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A Herglotz wavefunction method for solving the inverse Cauchy problem connected with the Helmholtz equation

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ABSTRACT

This paper is concerned with the Cauchy problem connected with the Helmholtz equation. On the basis of the denseness of Herglotz wavefunctions, we propose a numerical method for obtaining an approximate solution to the problem. We analyze the convergence and stability with a suitable choice of regularization method. Numerical experiments are also presented to show the effectiveness of our method.

1. Introduction

Cauchy problems connected with the Helmholtz equation arise in many areas of science and engineering [1–4], such as wave propagation, vibration, electromagnetic scattering and so on. For Cauchy problems, the boundary conditions are incomplete, which will lead to some inverse problems. It is well known that they are ill-posed, i.e. the solutions do not depend continuously on Cauchy data and a small perturbation in the data may result in a large change in the solution [5–7]. There are some numerical methods in the literature for solving the Cauchy problem connected with the Helmholtz equation. We refer the reader to [4,8–12] for the alternating iterative boundary element method, the conjugate gradient boundary element method, the boundary knot method, and the methods of fundamental solutions. Study on the moment method and boundary particle method can be found in [13,14].

The main purpose of this paper is to provide a simple and effective numerical method for solving Cauchy problems connected with the Helmholtz equation. The main idea is to approximate the solution to the Cauchy problem by some Herglotz wavefunction, which is a solution of the Helmholtz equation

$$\Delta v_g + k^2 v_g = 0, \quad \text{in } \mathbb{R}^2 \tag{1.1}$$

of the form

$$v_g(x) := \int_\Omega e^{\mathbf{i}kx\cdot d} g(d)\mathrm{d}s(d) \tag{1.2}$$

where Ω is the unit circle and $g \in L^2(\Omega)$. An illuminating approximation property of Herglotz wavefunctions is that Herglotz wavefunctions are a dense set of solutions to the Helmholtz equation in the Sobolev space $H^1(D)$ (see [15, Theorem 2.6]). With this idea in mind, we derive two integral equations for the kernel g on the specified boundary, which can be solved by regularization methods. Then the data for the solution on the unspecified boundary can be obtained by simple calculation of the Herglotz wavefunction.

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We emphasize that our algorithm can be easily implemented. Only two first-kind integral equations need to be solved. By computing the values of the Herglotz wavefunction, we can achieve the numerical approximation of the solution not only on the unspecified boundary but also in the whole domain. The corresponding error estimates can be given directly. We think that our method can also be developed to solve some inverse problems for the Maxwell equations which will be reported elsewhere.

The outline of this paper is as follows. In Section 2, we present the Herglotz wavefunction approximation of the solution to the Cauchy problem, and two integral equations for the kernel function. In Section 3, we solve the integral equations by the Tikhonov regularization method using the Morozov principle, and analyze the convergence and stability. Finally, several numerical examples are included to show the effectiveness of our method.

2. Herglotz wavefunction approximation

Let $D \subset \mathbb{R}^2$ be a bounded and simply connected domain with a regular boundary ∂D, and Γ be a portion of the boundary ∂D.

Consider the following Cauchy problem: Given Cauchy data f_D and f_N on Γ, find u on $\Sigma = \partial D \setminus \Gamma$ such that u satisfies

$$\Delta u + k^2 u = 0, \quad \text{in } D, \tag{2.1}$$

$$u = f_D, \quad \text{on } \Gamma, \tag{2.2}$$

$$\frac{\partial u}{\partial n} = f_N, \quad \text{on } \Gamma, \tag{2.3}$$

where n is the unit normal to the boundary ∂D directed into the exterior of D and the wavenumber k is a positive constant. Without loss of generality, we make the assumption on the measured data that $f_D \in H^1(\Gamma)$ and $f_N \in L^2(\Gamma)$, and suppose that the Cauchy problem has a unique solution u in $H^{3/2}(D)$.

Under the assumptions, the solution u can be approximated by some Herglotz wavefunction.

Theorem 2.1. *Let $u \in H^{3/2}(D)$ satisfy the Helmholtz equation; then for every $\varepsilon > 0$, there exists a Herglotz wavefunction v_ψ of the form*

$$v_\psi(x) := \int_\Omega e^{\mathbf{i}kx\cdot d}\psi(d)\mathrm{d}s(d), \quad x \in \mathbb{R}^2 \tag{2.4}$$

for some $\psi \in L^2(\Omega)$, such that

$$\|v_\psi - u\|_{H^1(D)} \le \varepsilon \tag{2.5}$$

and

$$\|v_\psi - u\|_{L^2(\partial D)} + \left\| \frac{\partial v_\psi}{\partial n} - \frac{\partial u}{\partial n} \right\|_{L^2(\partial D)} \le \varepsilon. \tag{2.6}$$

Proof. Let $\tilde{f} = \frac{\partial u}{\partial n} + \mathbf{i}u$ on ∂D. From the assumptions, it is readily seen that $\tilde{f} \in L^2(\partial D)$ and $u \in H^1(D)$ is the solution to the variational problem

$$\int_D [\nabla u \cdot \nabla \overline{w} - k^2 u\overline{w}]\mathrm{d}x + \mathbf{i}\int_{\partial D} u\overline{w}\mathrm{d}s = \int_{\partial D} \tilde{f}\overline{w}\mathrm{d}s, \quad \forall w \in H^1(D).$$

It is sufficient to prove that the traces $\frac{\partial v_g}{\partial n} + \mathbf{i}v_g$ on ∂D are dense in $L^2(\partial D)$. Let $\varphi \in L^2(\partial D)$ satisfy

$$\int_{\partial D} \varphi \overline{\left(\frac{\partial v_g}{\partial n} + \mathbf{i}v_g\right)}\mathrm{d}s = 0$$

for all $g \in L^2(\Omega)$. Then, by substituting in the form of v_g and interchanging the orders of integration we have

$$\int_\Omega g(d)\left\{\int_{\partial D}\varphi(y)\left[\frac{\partial}{\partial n(y)}e^{-\mathbf{i}ky\cdot d} + \mathbf{i}e^{-\mathbf{i}ky\cdot d}\right]\mathrm{d}s(y)\right\}\mathrm{d}s(d) = 0$$

for all $g \in L^2(\Omega)$, and thus

$$\int_{\partial D}\varphi(y)\left[\frac{\partial}{\partial n(y)}e^{-\mathbf{i}ky\cdot d} + \mathbf{i}e^{-\mathbf{i}ky\cdot d}\right]\mathrm{d}s(y) = 0 \qquad \forall d \in \Omega.$$

This means that the single- and double-layer potential

$$w(x) = \int_{\partial D} \varphi(y) \left[\frac{\partial}{\partial n(y)} \Phi(x, y) + \mathbf{i}\Phi(x, y) \right] ds(y)$$

has vanishing far field pattern $w_\infty = 0$, where $\Phi(x, y) = \frac{\mathbf{i}}{4} H_0^1(k|x - y|)$ is the fundamental solution to the Helmholtz equation, with H_0^1 being the Hankel function of the first kind of order 0. Therefore, by Rellich's lemma we know that $w = 0$ in $\mathbb{R}^2 \setminus \bar{D}$. The L^2 jump relation implies $\varphi + K\varphi + \mathbf{i}S\varphi = 0$ on ∂D. Since the operator $I + K + \mathbf{i}S$ has a trivial nullspace in $L^2(\partial D)$ (see [16, Theorem 3.20]), it can be seen that $\varphi = 0$.

Now, from the denseness we know that for any $\varepsilon > 0$, there exists a Herglotz wavefunction v_ψ of the form (2.4) for some $\psi \in L^2(\Omega)$, such that

$$\left\| \frac{\partial v_\psi}{\partial n} + \mathbf{i}v_\psi - \tilde{f} \right\|_{L^2(\partial D)} \leq \varepsilon.$$

Then, the estimate (2.5) follows from the interior regularity results for the Helmholtz equation.

From the trace theorem and (2.5), it can be seen that

$$\|v_\psi - u\|_{L^2(\partial D)} \leq \|v_\psi - u\|_{H^{1/2}(\partial D)} \leq \gamma \|v_\psi - u\|_{H^1(D)} \leq \gamma\varepsilon, \tag{2.7}$$

where $\gamma > 0$ depends only on k and D. And, by the triangle inequality, we have that

$$\begin{aligned} \left\| \frac{\partial v_\psi}{\partial n} - \frac{\partial u}{\partial n} \right\|_{L^2(\partial D)} &\leq \left\| \frac{\partial v_\psi}{\partial n} + \mathbf{i}v_\psi - \tilde{f} \right\|_{L^2(\partial D)} + \|v_\psi - u\|_{L^2(\partial D)} \\ &\leq (\gamma + 1)\varepsilon. \end{aligned} \tag{2.8}$$

The inequalities (2.7) and (2.8) imply the estimate (2.6). □

Remark 2.1. In [15], the author proved the estimate $\left\| \partial_n v_\psi - \partial_n u \right\|_{H^{-1/2}(\partial D)} \leq C\varepsilon$. In Theorem 2.1, the estimate (2.6) with L^2-norm is given, which will be used in our numerical analysis.

From Theorem 2.1, we know that if ψ, any one of the kernel functions, is approximated, then the approximation of v_ψ, and therefore of u, can be obtained. Our aim here is to get the numerical approximation of ψ.

To this end, we define the trace operator $\mathcal{H} : L^2(\Omega) \to L^2(\Gamma) \times L^2(\Gamma)$ by

$$\mathcal{H}g(x) := \begin{pmatrix} \displaystyle\int_\Omega e^{\mathbf{i}kx\cdot d} g(d) ds(d) \\ \displaystyle\frac{\partial}{\partial n_x} \int_\Omega e^{\mathbf{i}kx\cdot d} g(d) ds(d) \end{pmatrix}, \quad x \in \Gamma. \tag{2.9}$$

Then, the following property of the operator $\mathcal{H}$ holds.

Theorem 2.2. *The operator $\mathcal{H} : L^2(\Omega) \to L^2(\Gamma) \times L^2(\Gamma)$ defined by (2.9) is compact and injective.*

Proof. From the embedding theorem, we know that the operator $\mathcal{H}$ is compact. Next, let $\mathcal{H}g = 0$. This means that there exists a Herglotz wavefunction v_g (with kernel g) such that $v_g|_\Gamma = 0$ and $\frac{\partial v_g}{\partial n}|_\Gamma = 0$. From the Green formula and the analyticity of the Herglotz wavefunctions, it can be seen that $v_g = 0$ in $\mathbb{R}^2$ and then $g = 0$. Therefore the operator $\mathcal{H}$ is injective. □

Now, we turn to introducing our numerical algorithm. First, function ϕ is obtained by solving the following integral equations:

$$\mathcal{H}\phi(x) = f(x), \quad x \in \Gamma, \tag{2.10}$$

where $f = (f_D, f_N)^T \in L^2(\Gamma) \times L^2(\Gamma)$, which will approximate the kernel function ψ. And then the Herglotz wavefunction v_ϕ defined by

$$v_\phi(x) := \int_\Omega e^{\mathbf{i}kx\cdot d} \phi(d) ds(d), \quad x \in \mathbb{R}^2$$

will be the approximation of v_ψ, and therefore of the solution.

Remark 2.2. In general, Eqs. (2.10) are not solvable since we cannot assume that the Cauchy data f, especially the measured noisy data f^δ, are in the range $\mathcal{H}(L^2(\Omega))$ of $\mathcal{H}$. Therefore, we will solve Eqs. (2.10) by a regularization method in the next section, and then give the error estimates.

3. A regularization method for solving the integral equations

In this section, we will use the Tikhonov regularization method together with the Morozov discrepancy principle to solve the integral equations (2.10), and then give the error estimates and convergence results.

Due to the ill-posedness, we need to consider the perturbed equations

$$\mathcal{H}\phi^{\delta} = f^{\delta}. \tag{3.1}$$

Here $f^{\delta} \in L^2(\Gamma) \times L^2(\Gamma)$ are measured noisy data satisfying

$$\|f - f^{\delta}\|_{L^2(\Gamma)\times L^2(\Gamma)} \leq \delta.$$

The Tikhonov regularization of integral system (3.1) is adopted to solve the following equations:

$$\alpha\phi_{\alpha}^{\delta} + \mathcal{H}^*\mathcal{H}\phi_{\alpha}^{\delta} = \mathcal{H}^*f^{\delta}.$$

By introducing the regularization operators

$$R_{\alpha} := (\alpha I + \mathcal{H}^*\mathcal{H})^{-1}\mathcal{H}^* \quad \text{for } \alpha > 0,$$

we can achieve the regularized solution $\phi_{\alpha}^{\delta} = R_{\alpha}f^{\delta}$ of Eqs. (3.1). We choose the regularization parameter α by using the Morozov discrepancy principle, and then we have the following result.

Theorem 3.1. *Let ε be a positive constant and $\delta + \varepsilon < \|f^{\delta}\|_{L^2(\Gamma)\times L^2(\Gamma)}$. Let the Tikhonov solution $\phi_{\alpha(\delta)}^{\delta}$ satisfy $\|\mathcal{H}\phi_{\alpha(\delta)}^{\delta} - f^{\delta}\|_{L^2(\Gamma)\times L^2(\Gamma)} = \delta + \varepsilon$ for all $\delta \in (0, \delta_0)$, and $\psi = \mathcal{H}^*z \in \mathcal{H}^*(L^2(\Gamma) \times L^2(\Gamma))$ with $\|z\|_{L^2(\Gamma)\times L^2(\Gamma)} \leq E$. Then*

$$\left\|\phi_{\alpha(\delta)}^{\delta} - \psi\right\|_{L^2(\Omega)} \leq 2\sqrt{(\delta+\varepsilon)E}. \tag{3.2}$$

Here $\psi \in L^2(\Omega)$ is the kernel of some Herglotz function v_{ψ} which satisfies the approximation properties in Theorem 2.1.

Proof. From Theorem 2.1, for every $\varepsilon > 0$ there exists some $\psi \in L^2(\Omega)$ such that

$$\|\mathcal{H}\psi - f\|_{L^2(\Gamma)\times L^2(\Gamma)} \leq \varepsilon.$$

This means that ψ satisfies the equations

$$\mathcal{H}\psi(x) = f_{\varepsilon}(x), \quad x \in \Gamma,$$

for some $f_{\varepsilon} \in L^2(\Gamma) \times L^2(\Gamma)$ with $\|f_{\varepsilon} - f\|_{L^2(\Gamma)\times L^2(\Gamma)} \leq \varepsilon$. Further, we have

$$\|f_{\varepsilon} - f^{\delta}\|_{L^2(\Gamma)\times L^2(\Gamma)} \leq \|f - f^{\delta}\|_{L^2(\Gamma)\times L^2(\Gamma)} + \|f_{\varepsilon} - f\|_{L^2(\Gamma)\times L^2(\Gamma)} \leq \delta + \varepsilon,$$

and thus $\|f_{\varepsilon} - f^{\delta}\|_{L^2(\Gamma)\times L^2(\Gamma)} \leq \delta + \varepsilon \leq \|f^{\delta}\|_{L^2(\Gamma)\times L^2(\Gamma)}$. Now, the statement follows directly from Theorem 2.17 in [17]. □

Remark 3.1. From Theorem 2.1, it can be seen that for $\varepsilon > 0$ the kernel function ψ is not unique. In Theorem 3.1, any one of the kernel functions can be chosen. One can prove that for two different kernel functions ψ_1 and ψ_2, the same estimate (3.2) can be derived.

From Theorem 3.1, the approximation of the kernel function is obtained. Now, define the Herglotz wavefunction $v_{\phi_{\alpha(\delta)}^{\delta}}$ in the form

$$v_{\phi_{\alpha(\delta)}^{\delta}}(x) := \int_{\Omega} e^{\mathbf{i}kx\cdot d}\phi_{\alpha(\delta)}^{\delta}(d)\mathrm{d}s(d), \quad x \in \mathbb{R}^2.$$

Then we have the following main result in this paper.

Theorem 3.2. *Let the assumptions in Theorem 3.1 hold. Then*

$$\left\|v_{\phi_{\alpha(\delta)}^{\delta}} - u\right\|_{H^1(D)} \leq C_1(\delta+\varepsilon)^{\frac{1}{2}}. \tag{3.3}$$

Moreover, the following estimate on boundary Σ holds:

$$\left\|v_{\phi_{\alpha(\delta)}^{\delta}} - u\right\|_{L^2(\Sigma)} + \left\|\frac{\partial v_{\phi_{\alpha(\delta)}^{\delta}}}{\partial n} - \frac{\partial u}{\partial n}\right\|_{L^2(\Sigma)} \leq C_2(\delta+\varepsilon)^{\frac{1}{2}}. \tag{3.4}$$

The positive constants C_1 and C_2 depend only on k, D and E.

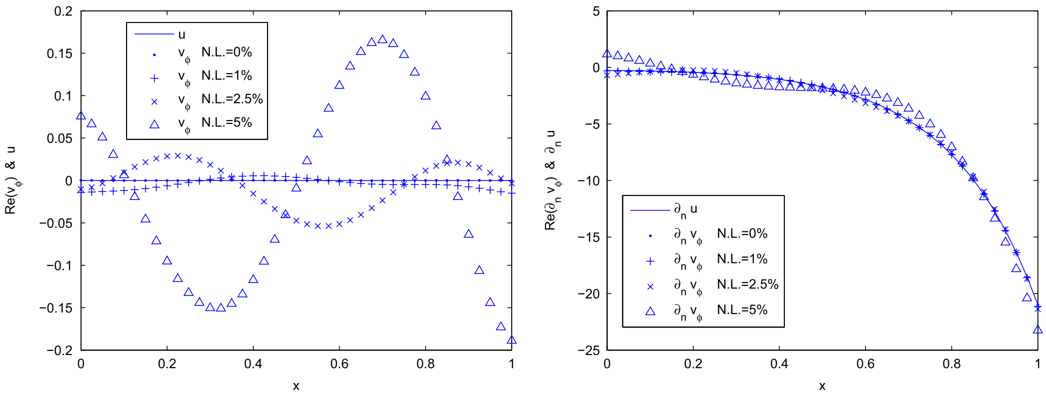

Fig. 1. The exact solution u and the numerical solution v_ϕ on Σ with $k = 5$ for Example 1.

Proof. From the Cauchy–Schwarz inequality, for any $x \in \mathbb{R}^2$, we have

$$\begin{aligned}\left|\left(v_{\phi^\delta_{\alpha(\delta)}} - v_\psi\right)(x)\right| &= \left|\int_\Omega e^{\mathbf{i}kx\cdot d}\left(\phi^\delta_{\alpha(\delta)} - \psi\right)(d)\mathrm{d}s(d)\right| \\ &\le |\Omega|^{\frac{1}{2}}\left\|\phi^\delta_{\alpha(\delta)} - \psi\right\|_{L^2(\Omega)}, \\ \left|\nabla\left(v_{\phi^\delta_{\alpha(\delta)}} - v_\psi\right)(x)\right| &= \left|\int_\Omega \mathbf{i}kde^{\mathbf{i}kx\cdot d}\left(\phi^\delta_{\alpha(\delta)} - \psi\right)(d)\mathrm{d}s(d)\right| \\ &\le k|\Omega|^{\frac{1}{2}}\left\|\phi^\delta_{\alpha(\delta)} - \psi\right\|_{L^2(\Omega)}.\end{aligned}$$

This yields

$$\left\|v_{\phi^\delta_{\alpha(\delta)}} - v_\psi\right\|_{H^1(D)} \le C\left\|\phi^\delta_{\alpha(\delta)} - \psi\right\|_{L^2(\Omega)}.$$

By using the triangle inequality, (2.5) and (3.2), we get the estimate (3.3).

Similarly, from (2.6) and (3.2) we have

$$\left\|\frac{\partial v_{\phi^\delta_{\alpha(\delta)}}}{\partial n} - \frac{\partial u}{\partial n}\right\|_{L^2(\Sigma)} \le C(\delta + \varepsilon)^{\frac{1}{2}}.$$

From the trace theorem and (3.3), we know that

$$\left\|v_{\phi^\delta_{\alpha(\delta)}} - u\right\|_{L^2(\Sigma)} \le \left\|v_{\phi^\delta_{\alpha(\delta)}} - u\right\|_{H^{1/2}(\Sigma)} \le C(\delta + \varepsilon)^{\frac{1}{2}}.$$

The inequalities imply the estimate (3.4). □

4. Numerical examples

In this section, we report two examples to demonstrate the competitiveness of our algorithm. The implementation of the algorithm is based on the MATLAB software. Since ε can be any positive constant, we make the assumption that $\varepsilon \le 10^{-8}$ which makes ε negligible compared with the discretization errors.

Example 1. To test our code, consider the case in which the exact solution to the Cauchy problem is $u(x) = \frac{1}{2k^2}\sin(\sqrt{2}kx_2)(e^{kx_1} + e^{-kx_1})$ (see [13]). Let $D = \{(x_1, x_2)|(x_1 - 0.5)^2 + x_2^2 < 0.5^2, x_2 > 0\}$, $\Gamma = \{(x_1, x_2)|(x_1 - 0.5)^2 + x_2^2 = 0.5^2, x_2 \ge 0\}$ and $\Sigma = \partial D \setminus \Gamma = \{(x_1, x_2)|0 \le x_1 \le 1, x_2 = 0\}$.

Table 1 gives the regularization parameters α chosen by using the Morozov discrepancy principle for noisy data (0%–10% noise). Table 2 presents the corresponding L^2 errors and relative L^2 errors for the approximation of u and $\frac{\partial u}{\partial n}$ on boundary Σ. Visually, Fig. 1 shows the numerical solution for wavenumber $k = 5$ with various levels of noise. From the figure and the tables it can be seen that the numerical solution is a stable approximation to the exact solution. From Fig. 1 and Table 2, it should be noted that the numerical solution converges to the exact solution as the level of noise decreases.

For illustrating the effectiveness of our proposed method (HWM), we give the data comparing the accuracy errors with those of the fundamental solution method (FSM) in Table 3. From the data we can see that our proposed method can give a high order of accuracy.

Table 1
Regularization parameter α for Example 1.

Noise level (N.L.) (%)	α		
	$k = 1$	$k = 3$	$k = 5$
0	1.5×10^{-16}	8.4×10^{-17}	1.0×10^{-16}
1	2.6×10^{-6}	1.4×10^{-7}	3.3×10^{-7}
2.5	3.9×10^{-5}	5.6×10^{-6}	6.1×10^{-6}
5	1.7×10^{-4}	1.2×10^{-4}	1.2×10^{-4}
10	1.0×10^{-3}	1.8×10^{-3}	1.7×10^{-3}

Table 2
Errors for boundary data with different noise levels for Example 1.

Noise level (N.L.) (%)	$\left\| v_{\phi_\alpha^\delta} - u \right\|_{L^2(\sum)}$			$\frac{\left\| \partial_n v_{\phi_\alpha^\delta} - \partial_n u \right\|_{L^2(\sum)}}{\left\| \partial_n u \right\|_{L^2(\sum)}}$		
	$k = 1$	$k = 3$	$k = 5$	$k = 1$	$k = 3$	$k = 5$
0	1.3×10^{-7}	1.2×10^{-6}	3.3×10^{-5}	2.1×10^{-6}	2.2×10^{-5}	1.9×10^{-4}
1	2.5×10^{-4}	6.1×10^{-4}	2.4×10^{-3}	3.0×10^{-3}	1.2×10^{-2}	1.5×10^{-2}
2.5	8.9×10^{-4}	2.7×10^{-3}	9.5×10^{-3}	5.9×10^{-3}	2.7×10^{-2}	4.9×10^{-2}
5	1.7×10^{-3}	4.7×10^{-3}	3.2×10^{-2}	1.6×10^{-2}	5.8×10^{-2}	1.2×10^{-1}
10	6.6×10^{-3}	1.1×10^{-2}	6.1×10^{-2}	3.3×10^{-2}	1.0×10^{-1}	1.6×10^{-1}

Table 3
Comparison of the accuracy errors with those of the fundamental solution method (FSM) for Example 1.

Noise level (%)		$k = 3$		$k = 5$	
		err(u)	err($\frac{\partial u}{\partial n}$)	err(u)	err($\frac{\partial u}{\partial n}$)
1	FSM	7.9×10^{-3}	2.2×10^{-2}	2.0×10^{-2}	2.4×10^{-2}
	HWM	6.1×10^{-4}	1.2×10^{-2}	2.4×10^{-3}	1.5×10^{-2}
2.5	FSM	1.2×10^{-2}	3.4×10^{-2}	4.6×10^{-2}	5.5×10^{-2}
	HWM	2.7×10^{-3}	1.2×10^{-2}	9.5×10^{-3}	4.9×10^{-2}

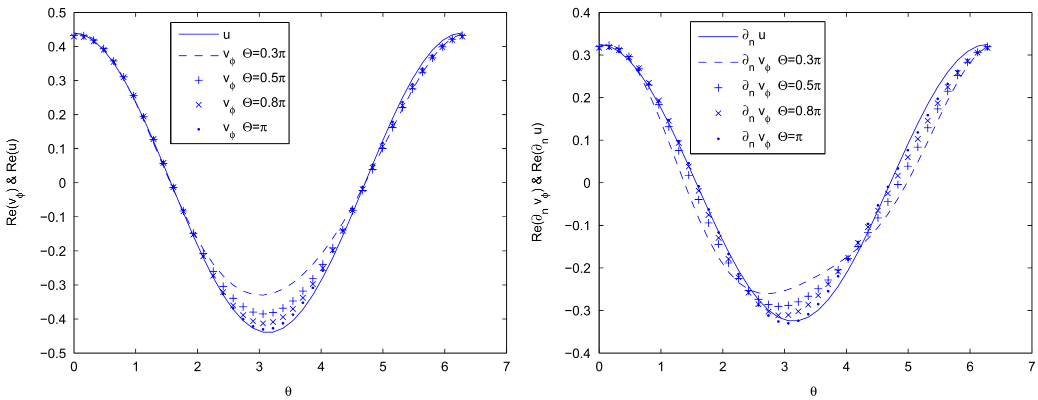

Fig. 2. The exact solution u and the numerical solution v_ϕ on ∂D for $k = 1$, 5% noise and different values of Θ for Example 2.

Example 2. Consider the unit disc $D = \{(x_1, x_2) | x_1^2 + x_2^2 < 1\}$. Let $\Gamma = \{x \in \partial D | 0 < \theta(x) < \Theta\}$ and $\Sigma = \partial D \setminus \Gamma = \{x \in \partial D | \Theta < \theta(x) < 2\pi\}$, where $\theta(x)$ is the polar angle of x and Θ is a specified angle. In this example, we observe the effect of Θ on the numerical solution. Choose $u(x) = J_1(kr)e^{\mathbf{i}\theta}$ as the exact solution, where J_1 is the Bessel function of order 1.

Figs. 2 and 3 compare the numerical results on the whole boundary ∂D for different values of Θ, where the wavenumber $k = 1$ and the noise level N.L. = 5% in Fig. 2, and the wavenumber $k = 5$ and the noise level N.L. = 2% in Fig. 3. From these figures, it is readily seen that the numerical approximation for larger Θ is more stable and accurate. The same conclusion can be drawn from Tables 4–6 which present the regularization parameters and the relative L^2 errors for the numerical solutions on boundary Σ. From Tables 4, 5 and Fig. 2, it is observed that smaller Θ can be chosen for low wavenumber, and for high wavenumber Θ should be larger.

From the numerical results presented in this section, it can be concluded that the Herglotz wavefunction method proposed in this paper is stable and effective.

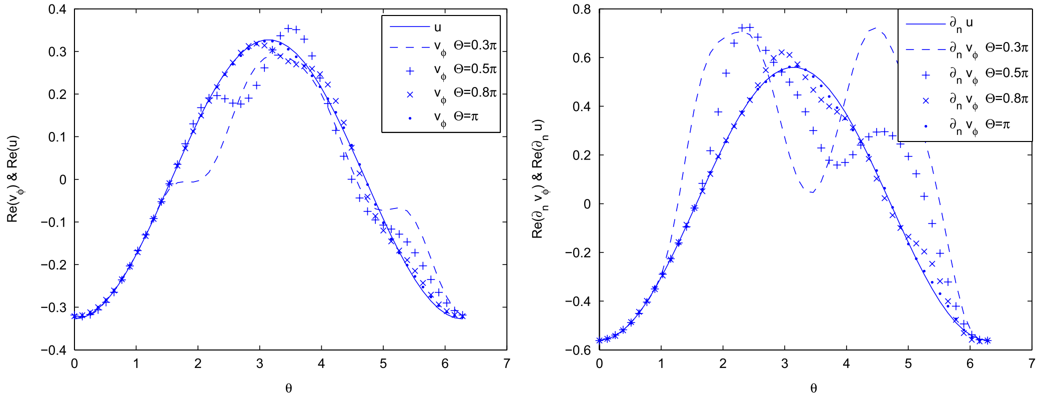

Fig. 3. The exact solution u and the numerical solution v_ϕ on ∂D for $k = 5$, 2% noise and different values of Θ for Example 2.

Table 4
Regularization parameter α for Example 2.

Noise level (N.L.) (%)	α					
	$\Theta = \pi/2$		$\Theta = \pi$		$\Theta = 3\pi/2$	
	$k = 0.5$	$k = 5$	$k = 0.5$	$k = 5$	$k = 0.5$	$k = 5$
0	3.0×10^{-10}	7.1×10^{-12}	2.2×10^{-9}	2.0×10^{-9}	7.8×10^{-9}	8.3×10^{-9}
1	1.1×10^{-3}	2.5×10^{-3}	2.8×10^{-3}	1.5×10^{-2}	3.4×10^{-3}	2.2×10^{-2}
2.5	2.6×10^{-3}	8.1×10^{-3}	4.6×10^{-3}	3.1×10^{-2}	8.3×10^{-3}	5.3×10^{-2}
5	9.8×10^{-3}	2.1×10^{-2}	2.0×10^{-2}	5.6×10^{-2}	2.4×10^{-2}	9.7×10^{-2}
10	2.7×10^{-2}	6.8×10^{-2}	2.6×10^{-2}	1.5×10^{-1}	2.3×10^{-2}	2.0×10^{-1}

Table 5
Relative L^2 errors on boundary Σ for different values of Θ and noise levels for Example 2.

Noise level (N.L.) (%)	$\left\| v_{\phi_\alpha^\delta} - u \right\|_{L^2(\Sigma)} \Big/ \|u\|_{L^2(\Sigma)}$					
	$\Theta = \pi/2$		$\Theta = \pi$		$\Theta = 3\pi/2$	
	$k = 0.5$	$k = 5$	$k = 0.5$	$k = 5$	$k = 0.5$	$k = 5$
0	4.6×10^{-7}	5.2×10^{-4}	4.8×10^{-8}	1.2×10^{-6}	1.4×10^{-8}	1.1×10^{-8}
1	8.5×10^{-3}	1.8×10^{-1}	4.4×10^{-3}	2.1×10^{-2}	3.2×10^{-3}	8.7×10^{-3}
2.5	1.5×10^{-2}	2.3×10^{-1}	9.5×10^{-3}	4.1×10^{-2}	7.8×10^{-3}	2.6×10^{-2}
5	6.2×10^{-2}	3.0×10^{-1}	3.7×10^{-2}	6.3×10^{-2}	3.1×10^{-2}	4.5×10^{-2}
10	9.7×10^{-2}	3.4×10^{-1}	4.3×10^{-2}	1.2×10^{-1}	4.1×10^{-2}	5.4×10^{-2}

Table 6
Relative L^2 errors on boundary Σ for different values of Θ and noise levels for Example 2.

Noise level (N.L.) (%)	$\left\| \partial_n v_{\phi_\alpha^\delta} - \partial_n u \right\|_{L^2(\Sigma)} \Big/ \|\partial_n u\|_{L^2(\Sigma)}$					
	$\Theta = \pi/2$		$\Theta = \pi$		$\Theta = 3\pi/2$	
	$k = 0.5$	$k = 5$	$k = 0.5$	$k = 5$	$k = 0.5$	$k = 5$
0	1.0×10^{-6}	1.0×10^{-3}	1.1×10^{-7}	3.4×10^{-6}	1.9×10^{-8}	3.5×10^{-8}
1	1.0×10^{-2}	3.5×10^{-1}	4.4×10^{-3}	3.0×10^{-2}	3.7×10^{-3}	2.1×10^{-2}
2.5	1.8×10^{-2}	5.4×10^{-1}	1.1×10^{-2}	5.8×10^{-2}	8.8×10^{-3}	4.9×10^{-2}
5	7.0×10^{-2}	6.6×10^{-1}	4.0×10^{-2}	1.0×10^{-1}	2.6×10^{-2}	1.1×10^{-1}
10	1.1×10^{-1}	6.9×10^{-1}	4.3×10^{-2}	2.0×10^{-1}	3.6×10^{-2}	1.4×10^{-1}

5. Conclusions

In this paper, we study the application of the Herglotz wavefunction in solving the Cauchy problem connected with the Helmholtz equation. The numerical method was based on the Tikhonov regularization method together with the Morozov discrepancy principle. Convergence and stability are analyzed with a suitable choice of a regularization method. The method does not require interior or surface meshing, which makes it extremely attractive for solving problems with complicated boundaries, and the function value at any point in the domain can be obtained. The proposed method is more stable with more Cauchy data. The numerical examples indicate that the effectiveness of the numerical solution depends on the domain and the wavenumber. We think that our method will work in the three-dimensional case, and the study of this constitutes our future work.

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