C \times (0, T), \\
&\mu_0 \mathbf{N} \boldsymbol{\alpha} - \int_\Gamma \mathbf{Z} \left(\boldsymbol{A}_C \times \boldsymbol{n}_C \right) \mathrm{d}\zeta = \mu_0 \mathbf{N} \boldsymbol{\alpha}_0 - \int_\Gamma \mathbf{Z} \left(\boldsymbol{A}_{C,0} \times \boldsymbol{n}_C \right) \mathrm{d}\zeta, \\
&\Delta \psi_I = 0 \quad \text{in } \Omega_I \times [0, T], \\
&\left(\boldsymbol{\mu}_C^{-1} \mathbf{curl}\, \boldsymbol{A}_C \right) \times \boldsymbol{n}_C + \left(\nabla \psi_I + \mathbf{Z}^{\mathrm{t}} \boldsymbol{\alpha} \right) \times \boldsymbol{n}_I = \mathbf{0} \quad \text{on } \Gamma \times [0, T], \\
&\mathbf{curl}\, \boldsymbol{A}_C \cdot \boldsymbol{n}_C + \mu_0 \nabla \psi_I \cdot \boldsymbol{n}_I = 0 \quad \text{on } \Gamma \times [0, T], \\
&\boldsymbol{A}_C(\boldsymbol{x}, 0) = \boldsymbol{A}_{C,0} \quad \text{in } \Omega_C.
\end{aligned} \tag{9}$$

Eqs. $(9)_4$ and $(9)_5$ come from the fact that $\boldsymbol{H} \in \boldsymbol{H}(\mathbf{curl}\,; \mathbb{R}^3)$ and $\boldsymbol{\mu H} \in \boldsymbol{H}(\mathrm{div}\,; \mathbb{R}^3)$ and, hence, $\boldsymbol{H}_C \times \boldsymbol{n}_C = -\boldsymbol{H}_I \times \boldsymbol{n}_I$ and $\boldsymbol{\mu}_C \boldsymbol{H}_C \cdot \boldsymbol{n}_C = -\mu_0 \boldsymbol{H}_I \cdot \boldsymbol{n}_I$ on Γ, respectively.

4. A FEM–BEM coupling variational formulation

In what follows we reduce problem (9) to the bounded domain Ω_C. To do this we will use Costabel's symmetric FEM–BEM coupling technique (cf. [12,13]). We introduce on Γ the single and double layer potentials, which are formally defined by

$$\begin{aligned}
&\mathcal{S} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad \mathcal{S}(\xi)(\boldsymbol{x}) := \int_\Gamma \frac{1}{4\pi \, |\boldsymbol{x} - \boldsymbol{y}|} \xi(\boldsymbol{y}) \, \mathrm{d}\zeta_{\boldsymbol{y}}, \\
&\mathcal{D} : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad \mathcal{D}(\eta)(\boldsymbol{x}) := \int_\Gamma \frac{\boldsymbol{x} - \boldsymbol{y}}{4\pi \, |\boldsymbol{x} - \boldsymbol{y}|^3} \cdot \eta(\boldsymbol{y}) \boldsymbol{n}_C(\boldsymbol{y}) \, \mathrm{d}\zeta_{\boldsymbol{y}},
\end{aligned}$$

respectively, and the hypersingular operator $\mathcal{H} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, which is formally defined as the following normal derivative:

$$\mathcal{H}(\eta)(\boldsymbol{x}) := -\nabla_{\boldsymbol{x}} \left(\int_\Gamma \frac{\boldsymbol{x} - \boldsymbol{y}}{4\pi \, |\boldsymbol{x} - \boldsymbol{y}|^3} \cdot \eta(\boldsymbol{y}) \, \boldsymbol{n}_C(\boldsymbol{y}) \, \mathrm{d}\zeta_{\boldsymbol{y}} \right) \cdot \boldsymbol{n}_C(\boldsymbol{x}).$$

Let us remark that the restrictions to the boundary as well as the normal derivative above have to be understood in a weak sense; for rigorous definitions see, for instance, [14]. The three operators are linear and bounded. Let $\mathcal{D}' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ denote the adjoint operator of $\mathcal{D}$.

In what follows, we recall some basic properties of these operators (see, e.g., [10,14] for the corresponding proofs).

Theorem 4.1. *Let $\varphi \in W^1(\Omega_I)$ be a harmonic function. Then, the following identities hold on Γ, where I denotes the identity operator:*

$$\begin{aligned}
&\left(\frac{1}{2} I - \mathcal{D} \right) (\varphi|_\Gamma) - \mathcal{S} \left(\frac{\partial \varphi}{\partial \boldsymbol{n}_I} \right) = 0, \\
&- \left(\frac{1}{2} I + \mathcal{D}' \right) \left(\frac{\partial \varphi}{\partial \boldsymbol{n}_I} \right) + \mathcal{H} \left(\varphi|_\Gamma \right) = 0.
\end{aligned}$$

Lemma 4.1. (i) *There exists $k_1 > 0$ such that*

$$\int_\Gamma \mathcal{S}(\eta) \eta \, \mathrm{d}\zeta \ge k_1 \, \|\eta\|^2_{-1/2,\Gamma} \quad \forall \eta \in H^{-1/2}(\Gamma).$$

(ii) *There exists* $k_2 > 0$ *such that*

$$\int_\Gamma \mathcal{H}(\varphi)\varphi \, d\zeta \geq k_2 \|\varphi\|^2_{1/2,\Gamma} \quad \forall \varphi \in H_0^{1/2}(\Gamma),$$

where

$$H_0^{1/2}(\Gamma) := \left\{\varphi \in H^{1/2}(\Gamma) : \int_\Gamma \varphi \, d\zeta = 0\right\}.$$

Lemma 4.2. $\mathcal{H}(1) = 0$, $\mathcal{D}(1) = -1/2$ *and* $\int_\Gamma \mathcal{H}(\eta)\, d\zeta = 0 \; \forall \eta \in H^{1/2}(\Gamma)$.

Here and thereafter, for the ease of notation, we use the integration symbol on Γ instead of the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$; namely, $\int_\Gamma \mathcal{H}(\eta)\, d\zeta = \langle \mathcal{H}(\eta), 1\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}$.

Theorem 4.2. *The linear operator* $\mathcal{H} : H^{1/2}(\Gamma)/\mathbb{R} \to H_0^{-1/2}(\Gamma)$, *where*

$$H_0^{-1/2}(\Gamma) := \left\{\eta \in H^{-1/2}(\Gamma) : \int_\Gamma \eta \, d\zeta = 0\right\},$$

defines an isomorphism.

Let $(\boldsymbol{A}_C, \psi_I, \boldsymbol{\alpha})$ satisfying (9). Let $\psi(t) := \psi_I|_\Gamma(t) - c(t)$, where $c : [0, T] \to \mathbb{R}$ is such that $\psi(t) \in H_0^{1/2}(\Gamma)$. By using $(9)_3$ and $(9)_5$, according to Theorem 4.1 and Lemma 4.2, for all $t \in [0, T]$ we have

$$-\frac{1}{2}\psi - \mathcal{D}(\psi) + \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\, \boldsymbol{A}_C \cdot \boldsymbol{n}_C) = -\psi_I \quad \text{on } \Gamma, \tag{10}$$

$$\frac{1}{2\mu_0}\mathbf{curl}\, \boldsymbol{A}_C \cdot \boldsymbol{n}_C + \frac{1}{\mu_0}\mathcal{D}'(\mathbf{curl}\, \boldsymbol{A}_C \cdot \boldsymbol{n}_C) + \mathcal{H}(\psi) = 0 \quad \text{on } \Gamma. \tag{11}$$

The following is a variational formulation of problem (9), where

$\mathcal{V} := \boldsymbol{H}(\mathbf{curl}\,; \Omega_C)$.

Find $\boldsymbol{A}_C \in L^2(0, T; \mathcal{V}) \cap H^1(0, T; (L^2(\Omega_C))^3)$, $\psi \in L^2(0, T; H_0^{1/2}(\Gamma))$ and $\boldsymbol{\alpha} \in L^2(0, T; \mathbb{R}^L)$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_C} \sigma \boldsymbol{A}_C \cdot \boldsymbol{w}_C \, d\boldsymbol{x} + \int_{\Omega_C} \mu_C^{-1}\mathbf{curl}\, \boldsymbol{A}_C \cdot \mathbf{curl}\, \boldsymbol{w}_C \, d\boldsymbol{x} \\
&\quad + \int_\Gamma \left[-\frac{1}{2}\psi - \mathcal{D}(\psi) + \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\, \boldsymbol{A}_C \cdot \boldsymbol{n}_C)\right]\mathbf{curl}\, \boldsymbol{w}_C \cdot \boldsymbol{n}_C \, d\zeta + \boldsymbol{\alpha}^t \int_\Gamma \mathbf{Z}(\boldsymbol{w}_C \times \boldsymbol{n}_C)\, d\zeta = \int_{\Omega_C} \boldsymbol{J} \cdot \boldsymbol{w}_C \, d\boldsymbol{x}, \\
&\int_\Gamma \left[\frac{1}{2}\mathbf{curl}\, \boldsymbol{A}_C \cdot \boldsymbol{n}_C + \mathcal{D}'(\mathbf{curl}\, \boldsymbol{A}_C \cdot \boldsymbol{n}_C) + \mu_0\mathcal{H}(\psi)\right]\eta \, d\zeta = 0, \\
&\mu_0\boldsymbol{\beta}^t\mathbf{N}\boldsymbol{\alpha} - \boldsymbol{\beta}^t\int_\Gamma \mathbf{Z}(\boldsymbol{A}_C \times \boldsymbol{n}_C)\, d\zeta = \mu_0\boldsymbol{\beta}^t\mathbf{N}\boldsymbol{\alpha}_0 - \boldsymbol{\beta}^t\int_\Gamma \mathbf{Z}\left(\boldsymbol{A}_{C,0} \times \boldsymbol{n}_C\right) d\zeta,
\end{aligned} \tag{12}$$

for all $\boldsymbol{w}_C \in \mathcal{V}$, $\eta \in H_0^{1/2}(\Gamma)$ and $\boldsymbol{\beta} \in \mathbb{R}^L$, with

$\boldsymbol{A}_C(0) = \boldsymbol{A}_{C,0} \quad \text{in } \Omega_C$.

In fact, to derive $(12)_1$, we have multiplied $(9)_1$ by $\boldsymbol{w}_C$, integrated by parts in Ω_C and used $(9)_4$, the identity

$$\int_\Gamma \boldsymbol{n}_I \times \nabla\psi_I \cdot \boldsymbol{w}_C \, d\zeta = \int_\Gamma \psi_I \mathbf{curl}\, \boldsymbol{w}_C \cdot \boldsymbol{n}_C \, d\zeta, \tag{13}$$

(which in its turn follows by integration by parts, too) and (10). On the other hand, Eqs. $(12)_2$ and $(12)_3$ follow directly from (11) and $(9)_2$.

For the theoretical analysis it is convenient to eliminate $\boldsymbol{\alpha}$ and ψ from the previous formulation. With this aim, we introduce the linear operator $\mathbb{T} : \mathcal{V} \to \mathbb{R}^L$ defined by

$$\mathbb{T}(\boldsymbol{w}_C) := \int_\Gamma \mathbf{Z}(\boldsymbol{w}_C \times \boldsymbol{n}_C)\, d\zeta.$$

We eliminate $\boldsymbol{\alpha}$ from $(12)_3$ and replace it in $(12)_1$. Then, the fourth term of this equation reads

$$\boldsymbol{\alpha}^t \int_\Gamma \mathbf{Z}(\boldsymbol{w}_C \times \boldsymbol{n}_C)\, d\zeta = \left(\mathbb{T}(\boldsymbol{w}_C)\right)^t\boldsymbol{\alpha} = \mu_0^{-1}\left(\mathbb{T}(\boldsymbol{w}_C)\right)^t\mathbf{N}^{-1}\mathbb{T}(\boldsymbol{A}_C) + \left(\mathbb{T}(\boldsymbol{w}_C)\right)^t\boldsymbol{\alpha}_0 - \mu_0^{-1}\left(\mathbb{T}(\boldsymbol{w}_C)\right)^t\mathbf{N}^{-1}\mathbb{T}(\boldsymbol{A}_{C,0}).$$

Moreover, we introduce the operator $\mathcal{R} : H_0^{-1/2}(\Gamma) \to H_0^{1/2}(\Gamma)$ given by

$$\int_\Gamma \mathcal{H}(\mathcal{R}(\xi))\eta \,\mathrm{d}\zeta = \int_\Gamma \xi\eta \,\mathrm{d}\zeta \quad \forall \eta \in H_0^{1/2}(\Gamma),\ \forall \xi \in H_0^{-1/2}(\Gamma). \tag{14}$$

It is straightforward to show, from Lemma 4.1(ii) and the Lax–Milgram lemma, that $\mathcal{R}$ is well defined and bounded. Therefore, the second equation of (12) may be equivalently written

$$\psi = -\mu_0^{-1}\mathcal{R}\left(\frac{1}{2}\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C + \mathcal{D}'(\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C)\right).$$

Consequently, (12) admits the following equivalent reduced form:

Find $\boldsymbol{A}_C \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;(L^2(\Omega_C))^3)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A}_C(t),\boldsymbol{w}_C)_{\boldsymbol{\sigma}} + \mathcal{A}(\boldsymbol{A}_C(t),\boldsymbol{w}_C) + \mathcal{B}(\boldsymbol{A}_C(t),\boldsymbol{w}_C) = (\boldsymbol{J}(t),\boldsymbol{w}_C)_{0,\Omega_C} + \boldsymbol{g}(\boldsymbol{w}_C) \tag{15}$$

for all $\boldsymbol{w}_C \in \mathcal{V}$, with

$$\boldsymbol{A}_C(0) = \boldsymbol{A}_{C,0} \quad \text{in } \Omega_C,$$

where

$$(\boldsymbol{H},\boldsymbol{G})_{\boldsymbol{\sigma}} := \int_{\Omega_C} \sigma \boldsymbol{H}\cdot\boldsymbol{G}\,\mathrm{d}\boldsymbol{x} \quad \forall \boldsymbol{H},\boldsymbol{G} \in (L^2(\Omega_C))^3,$$

$$\mathcal{A} : \mathcal{V}\times\mathcal{V} \to \mathbb{R}, \qquad \mathcal{A}(\boldsymbol{H},\boldsymbol{G}) := \int_{\Omega_C} \mu_C^{-1}\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{G}\,\mathrm{d}\boldsymbol{x} + \mu_0^{-1}\int_\Gamma \mathcal{S}(\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{n}_C)\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}_C\,\mathrm{d}\zeta,$$

$$\mathcal{B} : \mathcal{V}\times\mathcal{V} \to \mathbb{R}, \qquad \mathcal{B}(\boldsymbol{H},\boldsymbol{G}) := \mu_0^{-1}\int_\Gamma \mathcal{K}(\boldsymbol{G})\,\mathcal{R}(\mathcal{K}(\boldsymbol{H}))\,\mathrm{d}\zeta + \mu_0^{-1}\big(\mathbb{T}(\boldsymbol{G})\big)^{\mathrm{t}}\mathbf{N}^{-1}\mathbb{T}(\boldsymbol{H}),$$

$$\mathcal{K} : \mathcal{V} \to H_0^{-1/2}(\Gamma), \qquad \mathcal{K}(\boldsymbol{H}) := \frac{1}{2}\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{n}_C + \mathcal{D}'(\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{n}_C),$$

$$\boldsymbol{g} : \mathcal{V} \to \mathbb{R}, \qquad \boldsymbol{g}(\boldsymbol{H}) := \mu_0^{-1}\big(\mathbb{T}(\boldsymbol{H})\big)^{\mathrm{t}}\mathbf{N}^{-1}\mathbb{T}(\boldsymbol{A}_{C,0}) - \big(\mathbb{T}(\boldsymbol{H})\big)^{\mathrm{t}}\boldsymbol{\alpha}_0.$$

Notice that $\mathcal{A}$ and $\mathcal{B}$ are bounded, symmetric and non-negative definite bilinear forms.

Remark 4.1. The norm $\|\cdot\|_{0,\Omega_C}$ is equivalent to $\|\cdot\|_{\boldsymbol{\sigma}}$ and, therefore, $\|\cdot\|_{\mathcal{V}}$ is equivalent to $\|\cdot\|_{\boldsymbol{\sigma}} + \|\mathbf{curl}\,(\cdot)\|_{0,\Omega_C}$.

4.1. Existence and uniqueness

As shown in the following lemma, problem (15) is well posed.

Lemma 4.3. *There exists a unique solution to* (15) *and*

$$\|\boldsymbol{A}_C\|^2_{L^\infty(0,T;\mathcal{V})} + \|\partial_t\boldsymbol{A}_C\|^2_{L^2(0,T;(L^2(\Omega_C))^3)} \le C\left\{\|\boldsymbol{J}\|^2_{L^2(0,T;(L^2(\Omega_C))^3)} + \left\|\boldsymbol{A}_{C,0}\right\|^2_{\mathcal{V}} + |\boldsymbol{\alpha}_0|^2\right\} \tag{16}$$

for some constant $C > 0$, independent of the problem data $\boldsymbol{J}$, $\boldsymbol{A}_{C,0}$ and $\boldsymbol{\alpha}_0$.

Proof. The classical theory for parabolic problems (see, for instance, [11]) allows us to show that problem (15) has a unique solution $\boldsymbol{A}_C \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}')$. Moreover, since $\boldsymbol{A}_{C,0} \in \mathcal{V}$ and the right hand side of (15) is the sum of two terms, $(\boldsymbol{J}(t),\boldsymbol{w}_C)_{0,\Omega_C}$ with $\boldsymbol{J} \in L^2(0,T;(L^2(\Omega_C))^3)$ and $\boldsymbol{g}(\boldsymbol{w}_C)$ with $\boldsymbol{g} \in \mathcal{V}'$ independent of t, it is straightforward to show that actually $\partial_t\boldsymbol{A}_C \in L^2(0,T;(L^2(\Omega_C))^3)$ and the estimate (16) holds true (in fact, we may proceed as in the proof of Theorem 7.1.5 from [15] for the first term, and use Theorem A.1 from [16] for the second one). $\square$

Remark 4.2. Problems (12) and (15) are actually equivalent. In fact, for $\boldsymbol{A}_C$ being a solution of (15), if we define $\psi := -\mu_0^{-1}\mathcal{R}(\mathcal{K}(\boldsymbol{A}_C))$ and $\boldsymbol{\alpha} := \boldsymbol{\alpha}_0 + \mu_0^{-1}\mathbf{N}^{-1}(\mathbb{T}(\boldsymbol{A}_C) - \mathbb{T}(\boldsymbol{A}_{C,0}))$, then $(\boldsymbol{A}_C,\psi,\boldsymbol{\alpha})$ is a solution of (12). Moreover this problem has a unique solution, because $\boldsymbol{A}_C$ has to be the unique solution of (15) and ψ and $\boldsymbol{\alpha}$ are determined via $(12)_2$ and $(12)_3$, respectively.

Problems (9) and (12) are also equivalent. In fact, we derived (12) from (9). In what follows, we show the converse implication:

Theorem 4.3. *Let $(\boldsymbol{A}_C,\psi,\boldsymbol{\alpha})$ be the solution to problem* (12)*. Then, there exists $\psi_I \in L^2(0,T;W^1(\Omega_I))$ and a function $c : [0,T] \to \mathbb{R}$ such that $\psi = \psi_I|_\Gamma - c$ and $(\boldsymbol{A}_C,\psi_I,\boldsymbol{\alpha})$ satisfies* (9)*.*

Proof. Testing $(12)_1$ with $\boldsymbol{w}_C \in (\mathcal{C}_0^\infty(\Omega_C))^3$ we obtain

$$\sigma \partial_t \boldsymbol{A}_C + \mathbf{curl}\,\left(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\boldsymbol{A}_C\right) = \boldsymbol{J} \quad \text{in } \Omega_C \tag{17}$$

a.e. in $[0, T]$. Then, testing $(12)_2$ with $\eta \in H^{1/2}(\Gamma)$ and using Lemma 4.2 we have

$$\frac{1}{2}\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C + \mathcal{D}'(\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C) + \mu_0 \mathcal{H}(\psi) = 0 \quad \text{on } \Gamma. \tag{18}$$

Now, let $\psi_I \in W^1(\Omega_I)$ be the solution of the following problem:

$$\begin{aligned} &\Delta \psi_I = 0 \quad \text{in } \Omega_I, \\ &\mu_0 \nabla \psi_I \cdot \boldsymbol{n}_I = -\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C \quad \text{on } \Gamma. \end{aligned} \tag{19}$$

Since $\psi_I \in W^1(\Omega_I)$ is a harmonic function, Theorem 4.1 ensures that

$$\begin{aligned} &\frac{1}{2}\psi_I|_\Gamma - \mathcal{D}(\psi_I|_\Gamma) + \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C) = 0, \\ &\frac{1}{2}\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C + \mathcal{D}'(\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C) + \mu_0 \mathcal{H}(\psi_I|_\Gamma) = 0. \end{aligned} \tag{20}$$

Subtracting (18) from $(20)_2$, we obtain $\mathcal{H}(\psi - \psi_I) = 0$ on Γ. Therefore, we conclude from Theorem 4.2 that $\psi_I(t) = \psi(t) + c(t)$ on Γ, where, for each $t \in [0, T]$, $c(t)$ is a constant. As a consequence, from $(20)_1$ we have

$$-\frac{1}{2}\psi|_\Gamma - \mathcal{D}(\psi|_\Gamma) + \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C) = -\frac{1}{2}(\psi_I|_\Gamma - c) - \mathcal{D}(\psi_I|_\Gamma - c) + \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\,\boldsymbol{A}_C \cdot \boldsymbol{n}_C) = -\psi_I|_\Gamma.$$

Now, replacing this equality in $(12)_1$, using (13) and testing with $\boldsymbol{w}_C \in \boldsymbol{H}(\mathbf{curl}\,; \Omega_C)$, we obtain

$$\left(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\boldsymbol{A}_C\right) \times \boldsymbol{n}_C + \left(\nabla \psi_I + \mathbf{Z}^{\mathrm{t}}\boldsymbol{\alpha}\right) \times \boldsymbol{n}_I = \mathbf{0} \quad \text{on } \Gamma.$$

Let us emphasize that the first term on the left hand side is well defined in $H^{-1/2}(\Gamma)$, since $\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\boldsymbol{A}_C \in \boldsymbol{H}(\mathbf{curl}\,; \Omega_C)$, which in turn follows because of (17) and the facts that $\boldsymbol{J} \in L^2(0, T; (L^2(\Omega_C))^3)$ and the solution to problem (12) satisfies $\partial_t \boldsymbol{A}_C \in L^2(0, T; (L^2(\Omega_C))^3)$. Finally $(9)_2$ and $(9)_3$ follow from $(12)_3$ and the initial condition of problem (12), respectively. □

5. Fully-discrete scheme

Let $\{\mathcal{T}_h(\Omega_C)\}_h$ be a regular family of tetrahedral meshes of Ω_C. As usual, h stands for the largest diameter of the tetrahedra K in $\mathcal{T}_h(\Omega_C)$. Furthermore, we consider the corresponding family of triangulations induced on Γ, $\{\mathcal{T}_h(\Gamma)\}_h$. Let $N \in \mathbb{N}$, $\Delta t := T/N$ and $t_n = n\Delta t$, $n = 0, \ldots, N$.

We define a fully-discrete version of (12) by means of Nédélec finite elements. The local representation on K of the lowest-order Nédélec finite element is given by

$$\mathcal{N}(K) := \left\{\boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b} :\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3,\ \boldsymbol{x} \in K\right\}.$$

The corresponding global space $\boldsymbol{\mathcal{V}}_h$ is the space of vector fields that are locally in $\mathcal{N}(K)$ for all K in Ω_C and globally in $\boldsymbol{\mathcal{V}} = \boldsymbol{H}(\mathbf{curl}\,; \Omega_C)$. Moreover, we define

$$\mathcal{L}_h(\Gamma) := \left\{\eta \in H_0^{1/2}(\Gamma) :\ \eta|_F \in \mathbb{P}_1(F)\ \forall F \in \mathcal{T}_h(\Gamma)\right\},$$

which approximates the space $H_0^{1/2}(\Gamma)$, where $\mathbb{P}_k(F)$ is the set of polynomial functions defined in F of degree not greater than k.

When Ω_C is not simply connected, problem (12) involves the matrices $\mathbf{N}$ and $\mathbf{Z}$ defined by (7) and (8), respectively. To compute these matrices we also need to approximate numerically the basis $\{\widetilde{\nabla} z_k\}_{k=1}^L$ of the harmonic Neumann vector-fields space $\mathbb{H}(\Omega_I)$. A similar need arose in [7], where the authors proposed a coupled FEM–BEM method to compute the entries of a matrix $\mathbf{N}_h$ approximating $\mathbf{N}$. For the sake of completeness, in what follows, we briefly describe the method introduced in [7] to approximate $\mathbf{N}$ and the corresponding error estimate proved in this reference.

Consider a convex polyhedron Ω such that $\overline{\Omega}_C \cup \left(\bigcup_{k=1}^L \overline{\Sigma}_k^{\mathrm{ext}}\right) \subset \Omega$. Set

$$\mathcal{Q}^0 := \Omega \setminus \left\{\overline{\Omega}_C \cup \left(\bigcup_{k=1}^L \overline{\Sigma}_k^{\mathrm{ext}}\right)\right\}, \qquad \mathcal{Q} := \Omega \setminus \overline{\Omega}_C \quad \text{and} \quad \Lambda := \partial\Omega.$$

From (4), $\boldsymbol{p}_k := \widetilde{\nabla} z_k|_{\mathcal{Q}}$, $k = 1, \ldots, L$, belong to the closed subspace of $\boldsymbol{H}(\mathrm{div}; \mathcal{Q})$

$$\mathcal{Y} := \left\{\boldsymbol{q} \in (L^2(\mathcal{Q}))^3 :\ \mathrm{div}\,\boldsymbol{q} = 0 \text{ in } \mathcal{Q} \text{ and } \boldsymbol{q} \cdot \boldsymbol{n}_I = 0 \text{ on } \Gamma\right\}$$

and satisfies the variational equation

$$\int_{\mathcal{Q}} \boldsymbol{p}_k \cdot \boldsymbol{q}\,\mathrm{d}\boldsymbol{x} - \int_{\Sigma_k^{\mathrm{ext}}} \boldsymbol{q}\cdot\boldsymbol{n}_k\,\mathrm{d}\zeta + \int_{\Lambda} \boldsymbol{q}\cdot\boldsymbol{n} z_k\,\mathrm{d}\zeta \quad \forall \boldsymbol{q}\in\mathcal{Y},$$

where $\boldsymbol{n}$ correspond to the unit normal vector on Λ outer to $\mathcal{Q}$. Furthermore, as z_k is harmonic in $\mathbb{R}^3\setminus\overline{\Omega}$, the last equation may be coupled with boundary integral equations relating z_k and its normal derivative $\boldsymbol{p}_k\cdot\boldsymbol{n}$ on Λ. This leads to the following weak formulation (see [17] for more details).

Find $\boldsymbol{p}_k\in\mathcal{Y}$ and $\phi_k\in H^{1/2}(\Lambda)/\mathbb{R}$ such that

$$\begin{aligned}
&\int_{\mathcal{Q}} \boldsymbol{p}_k\cdot\boldsymbol{q}\,\mathrm{d}\boldsymbol{x} + \int_{\Lambda} \mathcal{S}(\boldsymbol{p}_k\cdot\boldsymbol{n})\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}\zeta - \int_{\Lambda}\left[\frac{1}{2}\phi_k + \mathcal{D}(\phi_k)\right]\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}\zeta = \int_{\Sigma_k^{\mathrm{ext}}} \boldsymbol{q}\cdot\boldsymbol{n}_k\,\mathrm{d}\zeta,\\
&\int_{\Lambda}\left[\frac{1}{2}\chi + \mathcal{D}(\chi)\right]\boldsymbol{p}_k\cdot\boldsymbol{n}\,\mathrm{d}\zeta + \int_{\Lambda}\mathcal{H}(\phi_k)\chi\,\mathrm{d}\zeta = 0,
\end{aligned} \tag{21}$$

for all functions $\boldsymbol{q}\in\mathcal{Y}$ and $\chi\in H^{1/2}(\Lambda)/\mathbb{R}$. The variable ϕ_k represents (up to and additive constant) the trace of z_k on Λ. Now, consider a regular family of triangulations $\{\mathcal{T}_h(\mathcal{Q})\}_h$ of $\mathcal{Q}$ by tetrahedra K of diameter no greater than $h>0$. Assume that, for each h, the set $\mathcal{T}_h(\Omega_C)\cup\mathcal{T}_h(\mathcal{Q})$ is a triangulation of Ω. This implies that the triangulation induced by $\mathcal{T}_h(\mathcal{Q})$ on Γ is identical to $\mathcal{T}_h(\Gamma)$. It can be assumed, without loss of generality, that, for each mesh, the cutting surfaces Σ_k^{ext} are union of faces of tetrahedra in $\mathcal{T}_h(\mathcal{Q})$. Finally, denote by $\mathcal{T}_h(\Lambda)$ the triangulation induced by $\mathcal{T}_h(\mathcal{Q})$ on Λ.

Consider a conforming discretization of $\boldsymbol{H}(\mathrm{div};\mathcal{Q})$:

$$\mathcal{RT}_h(\mathcal{Q}) := \{\boldsymbol{q}\in\boldsymbol{H}(\mathrm{div};\mathcal{Q}) : \boldsymbol{q}|_K\in\mathcal{RT}(K)\ \forall K\in\mathcal{T}_h(\mathcal{Q})\},$$

$\mathcal{RT}(K) := \left\{a\boldsymbol{x}+\boldsymbol{b} : a\in\mathbb{R}, \boldsymbol{b}\in\mathbb{R}^3, \boldsymbol{x}\in K\right\}$ being the lowest-order Raviart–Thomas element. The following is a convenient way of discretizing problem (21) (for more details, see [7]):

Find $\boldsymbol{p}_{kh}\in\mathcal{RT}_h^0(\mathcal{Q})$, $\phi_{kh}\in\Phi_h/\mathbb{R}$ and $\beta_{kh}\in M_h$ such that

$$\begin{aligned}
&\int_{\mathcal{Q}} \boldsymbol{p}_{kh}\cdot\boldsymbol{q}\,\mathrm{d}\boldsymbol{x} + \int_{\Lambda}\mathcal{S}(\boldsymbol{p}_{kh}\cdot\boldsymbol{n})\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}\zeta - \int_{\Lambda}\left[\frac{1}{2}\phi_{kh} + \mathcal{D}(\phi_{kh})\right]\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}\zeta + \int_{\mathcal{Q}}\beta_{kh}\mathrm{div}\,\boldsymbol{q}\,\mathrm{d}\boldsymbol{x} = \int_{\Sigma_k^{\mathrm{ext}}}\boldsymbol{q}\cdot\boldsymbol{n}_k\,\mathrm{d}\zeta,\\
&\int_{\Lambda}\left[\frac{1}{2}\chi + \mathcal{D}(\chi)\right]\boldsymbol{p}_{kh}\cdot\boldsymbol{n}\,\mathrm{d}\zeta + \int_{\Lambda}\mathcal{S}(\mathbf{curl}_{\tau}\phi_{kh})\mathbf{curl}_{\tau}\chi\,\mathrm{d}\zeta = 0,\\
&\int_{\mathcal{Q}}\mathrm{div}\,\boldsymbol{p}_{kh} v\,\mathrm{d}\boldsymbol{x} = 0,
\end{aligned} \tag{22}$$

for all functions $\boldsymbol{q}\in\mathcal{RT}_h^0(\mathcal{Q})$, $\chi\in\Phi_h/\mathbb{R}$ and $v\in M_h$, where

$$\begin{aligned}
\mathcal{RT}_h^0(\mathcal{Q}) &:= \{\boldsymbol{q}\in\mathcal{RT}_h(\mathcal{Q}) : \boldsymbol{q}|_{\Gamma}\cdot\boldsymbol{n}_I = 0\},\\
\Phi_h &:= \{\eta\in\mathcal{C}^0(\Lambda) : \eta|_F\in\mathbb{P}_1(F)\ \ \forall F\in\mathcal{T}_h(\Lambda)\},\\
M_h &:= \{v\in L^2(\mathcal{Q}) : v|_K\in\mathbb{P}_0(K)\ \ \forall K\in\mathcal{T}_h(\mathcal{Q})\}.
\end{aligned}$$

Moreover, $\mathbf{curl}_{\tau}$ denotes the surface curl on Λ (see, for instance, [1, Section A.1]).

We know from [17] that (22) is a well posed problem. Once the functions $\boldsymbol{p}_{kh}$, $1\le k\le L$, are computed, the matrix $\mathbf{N}$ can be approximated by

$$\mathbf{N}_h := \left(\int_{\Sigma_j^{\mathrm{ext}}}\boldsymbol{p}_{kh}\cdot\boldsymbol{n}_j\,\mathrm{d}\zeta\right)_{1\le k,j\le L}. \tag{23}$$

Note that this matrix is symmetric and positive definite. Error estimates for the approximation $\mathbf{N}_h$ of $\mathbf{N}$ has been obtained in [7]. With this end, an additional regularity result has been also proved therein. In the sequel, we denote by $s_{\mathcal{Q}}\in(1/2,1)$ the exponent of maximal regularity in $\mathcal{Q}$ of the solution of the Laplace operator with $L^2(\mathcal{Q})$ right-hand side and homogeneous Neumann boundary data.

Theorem 5.1. *If $(\boldsymbol{p}_k,\phi_k)$ is the solution to problem (21), $k=1,\dots,L$, then $\boldsymbol{p}_k\in(H^s(\mathcal{Q}))^3$ for all $s\in(1/2,s_{\mathcal{Q}})$.*

Proof. See [7, Theorem 7.1]. □

Finally we recall the error estimates obtained in [7]. Here and thereafter C denotes a generic positive constant not necessarily the same at each occurrence, but always independent of the mesh size h and the time step Δt.

Theorem 5.2. *Problems (21) and (22) are well posed and*

$$\|\boldsymbol{p}_k-\boldsymbol{p}_{kh}\|_{0,\mathcal{Q}} + \|\phi_k-\phi_{kh}\|_{H^{1/2}(\Lambda)/\mathbb{R}} \le Ch^s\left\{\|\boldsymbol{p}_k\|_{s,\mathcal{Q}} + \|\phi_k\|_{s+1/2,\Lambda}\right\}$$

holds, with s as in Theorem 5.1.

Proof. See [7, Theorem 7.2]. □

Theorem 5.3. *There exists $h_0 > 0$ such that $\mathbf{N}_h$ is invertible for all $h \in (0, h_0)$. Moreover, the error estimate*

$$\|\mathbf{N} - \mathbf{N}_h\| + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\| \le Ch^s \max_{1\le k\le L} \left\{\|\boldsymbol{p}_k\|_{s,\mathcal{Q}} + \|\phi_k\|_{s+1/2,\Lambda}\right\}$$

holds, with s as in Theorem 5.1.

Proof. See [7, Corollary 7.3]. □

Notice that $\|\phi_k\|_{s+1/2,\Lambda}$ is clearly bounded, since ϕ_k is the trace on Λ of the solution z_k to problem (4).

To compute an approximation of the entries of $\mathbf{Z}$, we need to resort to a different strategy. In fact, the previous methods yields good approximation of $\boldsymbol{p}_k|_\Gamma \cdot \boldsymbol{n}_I = \widetilde{\nabla} z_k|_\Gamma \cdot \boldsymbol{n}_I$, but not of $\widetilde{\nabla} z_k|_\Gamma \times \boldsymbol{n}_I$ (which are the terms defining the entries of $\mathbf{Z}$). A similar situation happened in [7], too. However, in this case, we follow an alternative approach that we think is simpler.

It is easy to show that the solution of (4) satisfies the following variational formulation:

Find $z_k \in H^1(\mathcal{Q} \setminus \Sigma_k^{\text{ext}})/\mathbb{R}$ such that $[\![z_k]\!]_{\Sigma_k^{\text{ext}}} = 1$ and

$$\int_{\mathcal{Q}\setminus\Sigma_k^{\text{ext}}} \nabla z_k \cdot \nabla\varphi \, d\boldsymbol{x} = \int_\Lambda \boldsymbol{p}_k \cdot \boldsymbol{n}\, \varphi \, d\zeta \quad \forall \varphi \in H^1(\mathcal{Q})/\mathbb{R}. \tag{24}$$

We introduce

$$\mathcal{L}_h(\mathcal{Q}) := \left\{\theta \in H^1(\mathcal{Q}) : \theta|_K \in \mathbb{P}_1(K) \; \forall K \in \mathcal{T}_h(\mathcal{Q})\right\},$$
$$\mathcal{L}_h(\mathcal{Q} \setminus \Sigma_k^{\text{ext}}) := \left\{\theta \in H^1(\mathcal{Q} \setminus \Sigma_k^{\text{ext}}) : \theta|_K \in \mathbb{P}_1(K) \; \forall K \in \mathcal{T}_h(\mathcal{Q})\right\}$$

and consider the following discrete version of problem (24):

Find $z_{kh} \in \mathcal{L}_h(\mathcal{Q} \setminus \Sigma_k^{\text{ext}})/\mathbb{R}$ such that $[\![z_{kh}]\!]_{\Sigma_k^{\text{ext}}} = 1$ and

$$\int_{\mathcal{Q}\setminus\Sigma_k^{\text{ext}}} \nabla z_{kh} \cdot \nabla\varphi \, d\boldsymbol{x} = \int_\Lambda \boldsymbol{p}_{kh} \cdot \boldsymbol{n}\, \varphi \, d\zeta \quad \forall \varphi \in \mathcal{L}_h(\mathcal{Q})/\mathbb{R}. \tag{25}$$

Lemma 5.1. *Let z_k and z_{kh} be the solutions to problems (24) and (25), respectively. Then*

$$\|\widetilde{\nabla} z_k - \widetilde{\nabla} z_{kh}\|_{0,\mathcal{Q}} \le Ch^s,$$

with s as in Theorem 5.1.

Proof. Let $\widehat{z}_k \in \mathcal{C}^\infty(\mathcal{Q} \setminus \Sigma_k^{\text{ext}})$ be such that $[\![\widehat{z}_k]\!]_{\Sigma_k^{\text{ext}}} = 1$. Let $\widehat{z}_k^I$ be the Lagrange intepolant of $\widehat{z}_k$ in $\mathcal{Q} \setminus \Sigma_k^{\text{ext}}$. Notice that $[\![\widehat{z}_k^I]\!]_{\Sigma_k^{\text{ext}}} = 1$, too. We write

$$z_k = \widehat{z}_k + \bar{z}_k \quad \text{and} \quad z_{kh} = \widehat{z}_k^I + \bar{z}_{kh},$$

with $\bar{z}_k \in H^1(\mathcal{Q})/\mathbb{R}$ and $\bar{z}_{kh} \in \mathcal{L}_h(\mathcal{Q})/\mathbb{R}$. Substituting these expressions in problems (24) and (25), respectively, and using the first Strang lemma (see, for instance, [18, Theorem 4.4.1]), we obtain

$$\|\nabla\bar{z}_k - \nabla\bar{z}_{kh}\|_{0,\mathcal{Q}} \le C \inf_{\varphi\in\mathcal{L}_h(\mathcal{Q})/\mathbb{R}} \|\nabla\bar{z}_k - \nabla\varphi\|_{0,\mathcal{Q}}$$
$$+ C \sup_{\varphi\in\mathcal{L}_h(\mathcal{Q})/\mathbb{R}} \frac{\left|-\int_{\mathcal{Q}\setminus\Sigma_k^{\text{ext}}} \nabla(\widehat{z}_k - \widehat{z}_k^I)\cdot\nabla\varphi\, d\boldsymbol{x} + \int_\Lambda (\boldsymbol{p}_k - \boldsymbol{p}_{kh})\cdot\boldsymbol{n}\,\varphi\, d\zeta\right|}{\|\nabla\varphi\|_{0,\mathcal{Q}}}.$$

The second term on the right-hand side above is bounded as follows:

$$\left|-\int_{\mathcal{Q}\setminus\Sigma_k^{\text{ext}}} \nabla(\widehat{z}_k - \widehat{z}_k^I)\cdot\nabla\varphi\, d\boldsymbol{x} + \int_\Lambda (\boldsymbol{p}_k - \boldsymbol{p}_{kh})\cdot\boldsymbol{n}\,\varphi\, d\zeta\right|$$
$$\le \left\|\nabla\widehat{z}_k - \nabla\widehat{z}_k^I\right\|_{0,\mathcal{Q}\setminus\Sigma_j^{\text{ext}}} \|\nabla\varphi\|_{0,\mathcal{Q}} + C\,\|\boldsymbol{p}_k - \boldsymbol{p}_{kh}\|_{0,\mathcal{Q}}\,\|\nabla\varphi\|_{0,\mathcal{Q}},$$

where we have used that $\operatorname{div}\boldsymbol{p}_k = \operatorname{div}\boldsymbol{p}_{kh} = 0$ in $\mathcal{Q}$ and the fact that $\|\nabla(\cdot)\|_{0,\mathcal{Q}}$ is equivalent to $\|\cdot\|_{1,\mathcal{Q}}$ on $H^1(\mathcal{Q})/\mathbb{R}$.

On the other hand, from Theorem 5.1 we know that $\nabla\bar{z}_k|_\mathcal{Q} \in (H^s(\mathcal{Q}))^3$. Hence,

$$\inf_{\varphi\in\mathcal{L}_h(\mathcal{Q})/\mathbb{R}} \|\nabla\bar{z}_k - \nabla\varphi\|_{0,\mathcal{Q}} \le \left\|\nabla\bar{z}_k - \nabla\bar{z}_k^I\right\|_{0,\mathcal{Q}} \le Ch^s\,\|\nabla\bar{z}_k\|_{s,\mathcal{Q}}.$$

Thus, using the last two estimates and Theorem 5.2, we obtain

$$\|\nabla\bar{z}_k - \nabla\bar{z}_{kh}\|_{0,\mathcal{Q}} \le Ch^s \left\{\|\nabla\widehat{z}_k\|_{s,\mathcal{Q}\setminus\Sigma_k^{\text{ext}}} + \|\boldsymbol{p}_k\|_{s,\mathcal{Q}} + \|\phi_k\|_{s+1/2,\Lambda} + \|\nabla\bar{z}_k\|_{s,\mathcal{Q}}\right\}.$$

Therefore, as a consequence of Theorem 5.1,

$$\|\widetilde{\nabla} z_k - \widetilde{\nabla} z_{kh}\|_{0,\mathcal{Q}} \leq Ch^s$$

and we conclude the proof. □

Now, we are in a position to introduce the following full discretization of problem (12):
For $n = 1, \ldots, N$, find $(\boldsymbol{A}^n_{Ch}, \psi^n_h, \boldsymbol{\alpha}^n_h) \in \boldsymbol{\mathcal{V}}_h \times \mathcal{L}_h(\Gamma) \times \mathbb{R}^L$ such that

$$\begin{aligned}
&\int_{\Omega_C} \sigma \bar{\partial} \boldsymbol{A}^n_{Ch} \cdot \boldsymbol{w}_C \, \mathrm{d}\boldsymbol{x} + \int_{\Omega_C} \boldsymbol{\mu}_C^{-1} \mathbf{curl}\, \boldsymbol{A}^n_{Ch} \cdot \mathbf{curl}\, \boldsymbol{w}_C \, \mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Gamma \left[-\frac{1}{2}\psi^n_h - \mathcal{D}(\psi^n_h) + \frac{1}{\mu_0}\mathcal{S}(\mathbf{curl}\, \boldsymbol{A}^n_{Ch} \cdot \boldsymbol{n}_C) \right] \mathbf{curl}\, \boldsymbol{w}_C \cdot \boldsymbol{n}_C \, \mathrm{d}\zeta + (\boldsymbol{\alpha}^n_h)^{\mathrm{t}} \mathbb{T}_h(\boldsymbol{w}_C) = \int_{\Omega_C} \boldsymbol{J}(t_n) \cdot \boldsymbol{w}_C \, \mathrm{d}\boldsymbol{x}, \\
&\int_\Gamma \left[\frac{1}{2}\mathbf{curl}\, \boldsymbol{A}^n_{Ch} \cdot \boldsymbol{n}_C + \mathcal{D}'(\mathbf{curl}\, \boldsymbol{A}^n_{Ch} \cdot \boldsymbol{n}_C) + \mu_0 \mathcal{H}(\psi^n_h) \right] \eta \, \mathrm{d}\zeta = 0, \\
&\mu_0 \boldsymbol{\beta}^{\mathrm{t}} \mathbf{N}_h \boldsymbol{\alpha}^n_h - \boldsymbol{\beta}^{\mathrm{t}} \mathbb{T}_h(\boldsymbol{A}^n_{Ch}) = \mu_0 \boldsymbol{\beta}^{\mathrm{t}} \mathbf{N}_h \boldsymbol{\alpha}_0 - \boldsymbol{\beta}^{\mathrm{t}} \mathbb{T}_h(\boldsymbol{A}_{C,0}),
\end{aligned} \tag{26}$$

for all $(\boldsymbol{w}_C, \eta, \boldsymbol{\beta}) \in \boldsymbol{\mathcal{V}}_h \times \mathcal{L}_h(\Gamma) \times \mathbb{R}^L$, with

$$\boldsymbol{A}^0_{Ch} = \boldsymbol{A}_{Ch,0} \quad \text{in } \Omega_C,$$

where $\boldsymbol{A}_{Ch,0} \in \boldsymbol{\mathcal{V}}_h$ is an approximation of $\boldsymbol{A}_{C,0}$, $\bar{\partial}\boldsymbol{A}^n_{Ch} := (\boldsymbol{A}^n_{Ch} - \boldsymbol{A}^{n-1}_{Ch})/\Delta t$ and the linear and continuous operator $\mathbb{T}_h : \boldsymbol{\mathcal{V}} \to \mathbb{R}^L$ is defined by

$$\mathbb{T}_h(\boldsymbol{w}) := \int_\Gamma \mathbf{Z}_h \, (\boldsymbol{w} \times \boldsymbol{n}_C) \, \mathrm{d}\zeta, \quad \text{with } \mathbf{Z}_h := \left[\widetilde{\nabla} z_{1h} \quad \cdots \quad \widetilde{\nabla} z_{Lh} \right]^{\mathrm{t}}.$$

To prove the existence and uniqueness of solution to (26), first we proceed as in the continuous case and obtain a discrete form of problem (15). Let $\mathcal{R}_h : H_0^{-1/2}(\Gamma) \to \mathcal{L}_h(\Gamma)$ be the operator defined by

$$\int_\Gamma \mathcal{H}(\mathcal{R}_h(\xi))\eta \, \mathrm{d}\zeta = \int_\Gamma \xi\eta \, \mathrm{d}\zeta \quad \forall \eta \in \mathcal{L}_h(\Gamma), \ \forall \xi \in H_0^{-1/2}(\Gamma).$$

Note that this is a Galerkin discretization of the elliptic problem (14). Consequently, using the Galerkin orthogonality and the continuity and ellipticity of $\mathcal{H}$ (cf. Lemma 4.1(ii)), we have the following Cea estimate:

$$\|\mathcal{R}\xi - \mathcal{R}_h\xi\|_{1/2,\Gamma} \leq C \inf_{\eta \in \mathcal{L}_h(\Gamma)} \|\mathcal{R}\xi - \eta\|_{1/2,\Gamma} \quad \forall \xi \in H_0^{-1/2}(\Gamma). \tag{27}$$

Now, using again that $\psi^n_h := -\mu_0^{-1}\mathcal{R}_h\left(\mathcal{K}(\boldsymbol{A}^n_{Ch})\right)$ (cf. $(26)_2$) we obtain the following equivalent formulation of (26):
For $n = 1, \ldots, N$, find $\boldsymbol{A}^n_{Ch} \in \boldsymbol{\mathcal{V}}_h$ such that

$$\left(\bar{\partial}\boldsymbol{A}^n_{Ch}, \boldsymbol{w}_C\right)_{\boldsymbol{\sigma}} + \mathcal{A}(\boldsymbol{A}^n_{Ch}, \boldsymbol{w}_C) + \mathcal{B}_h(\boldsymbol{A}^n_{Ch}, \boldsymbol{w}_C) = (\boldsymbol{J}(t_n), \boldsymbol{w}_C)_{0,\Omega_C} + \boldsymbol{g}_h(\boldsymbol{w}_C) \tag{28}$$

for all $\boldsymbol{w}_C \in \boldsymbol{\mathcal{V}}_h$, with

$$\boldsymbol{A}^0_{Ch} = \boldsymbol{A}_{Ch,0} \quad \text{in } \Omega_C,$$

where

$$\begin{aligned}
&\mathcal{B}_h : \boldsymbol{\mathcal{V}}_h \times \boldsymbol{\mathcal{V}}_h \to \mathbb{R}, \qquad \mathcal{B}_h(\boldsymbol{H}, \boldsymbol{G}) := \mu_0^{-1} \int_\Gamma \mathcal{K}(\boldsymbol{G})\mathcal{R}_h(\mathcal{K}(\boldsymbol{H})) \, \mathrm{d}\zeta + \mu_0 \big(\mathbb{T}_h(\boldsymbol{G})\big)^{\mathrm{t}} \mathbf{N}_h^{-1} \mathbb{T}_h(\boldsymbol{H}), \\
&\boldsymbol{g}_h : \boldsymbol{\mathcal{V}}_h \to \mathbb{R}, \qquad \boldsymbol{g}_h(\boldsymbol{H}) := \mu_0^{-1} \big(\mathbb{T}_h(\boldsymbol{H})\big)^{\mathrm{t}} \mathbf{N}_h^{-1} \mathbb{T}_h(\boldsymbol{A}_{C,0}) - \big(\mathbb{T}_h(\boldsymbol{H})\big)^{\mathrm{t}} \boldsymbol{\alpha}_0.
\end{aligned}$$

Hence, at each iteration, we have to find $\boldsymbol{A}^n_{Ch} \in \boldsymbol{\mathcal{V}}_h$ such that

$$\left(\boldsymbol{A}^n_{Ch}, \boldsymbol{w}_C\right)_{\boldsymbol{\sigma}} + \Delta t \left[\mathcal{A}(\boldsymbol{A}^n_{Ch}, \boldsymbol{w}_C) + \mathcal{B}_h(\boldsymbol{A}^n_{Ch}, \boldsymbol{w}_C) \right] = \Delta t \left[(\boldsymbol{J}(t_n), \boldsymbol{w}_C)_{0,\Omega_C} + \boldsymbol{g}_h(\boldsymbol{w}_C) \right] + \left(\boldsymbol{A}^{n-1}_{Ch}, \boldsymbol{w}_C\right)_{\boldsymbol{\sigma}}. \tag{29}$$

Since $\mathcal{B}_h$ and $\mathcal{A}$ are non-negative definite, the existence and uniqueness of $\boldsymbol{A}^n_{Ch}$, $n = 1, \ldots, N$, is immediate.

Remark 5.1. It is easy to prove that if $\psi^n_h := -\mu_0^{-1}\mathcal{R}_h(\mathcal{K}(\boldsymbol{A}^n_{Ch}))$ as defined above and $\boldsymbol{\alpha}^n_h := \boldsymbol{\alpha}_0 + \mu_0^{-1}\mathbf{N}_h^{-1}(\mathbb{T}_h(\boldsymbol{A}^n_{Ch}) - \mathbb{T}_h(\boldsymbol{A}_{C,0}))$, then $(\boldsymbol{A}^n_{Ch}, \psi^n_h, \boldsymbol{\alpha}^n_h)$ is a solution of (26). This solution is unique, because $\mathcal{H}$ is elliptic in $\mathcal{L}_h(\Gamma) \subset H_0^{1/2}(\Gamma)$ and $\mathbf{N}_h$ is a symmetric and positive definite matrix.

5.1. Matrix form

To have it clear the kind of problem we have to solve in practice, we will write the fully discrete scheme (26) in matrix form. Let $\{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_J\}$ and $\{\lambda_1, \dots, \lambda_M\}$ be bases of $\mathcal{V}_h$ and $\mathcal{L}_h(\Gamma)$, respectively, and $\{\mathbf{e}_1, \dots, \mathbf{e}_L\}$ the canonical basis of $\mathbb{R}^L$. We write the solution $(\boldsymbol{A}^n_{Ch}, \psi^n_h, \boldsymbol{\alpha}^n_h)$, $n = 1, \dots, N$, to problem (26), in these bases:

$$\boldsymbol{A}^n_{Ch} = \sum_{j=1}^{J} a^n_j \boldsymbol{\phi}_j, \qquad \psi^n_h = \sum_{j=1}^{M} b^n_j \lambda_j, \qquad \boldsymbol{\alpha}^n_h = \sum_{j=1}^{L} c^n_j \mathbf{e}_j, \quad n = 1, \dots, N.$$

Analogously, we write

$$\boldsymbol{A}_{Ch,0} = \sum_{j=1}^{J} a^0_j \boldsymbol{\phi}_j \quad \text{and} \quad \boldsymbol{\alpha}_0 = \sum_{j=1}^{L} c^0_j \mathbf{e}_j.$$

We set $\mathbf{a}^n := (a^n_i)_{1\le i\le J}$, $\mathbf{c}^n := (c^n_i)_{1\le i\le L}$, with $n = 0, \dots, N$, and $\mathbf{b}^n := (b^n_i)_{1\le i\le M}$, with $n = 1, \dots, N$. We also set $\mathbf{F}^n := (F^n_i)_{1\le i\le J}$, $n = 1, \dots N$, where

$$F^n_i := \int_{\Omega_C} \boldsymbol{J}(t_n) \cdot \boldsymbol{\phi}_i \, d\boldsymbol{x}.$$

We introduce the matrices $\mathbf{W} := (W_{ij})_{1\le i,j\le J}$, $\mathbf{D} := (D_{ij})_{1\le i\le J, 1\le j\le M}$, $\mathbf{H} := (H_{ij})_{1\le i,j\le M}$, $\mathbf{R} := (R_{ij})_{1\le i,j\le J}$, $\mathbf{Q} := (Q_{ij})_{1\le i\le J, 1\le j\le L}$ and $\mathbf{S} := (S_{ij})_{1\le i,j\le J}$, where

$$W_{ij} := \int_{\Omega_C} \sigma \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j \, d\boldsymbol{x}, \qquad D_{ij} := \int_{\Gamma} \left[-\frac{1}{2}\lambda_j - \mathcal{D}(\lambda_j) \right] \mathbf{curl}\, \boldsymbol{\phi}_i \cdot \boldsymbol{n}_C \, d\zeta, \qquad H_{ij} := \int_{\Gamma} \mathcal{H}(\lambda_i)\lambda_j \, d\zeta,$$

$$R_{ij} := \int_{\Omega_C} \mu_C^{-1} \mathbf{curl}\, \boldsymbol{\phi}_i \cdot \mathbf{curl}\, \boldsymbol{\phi}_j \, d\boldsymbol{x}, \qquad Q_{ij} := \mathbf{e}^t_j \int_{\Gamma} \mathbf{Z}_h \left(\boldsymbol{\phi}_i \times \boldsymbol{n}_C\right) d\zeta, \qquad S_{ij} := \int_{\Gamma} \mathcal{S}(\mathbf{curl}\, \boldsymbol{\phi}_i \cdot \boldsymbol{n}_C) \mathbf{curl}\, \boldsymbol{\phi}_j \cdot \boldsymbol{n}_C \, d\zeta.$$

Hence, we write problem (26) in block matrix form as follows:

$$\begin{bmatrix} \mathbf{W} + \Delta t\,(\mathbf{R} + \mathbf{S}) & \Delta t\, \mathbf{D} & \Delta t\, \mathbf{Q} \\ \Delta t\, \mathbf{D}^t & -\Delta t\, \mathbf{H} & \mathbf{O} \\ \Delta t\, \mathbf{Q}^t & \mathbf{O} & -\Delta t\, \mathbf{N}_h \end{bmatrix} \begin{bmatrix} \mathbf{a}^n \\ \mathbf{b}^n \\ \mathbf{c}^n \end{bmatrix} = \begin{bmatrix} \Delta t\, \mathbf{F}^n + \mathbf{W}\mathbf{a}^{n-1} \\ \mathbf{O} \\ \Delta t \left(\mathbf{Q}^t \mathbf{a}^0 - \mathbf{N}_h \mathbf{c}^0 \right) \end{bmatrix}.$$

As already mentioned in Remark 5.1, problem (26) has a unique solution, so that the matrix on the left hand side is non singular.

Matrices $\mathbf{Z}_h$ and $\mathbf{N}_h$ are both readily obtained once the solution $\boldsymbol{p}_{kh}$ to problem (22) is computed. In what follows we write down the matrix form of this problem. Let $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_A\}$, $\{v_1, \dots, v_B\}$ and $\{w_1, \dots, w_C\}$ be bases of $\mathcal{RT}^0_h(\mathcal{Q})$, $\Phi_h/\mathbb{R}$ and M_h, respectively. Then, we write the solution of problem (22) in these bases as follows:

$$\boldsymbol{p}_{kh} = \sum_{j=1}^{A} \varphi_{kj} \boldsymbol{u}_j, \qquad \phi_{kh} = \sum_{j=1}^{B} \gamma_{kj} v_j \quad \text{and} \quad \beta_{kh} = \sum_{j=1}^{C} \eta_{kj} w_j.$$

Next, we define $\boldsymbol{\varphi}_k := (\varphi_{ki})_{1\le i\le A}$, $\boldsymbol{\gamma}_k := (\gamma_{ki})_{1\le i\le B}$, $\boldsymbol{\eta}_k := (\eta_{ki})_{1\le i\le C}$ and $\mathbf{G} := (G_i)_{1\le i\le A}$, where

$$G_i := \int_{\Sigma_k^{\text{ext}}} \boldsymbol{u}_i \cdot \boldsymbol{n}_k \, d\zeta.$$

Moreover, we introduce the matrices $\mathbf{U} := (U_{ij})_{1\le i,j\le A}$, $\mathbf{V} := (V_{ij})_{1\le i,j\le A}$, $\mathbf{K} := (K_{ij})_{1\le i\le A, 1\le j\le B}$, $\mathbf{E} := (E_{ij})_{1\le i\le A, 1\le j\le C}$ and $\mathbf{T} := (T_{ij})_{1\le i,j\le A}$, where

$$U_{ij} := \int_{\mathcal{Q}} \boldsymbol{u}_i \cdot \boldsymbol{u}_j \, d\boldsymbol{x}, \qquad V_{ij} := \int_{\Lambda} \mathcal{S}\,(\boldsymbol{u}_i \cdot \boldsymbol{n})\, \boldsymbol{u}_j \cdot \boldsymbol{n} \, d\zeta,$$

$$K_{ij} := -\int_{\Lambda} \left[\frac{1}{2} v_j + \mathcal{D}(v_j) \right] \boldsymbol{u}_i \cdot \boldsymbol{n} \, d\zeta, \qquad E_{ij} := \int_{\mathcal{Q}} w_j \operatorname{div} \boldsymbol{u}_i \, d\boldsymbol{x}, \qquad T_{ij} := \int_{\Lambda} \mathcal{S}\left(\mathbf{curl}_{\tau} v_j\right) \mathbf{curl}_{\tau} v_i \, d\zeta.$$

Then, problem (22) reads

$$\begin{bmatrix} \mathbf{U} + \mathbf{V} & \mathbf{K} & \mathbf{E} \\ \mathbf{K}^t & -\mathbf{T} & \mathbf{O} \\ \mathbf{E}^t & \mathbf{O} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_k \\ \boldsymbol{\gamma}_k \\ \boldsymbol{\eta}_k \end{bmatrix} = \begin{bmatrix} \mathbf{G} \\ \mathbf{O} \\ \mathbf{O} \end{bmatrix}.$$

It is proved in [17] that the matrix of the left hand side above is invertible. Finally, for a discussion on the efficient computation of all the singular integrals appearing above, we refer to [19].

As a conclusion, we have that problem (26) is actually solvable. Although it involves the solution of the auxiliary problem (22), this can be made off-line since it does not depend on time. Once it is solved, the time domain problem (26) involves only a vector field on the conducting domain and a scalar field on its boundary. Therefore, this approach allows to minimizing the number of degrees of freedom needed in the discretization.

6. Error estimates

For any $s \geq 0$, we consider the space

$$\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C) := \left\{ \boldsymbol{v} \in (H^s(\Omega_C))^3 : \mathbf{curl}\,\boldsymbol{v} \in (H^s(\Omega_C))^3 \right\}$$

endowed with the norm $\|\boldsymbol{v}\|^2_{\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C)} := \|\boldsymbol{v}\|^2_{s,\Omega_C} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{s,\Omega_C}$. It is well known that the Nédélec interpolation operator $I_h^{\mathcal{N}}\boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h$ is well defined for any $\boldsymbol{v} \in \boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C)$, with $s > 1/2$ (see, for instance, Lemma 4.7 of [9]). Moreover, for $1/2 < s \leq 1$, the following interpolation error estimate holds true (see Proposition 5.6 of [20]):

$$\left\| \boldsymbol{v} - I_h^{\mathcal{N}}\boldsymbol{v} \right\|_{\boldsymbol{\mathcal{V}}} \leq Ch^s \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C). \tag{30}$$

To simplify the notation, we introduce for any $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}$

$$\mathcal{G}_h(\boldsymbol{w}) := \|(\mathcal{R} - \mathcal{R}_h)\,\mathcal{K}(\boldsymbol{w})\|_{1/2,\Gamma}\,.$$

Lemma 6.1. *Let $(\boldsymbol{A}_C, \psi, \boldsymbol{\alpha})$ and $(\boldsymbol{A}^n_{Ch}, \psi^n_h, \boldsymbol{\alpha}^n_h)$ be the solutions to problems (12) and (26), respectively, the latter with initial data $\boldsymbol{A}^0_{Ch} := I_h^{\mathcal{N}}(\boldsymbol{A}_{C,0})$. Assume that $\boldsymbol{A}_C \in \mathcal{C}^1([0,T];\boldsymbol{\mathcal{V}}) \cap \mathcal{C}^0([0,T];\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C))$, with $s > 1/2$. Moreover, let $\boldsymbol{\rho}^n := \boldsymbol{A}_C(t_n) - I_h^{\mathcal{N}}\boldsymbol{A}_C(t_n)$, $\boldsymbol{\delta}^n := I_h^{\mathcal{N}}\boldsymbol{A}_C(t_n) - \boldsymbol{A}^n_{Ch}$ and $\boldsymbol{\tau}^n := \overline{\partial}\boldsymbol{A}_C(t_n) - \partial_t\boldsymbol{A}_C(t_n)$. Then, there exists $C > 0$, independent of h and Δt, such that*

$$\begin{aligned}
\max_{1\leq k\leq n} \left\|\boldsymbol{\delta}^k\right\|^2_{\boldsymbol{\mathcal{V}}} + \Delta t \sum_{k=1}^n \left\|\overline{\partial}\boldsymbol{\delta}^k\right\|^2_{\boldsymbol{\sigma}} \leq C \Bigg\{ &\Delta t \sum_{k=1}^n \Bigg[\left\|\overline{\partial}\boldsymbol{\rho}^k\right\|^2_{\boldsymbol{\mathcal{V}}} + \left\|\boldsymbol{\tau}^k\right\|^2_{\boldsymbol{\mathcal{V}}} + \mathcal{G}_h(\partial_t\boldsymbol{A}_C(t_k))^2 \\
&+ \left(\|\boldsymbol{A}_C(t_k)\|^2_{\boldsymbol{\mathcal{V}}} + \|\partial_t\boldsymbol{A}_C(t_k)\|^2_{\boldsymbol{\mathcal{V}}}\right) \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|^2_{0,\mathcal{Q}} \right.\\
&\left. + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2 \right) \Bigg] + \left(\left\|\boldsymbol{A}_{C,0}\right\|^2_{\boldsymbol{\mathcal{V}}} + |\boldsymbol{\alpha}_0|^2 \right) \\
&\times \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|^2_{0,\mathcal{Q}} + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2 \right) \\
&+ \max_{0\leq k\leq n} \left\|\boldsymbol{\rho}^k\right\|^2_{\boldsymbol{\mathcal{V}}} + \max_{0\leq k\leq n} \mathcal{G}_h(\boldsymbol{A}_C(t_k))^2 \Bigg\}.
\end{aligned}$$

Proof. It is straightforward to show that

$$\begin{aligned}
\left(\overline{\partial}\boldsymbol{\delta}^k, \boldsymbol{v}\right)_{\boldsymbol{\sigma}} + \mathcal{A}(\boldsymbol{\delta}^k, \boldsymbol{v}) + \mathcal{B}_h(\boldsymbol{\delta}^k, \boldsymbol{v}) = &- \left(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{v}\right)_{\boldsymbol{\sigma}} + \left(\boldsymbol{\tau}^k, \boldsymbol{v}\right)_{\boldsymbol{\sigma}} - \mathcal{A}(\boldsymbol{\rho}^k, \boldsymbol{v}) - \mathcal{B}_h(\boldsymbol{\rho}^k, \boldsymbol{v}) \\
&+ \mathcal{B}_h(\boldsymbol{A}_C(t_k), \boldsymbol{v}) - \mathcal{B}(\boldsymbol{A}_C(t_k), \boldsymbol{v}) + \mathbf{g}(\boldsymbol{v}) - \mathbf{g}_h(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h,
\end{aligned} \tag{31}$$

as well as the following inequalities:

$$\begin{aligned}
&\left(\overline{\partial}\boldsymbol{\delta}^k, \boldsymbol{\delta}^k\right)_{\boldsymbol{\sigma}} \geq \frac{1}{2\Delta t}\left(\left\|\boldsymbol{\delta}^k\right\|^2_{\boldsymbol{\sigma}} - \left\|\boldsymbol{\delta}^{k-1}\right\|^2_{\boldsymbol{\sigma}} \right), \\
&\mathcal{A}(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) \geq \mu_1^{-1} \left\|\mathbf{curl}\,\boldsymbol{\delta}^k\right\|^2_{0,\Omega_C}, \\
&\mathcal{B}(\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^k) - \mathcal{B}_h(\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^k) \leq C\,\|\boldsymbol{A}_C(t_k)\|_{\boldsymbol{\mathcal{V}}} \left\|\boldsymbol{\delta}^k\right\|_{\boldsymbol{\mathcal{V}}} \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}} + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\| \right) \\
&\qquad\qquad + C \left\|\mathbf{curl}\,\boldsymbol{\delta}^k\right\|_{0,\Omega_C} \mathcal{G}_h(\boldsymbol{A}_C(t_k)), \\
&\mathbf{g}(\boldsymbol{\delta}^k) - \mathbf{g}_h(\boldsymbol{\delta}^k) \leq C\left(\left\|\boldsymbol{A}_{C,0}\right\|_{\boldsymbol{\mathcal{V}}} + |\boldsymbol{\alpha}_0| \right) \left\|\boldsymbol{\delta}^k\right\|_{\boldsymbol{\mathcal{V}}} \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}} + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\| \right).
\end{aligned}$$

Constant μ_1 on the second inequality is an upper bound in Ω_C of the largest eigenvalue of $\boldsymbol{\mu}_C$. Hence, choosing $\boldsymbol{v} = \boldsymbol{\delta}^k$ in (31) and using that $\mathcal{B}_h$ is non-negative, the Cauchy–Schwarz inequality, Remark 4.1 and Young's inequality lead us to the following estimate:

$$\begin{aligned}
&\left\|\boldsymbol{\delta}^k\right\|^2_{\boldsymbol{\sigma}} - \left\|\boldsymbol{\delta}^{k-1}\right\|^2_{\boldsymbol{\sigma}} + \Delta t \mu_1^{-1} \left\|\mathbf{curl}\,\boldsymbol{\delta}^k\right\|^2_{0,\Omega_C} \leq \frac{\Delta t}{2T}\left\|\boldsymbol{\delta}^k\right\|^2_{\boldsymbol{\sigma}} + C\Delta t \Bigg[\left\|\overline{\partial}\boldsymbol{\rho}^k\right\|^2_{\boldsymbol{\sigma}} + \left\|\boldsymbol{\tau}^k\right\|^2_{\boldsymbol{\sigma}} + \left\|\boldsymbol{\rho}^k\right\|^2_{\boldsymbol{\mathcal{V}}} + \mathcal{G}_h(\boldsymbol{A}_C(t_k))^2 \\
&\quad + \|\boldsymbol{A}_C(t_k)\|^2_{\boldsymbol{\mathcal{V}}} \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|^2_{0,\mathcal{Q}} + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2 \right) \\
&\quad + \left(\left\|\boldsymbol{A}_{C,0}\right\|^2_{\boldsymbol{\mathcal{V}}} + |\boldsymbol{\alpha}_0|^2 \right) \left(\max_{1\leq i\leq L} \|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|^2_{0,\mathcal{Q}} + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2 \right) \Bigg].
\end{aligned} \tag{32}$$

Then, summing over k, using the discrete Gronwall's lemma (see, for instance, [21, Lemma 1.4.2]) and taking into account that $\boldsymbol{\delta}^0 = \mathbf{0}$, we obtain

$$\left\|\boldsymbol{\delta}^n\right\|_\sigma^2 \le C\left\{\Delta t \sum_{k=1}^n \left[\left\|\overline{\partial}\boldsymbol{\rho}^k\right\|_{\boldsymbol{\sigma}}^2 + \left\|\boldsymbol{\tau}^k\right\|_{\boldsymbol{\sigma}}^2 + \left\|\boldsymbol{\rho}^k\right\|_{\mathcal{V}}^2 + \mathcal{G}_h(\boldsymbol{A}_C(t_k))^2 \right.\right.$$
$$\left. + \|\boldsymbol{A}_C(t_k)\|_{\mathcal{V}}^2\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2\right)\right]$$
$$\left. + \left(\left\|\boldsymbol{A}_{C,0}\right\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2\right)\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2\right)\right\}$$

for $n = 1, \ldots, N$. Inserting the last inequality in (32) and summing over k we have the estimate

$$\left\|\boldsymbol{\delta}^n\right\|_{\boldsymbol{\sigma}}^2 + \Delta t\sum_{k=1}^n \left\|\mathbf{curl}\,\boldsymbol{\delta}^k\right\|_{0,\Omega_C}^2 \le C\left\{\Delta t \sum_{k=1}^n \left[\left\|\overline{\partial}\boldsymbol{\rho}^k\right\|_{\boldsymbol{\sigma}}^2 + \left\|\boldsymbol{\tau}^k\right\|_{\boldsymbol{\sigma}}^2 + \left\|\boldsymbol{\rho}^k\right\|_{\mathcal{V}}^2 + \mathcal{G}_h(\boldsymbol{A}_C(t_k))^2 \right.\right.$$
$$\left. + \|\boldsymbol{A}_C(t_k)\|_{\mathcal{V}}^2\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2\right)\right]$$
$$\left. + \left(\left\|\boldsymbol{A}_{C,0}\right\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2\right)\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2\right)\right\}. \tag{33}$$

Let us now take $\boldsymbol{v} = \overline{\partial}\boldsymbol{\delta}^k$ in (31). We have

$$\left\|\overline{\partial}\boldsymbol{\delta}^k\right\|_{\boldsymbol{\sigma}}^2 + \mathcal{A}(\boldsymbol{\delta}^k, \overline{\partial}\boldsymbol{\delta}^k) + \mathcal{B}_h(\boldsymbol{\delta}^k, \overline{\partial}\boldsymbol{\delta}^k) = -\left(\overline{\partial}\boldsymbol{\rho}^k, \overline{\partial}\boldsymbol{\delta}^k\right)_{\boldsymbol{\sigma}} + \left(\boldsymbol{\tau}^k, \overline{\partial}\boldsymbol{\delta}^k\right)_{\boldsymbol{\sigma}} + \mathcal{A}(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1}) + \mathcal{B}_h(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})$$
$$+ \mathcal{B}(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) + \mathcal{B}(\partial_t\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^{k-1})$$
$$- \mathcal{B}_h(\partial_t\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^{k-1}) + \boldsymbol{g}(\overline{\partial}\boldsymbol{\delta}^k) - \boldsymbol{g}_h(\overline{\partial}\boldsymbol{\delta}^k) - \frac{1}{\Delta t}\left(\gamma_k - \gamma_{k-1}\right), \tag{34}$$

where $\gamma_k := \mathcal{A}(\boldsymbol{\rho}^k, \boldsymbol{\delta}^k) + \mathcal{B}_h(\boldsymbol{\rho}^k, \boldsymbol{\delta}^k) - \mathcal{B}_h(\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^k) + \mathcal{B}(\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^k)$.

On the other hand, since $\mathcal{A}$ is non-negative definite and symmetric, it is easy to check that

$$\mathcal{A}(\boldsymbol{\delta}^k, \overline{\partial}\boldsymbol{\delta}^k) \ge \frac{1}{2\Delta t}\left[\mathcal{A}(\boldsymbol{\delta}^k, \boldsymbol{\delta}^k) - \mathcal{A}(\boldsymbol{\delta}^{k-1}, \boldsymbol{\delta}^{k-1})\right]$$

and similarly for $\mathcal{B}_h$. Using these inequalities in (34) together with Cauchy–Schwarz inequality, and, then, summing over k and recalling that $\mathcal{B}_h$ is non-negative, we deduce that

$$\frac{1}{2}\sum_{k=1}^n \left\|\overline{\partial}\boldsymbol{\delta}^k\right\|_{\boldsymbol{\sigma}}^2 + \frac{1}{2\Delta t}\mu_1^{-1}\left\|\mathbf{curl}\,\boldsymbol{\delta}^n\right\|_{0,\Omega_C}^2$$
$$\le \sum_{k=1}^n \left[\left\|\overline{\partial}\boldsymbol{\rho}^k\right\|_{\boldsymbol{\sigma}}^2 + \left\|\boldsymbol{\tau}^k\right\|_{\boldsymbol{\sigma}}^2\right] + \sum_{k=1}^n \left[\left|\mathcal{A}(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})\right| + \left|\mathcal{B}_h(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})\right| + \left|\mathcal{B}(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1})\right|\right.$$
$$\left. + \left|\mathcal{B}(\partial_t\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\partial_t\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^{k-1})\right|\right] + \frac{1}{\Delta t}\left|\boldsymbol{g}(\boldsymbol{\delta}^n) - \boldsymbol{g}_h(\boldsymbol{\delta}^n)\right| + \frac{1}{\Delta t}\left|\gamma_n\right|. \tag{35}$$

The following bounds are easy to obtain from Young's inequality and Remark 4.1:

$$\sum_{k=1}^n \left|\mathcal{A}(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})\right| \le \sum_{k=1}^n \left\|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\right\|_{0,\Omega_C}^2 + C\sum_{k=1}^n \left\|\mathbf{curl}\,\overline{\partial}\boldsymbol{\rho}^k\right\|_{0,\Omega_C}^2,$$

$$\sum_{k=1}^n \left|\mathcal{B}_h(\overline{\partial}\boldsymbol{\rho}^k, \boldsymbol{\delta}^{k-1})\right| \le \sum_{k=1}^n \left\|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\right\|_{0,\Omega_C}^2 + \sum_{k=1}^n \left\|\boldsymbol{\delta}^{k-1}\right\|_{\boldsymbol{\sigma}}^2 + C\sum_{k=1}^n \left\|\overline{\partial}\boldsymbol{\rho}^k\right\|_{\mathcal{V}}^2,$$

$$\sum_{k=1}^n \left|\mathcal{B}(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\boldsymbol{\tau}^k, \boldsymbol{\delta}^{k-1})\right| \le \sum_{k=1}^n \left\|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\right\|_{0,\Omega_C}^2 + \sum_{k=1}^n \left\|\boldsymbol{\delta}^{k-1}\right\|_{\boldsymbol{\sigma}}^2 + C\sum_{k=1}^n \left\|\boldsymbol{\tau}^k\right\|_{\mathcal{V}}^2,$$

$$\sum_{k=1}^n \left|\mathcal{B}(\partial_t\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^{k-1}) - \mathcal{B}_h(\partial_t\boldsymbol{A}_C(t_k), \boldsymbol{\delta}^{k-1})\right| \le \sum_{k=1}^n \left\|\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\right\|_{0,\Omega_C}^2 + \sum_{k=1}^n \left\|\boldsymbol{\delta}^{k-1}\right\|_{\boldsymbol{\sigma}}^2 + C\sum_{k=1}^n \mathcal{G}_h(\partial_t\boldsymbol{A}_C(t_k))^2$$
$$+ C\sum_{k=1}^n \|\partial_t\boldsymbol{A}_C(t_k)\|_{\mathcal{V}}^2\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i - \widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2 + \left\|\mathbf{N}^{-1} - \mathbf{N}_h^{-1}\right\|^2\right),$$

and

$$\begin{aligned}\left|\boldsymbol{g}(\delta^n)-\boldsymbol{g}_h(\delta^n)\right| &\le C\left(\left\|\boldsymbol{A}_{C,0}\right\|_{\mathcal{V}}^2+|\boldsymbol{\alpha}_0|^2\right)\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i-\widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2+\left\|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\right\|^2\right)\\&\quad+\frac{1}{8}\mu_1^{-1}\left\|\mathbf{curl}\,\delta^n\right\|_{0,\Omega_C}^2+\left\|\delta^n\right\|_{\sigma}^2,\\|\gamma_n| &\le \frac{1}{8}\mu_1^{-1}\left\|\mathbf{curl}\,\delta^n\right\|_{0,\Omega_C}^2+\left\|\delta^n\right\|_{\sigma}^2+C\left[\left\|\boldsymbol{\rho}^n\right\|_{\mathcal{V}}^2+\|\boldsymbol{A}_C(t_n)\|_{\mathcal{V}}^2\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i-\widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2+\left\|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\right\|^2\right)\right].\end{aligned}$$

Substituting all these inequalities in (35), using (33) and Remark 4.1, we obtain

$$\begin{aligned}&\Delta t\sum_{k=1}^{n}\left\|\bar{\partial}\delta^k\right\|_{\sigma}^2+\left\|\mathbf{curl}\,\delta^n\right\|_{0,\Omega_C}^2\\&\le C\Bigg\{\Delta t\sum_{k=1}^{n}\Bigg[\left\|\bar{\partial}\boldsymbol{\rho}^k\right\|_{\mathcal{V}}^2+\left\|\boldsymbol{\tau}^k\right\|_{\mathcal{V}}^2+\mathcal{g}_h(\partial_t(\boldsymbol{A}_C(t_k)))^2+\left\|\boldsymbol{\rho}^k\right\|_{\mathcal{V}}^2\\&\quad+\|\boldsymbol{A}_C(t_k)\|_{\mathcal{V}}^2\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i-\widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2+\left\|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\right\|^2\right)\\&\quad+\|\partial_t\boldsymbol{A}_C(t_k)\|_{\mathcal{V}}^2\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i-\widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2+\left\|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\right\|^2\right)\Bigg]\\&\quad+\left(\left\|\boldsymbol{A}_{C,0}\right\|_{\mathcal{V}}^2+|\boldsymbol{\alpha}_0|^2\right)\left(\max_{1\le i\le L}\|\widetilde{\nabla} z_i-\widetilde{\nabla} z_{ih}\|_{0,\mathcal{Q}}^2+\left\|\mathbf{N}^{-1}-\mathbf{N}_h^{-1}\right\|^2\right)+\left\|\boldsymbol{\rho}^n\right\|_{\mathcal{V}}^2+\mathcal{g}_h(\boldsymbol{A}_C(t_n))^2\Bigg\}.\end{aligned}$$

Combining this inequality with (33) and Remark 4.1, we end the proof. □

Lemma 6.2. *Let $(\boldsymbol{A}_C,\psi,\boldsymbol{\alpha})$ be the solution of (12). If we assume that $\boldsymbol{A}_C\in H^1(0,T;\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C))$, $1/2<s<s_{\mathcal{Q}}$, then $\psi\in H^1(0,T;H^{s+1/2}(\Gamma))$ and the following estimates hold true:*

$$\inf_{\eta\in\mathcal{L}_h(\Gamma)}\|\psi(t)-\eta\|_{1/2,\Gamma}\le Ch^s\,\|\mathbf{curl}\,\boldsymbol{A}_C(t)\|_{s,\Omega_C}\,, \tag{36}$$

$$\inf_{\eta\in\mathcal{L}_h(\Gamma)}\|\partial_t\psi(t)-\eta\|_{1/2,\Gamma}\le Ch^s\,\|\partial_t(\mathbf{curl}\,\boldsymbol{A}_C(t))\|_{s,\Omega_C}\,. \tag{37}$$

Proof. Let $(\boldsymbol{A}_C,\psi,\boldsymbol{\alpha})$ be the unique solution of (12). Let ψ_I be as in Theorem 4.3. As shown in that theorem, $\psi_I(t)=\psi(t)+c(t)$ with $c(t)\in\mathbb{R}$ and $t\in[0,T]$. Moreover, a.e. in $[0,T]$, $\psi_I|_{\mathcal{Q}}$ is the solution to

$$\begin{aligned}&-\Delta\psi_I=0\quad\text{in }\mathcal{Q},\\&\mu_0\frac{\partial\psi_I}{\partial\boldsymbol{n}_I}=-\mathbf{curl}\,\boldsymbol{A}_C\cdot\boldsymbol{n}_C\quad\text{on }\Gamma,\\&\psi_I|_{\Lambda}\in\mathcal{C}^{\infty}(\Lambda).\end{aligned} \tag{38}$$

Since $\boldsymbol{A}_C\in\mathcal{C}^0([0,T];\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C))$ with $1/2<s<s_{\mathcal{Q}}$ and Λ is the boundary of a convex polyhedron, by applying classical results for the Laplace equation (see [22]) we have that $\psi_I\in H^{s+1}(\mathcal{Q})$ and

$$\|\psi_I\|_{s+1,\mathcal{Q}}\le C\,\|\mathbf{curl}\,\boldsymbol{A}_C\cdot\boldsymbol{n}_C\|_{s-1/2,\Gamma}\le C\,\|\mathbf{curl}\,\boldsymbol{A}_C\|_{s,\Omega_C}\,. \tag{39}$$

Since $s>1/2$, the Lagrange interpolant ψ_I^I of ψ_I is well defined. Moreover, since ψ_I and ψ only differ in a constant,

$$\left(\psi_I-\psi_I^I\right)|_{\Gamma}=\psi-\psi^{I_\Gamma},$$

where $\psi^{I_\Gamma}\in\mathcal{L}_h(\Gamma)$ denotes the 2D Lagrange surface interpolant on Γ. Therefore, because of the trace theorem, standard estimates for the 3D Lagrange interpolant and (39), we have

$$\left\|\psi-\psi^{I_\Gamma}\right\|_{1/2,\Gamma}\le C\left\|\psi_I-\psi_I^I\right\|_{1,\mathcal{Q}}\le Ch^s\,\|\psi_I\|_{s+1,\mathcal{Q}}\le Ch^s\,\|\mathbf{curl}\,\boldsymbol{A}_C\|_{s,\Omega_C}\,.$$

Thus, we conclude (36).

To prove (37), we recall that ψ_I is the solution to problem (19) (cf. the proof of Theorem 4.3). Then, since $\boldsymbol{A}_C\in H^1(0,T;\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C))$, differentiating in time each equation in (19), we obtain an estimate analogous to (39) for $\partial_t\psi_I$. On the other hand, since $\psi_I(t)=\psi(t)+c(t)$ with

$$c(t)=\frac{1}{|\Gamma|}\int_{\Gamma}\psi_I(t)\,\mathrm{d}\zeta,$$

we have that $\partial_t\psi(t)=\partial_t\psi_I(t)-c'(t)$. Hence, the rest of the proof follows identically as above. □

Now we are in a position to conclude the following asymptotic error estimate for the fully discrete scheme.

Theorem 6.1. *Let $(\boldsymbol{A}_C, \psi, \boldsymbol{\alpha})$ and $(\boldsymbol{A}_{Ch}^n, \psi_h^n, \boldsymbol{\alpha}_h^n)$, $n = 1, \dots, N$, be the solutions to problem (12) and (26), respectively. Let us assume that $\boldsymbol{A}_C \in H^1(0, T; \boldsymbol{H}^s(\mathbf{curl}\,; \Omega_C)) \cap H^2(0, T; \boldsymbol{H}(\mathbf{curl}\,; \Omega_C))$ with $s \in (1/2, s_{\mathcal{Q}})$. Then, there exists $h_0 > 0$ such that, for all $h \in (0, h_0)$, the following estimate holds:*

$$
\begin{aligned}
&\max_{1\le n\le N} \left\|\boldsymbol{A}_C(t_n) - \boldsymbol{A}_{Ch}^n\right\|_{\mathcal{V}}^2 + \Delta t \sum_{n=1}^{N} \left\|\overline{\partial}(\boldsymbol{A}_C(t_n) - \boldsymbol{A}_{Ch}^n)\right\|_{\boldsymbol{\sigma}}^2 \\
&\le Ch^{2s}\left\{\int_0^T \|\partial_t \boldsymbol{A}_C(t)\|_{\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C)}^2 \,\mathrm{d}t + \max_{1\le n\le N} \|\partial_t(\mathbf{curl}\,\boldsymbol{A}_C(t_n))\|_{s,\Omega_C}^2\right. \\
&\quad + \max_{1\le n\le N}\left(\|\boldsymbol{A}_C(t_n)\|_{\mathcal{V}}^2 + \|\partial_t \boldsymbol{A}_C(t_n)\|_{\mathcal{V}}^2\right)\left(\max_{1\le k\le L}\|\widetilde{\nabla} z_k\|_{s,\mathcal{Q}}^2 + \|z_k\|_{s+1/2,\Lambda}^2\right) \\
&\quad + \left(\left\|\boldsymbol{A}_{C,0}\right\|_{\mathcal{V}}^2 + |\boldsymbol{\alpha}_0|^2\right)\left(\max_{1\le k\le L}\|\widetilde{\nabla} z_k\|_{s,\mathcal{Q}}^2 + \|z_k\|_{s+1/2,\Lambda}^2\right) \\
&\quad \left. + \max_{1\le n\le N} \|\boldsymbol{A}_C(t_n)\|_{\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C)}^2\right\} + (\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{A}_C(t)\|_{\mathcal{V}}^2 \,\mathrm{d}t \\
&\le C\left[(\Delta t)^2 + h^{2s}\right]\left(\|\boldsymbol{A}_C\|_{H^2(0,T;\boldsymbol{H}^s(\mathbf{curl}\ \Omega_C))}^2 + |\boldsymbol{\alpha}_0|^2\right),
\end{aligned}
$$

where z_k is the solution of problem (4), $k = 1, \dots, L$.

Proof. A Taylor expansion shows that

$$
\overline{\partial}\boldsymbol{A}_C(t_k) = \partial_t \boldsymbol{A}_C(t_k) + \frac{1}{\Delta t}\int_{t_{k-1}}^{t_k} (t_{k-1} - t)\, \partial_{tt}\boldsymbol{A}_C(t)\,\mathrm{d}t.
$$

Consequently,

$$
\sum_{k=1}^{n} \left\|\boldsymbol{\tau}^k\right\|_{\mathcal{V}}^2 \le \Delta t \int_0^T \|\partial_{tt}\boldsymbol{A}_C(t)\|_{\mathcal{V}}^2 \,\mathrm{d}t.
$$

Moreover, we have from (30),

$$
\sum_{k=1}^{n} \left\|\overline{\partial}\boldsymbol{\rho}^k\right\|_{\mathcal{V}}^2 \le \frac{1}{\Delta t}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \left\|\partial_t(I - I_h^{\mathcal{N}})\boldsymbol{A}_C(t)\right\|_{\mathcal{V}}^2 \,\mathrm{d}t \le \frac{Ch^{2s}}{\Delta t}\int_0^T \|\partial_t \boldsymbol{A}_C(t)\|_{\boldsymbol{H}^s(\mathbf{curl}\,;\Omega_C)}^2 \,\mathrm{d}t.
$$

We recall that $\psi(t) = -\mu_0^{-1}\mathcal{R}(\mathcal{K}(\boldsymbol{A}_C(t)))$ (cf. Remark 4.2). It follows from (27) that

$$
\begin{aligned}
\mathcal{G}_h(\boldsymbol{A}_C(t_n)) &\le \inf_{\eta\in\mathcal{L}_h(\Gamma)} \|\psi(t_n) - \eta\|_{1/2,\Gamma}^2, \\
\mathcal{G}_h(\partial_t\boldsymbol{A}_C(t_n)) &\le \inf_{\eta\in\mathcal{L}_h(\Gamma)} \|\partial_t\psi(t_n) - \eta\|_{1/2,\Gamma}^2.
\end{aligned}
$$

Thus, using Lemma 6.2, we obtain

$$
\begin{aligned}
\mathcal{G}_h(\boldsymbol{A}_C(t_n)) &\le Ch^s \|\mathbf{curl}\,\boldsymbol{A}_C(t_n)\|_{s,\Omega_C}, \\
\mathcal{G}_h(\partial_t\boldsymbol{A}_C(t_n)) &\le Ch^s \|\partial_t(\mathbf{curl}\,\boldsymbol{A}_C(t_n))\|_{s,\Omega_C}.
\end{aligned}
\tag{40}
$$

Hence, the results follows by writing $\boldsymbol{A}_C(t_n) - \boldsymbol{A}_{Ch}^n = \boldsymbol{\delta}^n + \boldsymbol{\rho}^n$ and using Lemmas 6.1 and 5.1, Theorem 5.3 and (30). □

Remark 6.1. Let us recall that $\psi(t_n) = -\mu_0^{-1}\mathcal{R}(\mathcal{K}(\boldsymbol{A}_C(t_n)))$ and $\psi_h^n = -\mu_0^{-1}\mathcal{R}_h(\mathcal{K}(\boldsymbol{A}_{Ch}^n))$. Therefore, using (40) and the uniform boundedness of $\mathcal{R}_h$ with respect to h, we obtain

$$
\left\|\psi(t_n) - \psi_h^n\right\|_{1/2,\Gamma} \le \mathcal{G}_h(\boldsymbol{A}_C(t_n)) + \left\|\mathcal{R}_h(\mathcal{K}(\boldsymbol{A}_C(t_n) - \boldsymbol{A}_{Ch}^n))\right\|_{1/2,\Gamma} \le C\left\{h^s \|\mathbf{curl}\,\boldsymbol{A}_C(t_n)\|_{s,\Omega_C} + \left\|\boldsymbol{A}_C(t_n) - \boldsymbol{A}_{Ch}^n\right\|_{\mathcal{V}}\right\}.
$$

Then, using Lemma 6.2 and Theorem 6.1, under the assumptions of the latter, we conclude that

$$
\Delta t \sum_{n=1}^{N} \left\|\psi(t_n) - \psi_h^n\right\|_{1/2,\Gamma}^2 \le C\left[h^{2s} + (\Delta t)^2\right].
$$

Moreover under the same assumptions, since $\boldsymbol{\alpha}(t_n) = \boldsymbol{\alpha}_0 - \mu_0^{-1}\mathbf{N}^{-1}(\mathbb{T}(\boldsymbol{A}_C(t_n) - \boldsymbol{A}_{C,0}))$ and $\boldsymbol{\alpha}_h^n = \boldsymbol{\alpha}_0 - \mu_0^{-1}\mathbf{N}_h^{-1}(g\mathbb{T}_h(\boldsymbol{A}_{Ch}^n - \boldsymbol{A}_{C,0}))$, from Theorem 5.3, Lemma 5.1 and Theorem 6.1, we also conclude that

$$\max_{1\le n\le N} \left|\boldsymbol{\alpha}(t_n) - \boldsymbol{\alpha}_h^n\right|^2 \le C\left[h^{2s} + (\Delta t)^2\right].$$

Acknowledgments

The authors gratefully thank Ana Alonso-Rodríguez and Alberto Valli for suggesting the problem and for many helpful discussions.

The first author was partially supported by MECESUP UCO0713 and a CONICYT Ph.D. fellowship at Universidad de Concepción, Chile. The second author was partially supported by BASAL project, CMM, Universidad de Chile.

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