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A numerical method for singularly perturbed turning point problems with an interior layer

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ABSTRACT

The objective of this paper is to present a numerical method for solving singularly perturbed turning point problems exhibiting an interior layer. The method is based on the asymptotic expansion technique and the reproducing kernel method (RKM). The original problem is reduced to interior layer and regular domain problems. The regular domain problems are solved by using the asymptotic expansion method. The interior layer problem is treated by the method of stretching variable and the RKM. Four numerical examples are provided to illustrate the effectiveness of the present method. The results of numerical examples show that the present method can provide very accurate approximate solutions.

1. Introduction

Singularly perturbed problems arise frequently in applications including geophysical fluid dynamics, oceanic and atmospheric circulation, chemical reactions, optimal control, etc. These problems are characterized by the presence of a small parameter that multiplies the highest order derivative, and they are stiff and there exists a boundary or interior layer where the solutions change rapidly.

The numerical treatment of such problems present some major computational difficulties due to the presence of boundary and interior layers. Recently, a large number of special purpose methods have been developed by various authors for singularly perturbed boundary value problems [1–11]. However, discussion on the numerical solutions of singularly perturbed turning point problems is rare. Phaneendra1, Reddy and Soujanya [1] proposed a non-iterative numerical integration method on a uniform mesh for dealing with singularly perturbed turning point problems. Rai and Sharma [2–4] discussed the numerical methods for solving singularly perturbed differential-difference equations with turning points. Natesan, Jayakumar and Vigo-Aguiar [5] introduced a parameter uniform numerical method for singularly perturbed turning point problems exhibiting boundary layers. Kadalbajoo, Arora and Gupta [10] developed a collocation method using artificial viscosity for solving a stiff singularly perturbed turning point problem having twin boundary layers.

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, amongst other fields [12–27,11]. Recently, based on reproducing kernel theory, the authors have discussed two-point boundary value problems, nonlocal boundary value problems and partial differential equations [18–27,11]. However, it is very difficult to extend the application of reproducing kernel theory to singularly perturbed differential equations. Geng [11] developed a method for solving a class of singularly perturbed boundary value problems based on the RKM and a proper transformation. Nevertheless, this method fails to solve singularly perturbed turning point problems.

In this paper, based on the RKM presented in [12,18], an effective numerical method shall be presented for solving singularly perturbed turning point problems exhibiting an interior layer.

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Consider the following singularly perturbed problems:

$$\begin{cases} \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), & -1 < x < 1, \\ u(-1) = \alpha, & u(1) = \gamma, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$, $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently smooth, and such that (1.1) has a unique solution.

The solution of (1.1) exhibits a layer behavior or turning point behavior depending upon the coefficient $a(x)$. The points of the domain where $a(x) = 0$ are known as turning points. The presence of the turning point results in a boundary or interior layer in the solution of the problem and is more difficult to handle as compared to the non-turning point case. In this paper, we consider the case in which the turning point results into an interior layer in the solution of the problem.

We consider problem (1.1) with the following assumptions

$$\begin{cases} a(0) = 0, & a'(0) > 0, \\ b(x) \geq b_0 > 0, & x \in [-1, 1], \\ |a'(x)| \geq \dfrac{|a'(0)|}{2}, & x \in [-1, 1]. \end{cases} \tag{1.2}$$

Under these assumptions the given turning point problem possesses a unique solution exhibiting interior layers at $x = 0$. Unlike boundary layers this layer occurs in the interior of the domains and is considerably weaker.

The rest of the paper is organized as follows. In the next section, the numerical technique for (1.1) is introduced. Error analysis is introduced in Section 3. The numerical examples are given in Section 4. Section 5 ends this paper with a brief conclusion.

2. Numerical method

Following the idea of [5,6], we divide the given interval $[-1, 1]$ into three subintervals $[-1, -K\varepsilon^\rho]$, $[-K\varepsilon^\rho, K\varepsilon^\rho]$ and $[K\varepsilon^\rho, 1]$, where K, ρ are positive real numbers. The subintervals $[-1, -K\varepsilon^\rho]$ and $[K\varepsilon^\rho, 1]$ represent the regular regions, and the interval $[-K\varepsilon^\rho, K\varepsilon^\rho]$ represents the interior layer region. The asymptotic approximation technique is used to solve (1.1) in the regular regions $[-1, -K\varepsilon^\rho]$ and $[K\varepsilon^\rho, 1]$. Then the values of the asymptotic approximation are used as the boundary conditions at the so-called transition points $x = \pm K\varepsilon^\rho$. In the interior layer region $[-K\varepsilon^\rho, K\varepsilon^\rho]$, (1.1) is solved by combining the method of stretching variable and the RKM. After solving both the regular and interior layer problems their solutions are combined to obtain an approximate solution to the original problem over the entire region $[-1, 1]$.

2.1. Solutions of the regular regions problems

Consider (1.1) in right regular region $[K\varepsilon^\rho, 1]$ and left regular region $[-1, -K\varepsilon^\rho]$. Let $U_{R,M}(x)$ and $U_{L,M}(x)$ be the straightforward asymptotic expansions in the intervals $[K\varepsilon^\rho, 1]$ and $[-1, -K\varepsilon^\rho]$ respectively.

$$U_{R,M}(x) = \sum_{k=0}^{M} \varepsilon^k u_k(x), \tag{2.1}$$

where $u_k(x)$ are solutions of the following equations

$$\begin{aligned} &a(x)u_0'(x) - b(x)u_0(x) = f(x), \qquad u_0(1) = \gamma, \\ &a(x)u_1'(x) - b(x)u_1(x) = -u_0''(x), \qquad u_1(1) = 0, \\ &a(x)u_2'(x) - b(x)u_2(x) = -u_1''(x), \qquad u_2(1) = 0, \\ &\ldots \\ &a(x)u_M'(x) - b(x)u_M(x) = -u_{M-1}''(x), \qquad u_M(1) = 0. \end{aligned} \tag{2.2}$$

$$U_{L,M}(x) = \sum_{k=0}^{M} \varepsilon^k v_k(x), \tag{2.3}$$

where $v_k(x)$ are solutions of the following equations

$$\begin{aligned} &a(x)v_0'(x) - b(x)v_0(x) = f(x), \qquad v_0(-1) = \alpha, \\ &a(x)v_1'(x) - b(x)v_1(x) = -v_0''(x), \qquad v_1(-1) = 0, \\ &a(x)v_2'(x) - b(x)v_2(x) = -v_1''(x), \qquad v_2(-1) = 0, \\ &\ldots \\ &a(x)v_M'(x) - b(x)v_M(x) = -v_{M-1}''(x), \qquad v_M(-1) = 0. \end{aligned} \tag{2.4}$$

2.2. Solution of the interior layer region problem

Consider (1.1) in the interior layer region $[-K\varepsilon^\rho, K\varepsilon^\rho]$

$$\begin{cases}\varepsilon\overline{u}''(x)+a(x)\overline{u}'(x)-b(x)\overline{u}(x)=f(x), & -K\varepsilon^\rho<x<K\varepsilon^\rho,\\ \overline{u}(-K\varepsilon^\rho)=\overline{\delta}_0\triangleq U_{L,M}(-K\varepsilon^\rho), & \overline{u}(K\varepsilon^\rho)=\overline{\delta}_1\triangleq U_{R,M}(K\varepsilon^\rho).\end{cases}\tag{2.5}$$

For the interior layer region, we scale $x=\varepsilon^\rho s$ with $\overline{y}(s)\equiv\overline{u}(x)$, then (2.5) becomes

$$\begin{cases}\varepsilon^{1-2\rho}\overline{y}''(s)+\varepsilon^{-\rho}a(\varepsilon^\rho s)\overline{y}'(s)-b(\varepsilon^\rho s)\overline{y}(s)=f(\varepsilon^\rho s), & -K<s<K,\\ \overline{y}(-K)=\overline{\delta}_0, & \overline{y}(K)=\overline{\delta}_1.\end{cases}\tag{2.6}$$

In the following, we shall show how to use the RKM to solve (2.6) in detail.

Setting $L\overline{y}(s)=\varepsilon^{1-2\rho}\overline{y}''(s)+\varepsilon^{-\rho}a(\varepsilon^\rho s)\overline{y}'(s)-b(\varepsilon^\rho s)\overline{y}(s)$ and $h(s)=f(\varepsilon^\rho s)$, (2.6) becomes

$$\begin{cases}L\overline{y}(s)=h(s), & -K<x<K,\\ y(-K)=\overline{\delta}_0, & y(K)=\overline{\delta}_1.\end{cases}\tag{2.7}$$

Introducing a new unknown function

$$\overline{Y}(s)=\overline{y}(s)-\phi(s)\tag{2.8}$$

where $\phi(s)=\overline{\gamma}_0+\overline{\gamma}_1 s$ and satisfies $\phi(-K)=\overline{\delta}_0$, $\phi(K)=\overline{\delta}_1$.

Problem (2.7) with inhomogeneous boundary conditions can be equivalently reduced to the problem of finding a function $\overline{Y}(s)$ satisfying

$$\begin{cases}L\overline{Y}(s)=\overline{F}(s), & -K<s<K,\\ \overline{Y}(-K)=0, & \overline{Y}(K)=0.\end{cases}\tag{2.9}$$

where $\overline{F}(s)=h(s)-L\phi(s)$.

To solve (2.9) using the RKM, first define and then construct a reproducing kernel space $W^4[-K,K]$ in which every function satisfies the homogeneous boundary conditions of (2.9).

Definition 2.1. $W^4[-K,K]=\{u(x)\mid u'''(x)$ is absolutely continuous, $u^{(4)}(x)\in L^2[-K,K], u(-K)=0, u(K)=0\}$. The inner product and norm in $W^4[-K,K]$ are given, respectively, by

$$(u(y),v(y))_4=u(-K)v(-K)+u'(-K)v'(-K)+u(K)v(K)+u'(K)v'(K)+\int_{-K}^{K}u^{(4)}v^{(4)}dy$$

and

$$\|u\|_4=\sqrt{(u,u)_4},\quad u,v\in W^4[-K,K].$$

Theorem 2.1. *$W^4[-K,K]$ is a reproducing kernel space and its reproducing kernel is*

$$k(x,y)=\begin{cases}k_1(x,y), & y\le x,\\ k_1(y,x), & y>x,\end{cases}\tag{2.10}$$

where $k_1(x,y)=\frac{1}{20160K^4}[26K^{11}+(-56x^2+6yx-56y^2)K^9+(35x^4-16yx^3+126y^2x^2-16y^3x+35y^4)K^7+2520K^6+(-7x^6+21yx^5-105y^2x^4+66y^3x^3-105y^4x^2+21y^5x-7y^6)K^5+2(x^7-7yx^6+21y^2x^5-35y^3x^4+35y^4x^3-21(y^5+60)x^2+7y(y^5+180)x-y^2(y^5+1260))K^4+xy(3x^6-7yx^5+21y^2x^4+21y^4x^2-7y^5x+3y^6)K^3-2520xy(x^2-yx+y^2)K^2-x^3y^3(x^4+y^4)K+2520x^3y^3]$.

In [18], we give the method of obtaining a reproducing kernel. For the proof, please refer to Appendix B in [18].

Definition 2.2. $W^1[-K,K]=\{u(x)\mid u(x)$ is an absolutely continuous real value function, $u'(x)\in L^2[-K,K]\}$. The inner product and norm in $W^1[0,1]$ are given, respectively, by

$$(u(y),v(y))_1=u(-K)v(-K)+\int_{-K}^{K}u'v'dy$$

and

$$\|u\|_1=\sqrt{(u,u)_1},\quad u,v\in W^1[-K,K].$$

Theorem 2.2. *$W^1[-K,K]$ is a reproducing kernel space and its reproducing kernel is*

$$\overline{k}(x,y)=\begin{cases}1+K+y, & y\le x,\\ 1+K+x, & y>x,\end{cases}\tag{2.11}$$

In (2.9), it is clear that $L : W^4[-K,K] \to W^1[-K,K]$ is a bounded linear operator. Put $\varphi_i(s) = \overline{k}(s,s_i)$ and $\psi_i(s) = L^*\varphi_i(s)$ where L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(s)\}_{i=1}^{\infty}$ of $W^4[-K,K]$ can be derived from the Gram–Schmidt orthogonalization process applied to $\{\psi_i(s)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(s) = \sum_{k=1}^{i} \beta_{ik}\psi_k(s), \quad (\beta_{ii} > 0,\ i = 1, 2, \ldots).$$

Theorem 2.3. *For (2.9), if $\{s_i\}_{i=1}^{\infty}$ is dense in $[-K,K]$, then $\{\psi_i(s)\}_{i=1}^{\infty}$ is a complete system of $W^4[-K,K]$ and $\psi_i(s) = L_t k(s,t)|_{t=s_i}$.*

Proof. Note here that

$$\begin{aligned}\psi_i(s) &= (L^*\varphi_i)(s) = ((L^*\varphi_i)(t), k(s,t)) \\ &= (\varphi_i(t), L_t k(s,t)) = L_t k(s,t)|_{t=s_i}.\end{aligned}$$

Hence, $\psi_i(s) \in W^4[-K,K]$.

For each fixed $u(s) \in W^4[-K,K]$, let $(u(s), \psi_i(s)) = 0$, $(i = 1, 2, \ldots)$, which means that

$$(u(s), (L^*\varphi_i)(s)) = (Lu(\cdot), \varphi_i(\cdot)) = (Lu)(s_i) = 0.$$

Since $\{s_i\}_{i=1}^{\infty}$ is dense in $[-K,K]$, $(Lu)(s) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. Completing the proof. □

Theorem 2.4. *If $\{s_i\}_{i=1}^{\infty}$ is dense in $[-K,K]$ and the solution of (2.9) is unique, then the solution of (2.9) is*

$$\overline{Y}(s) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\overline{F}(s_k)\overline{\psi}_i(s). \tag{2.12}$$

Proof. Applying Theorem 2.3, it follows that $\{\overline{\psi}_i(s)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W^4[-K,K]$.

Note that $(w(s), \varphi_i(s)) = w(s_i)$ for each $w(x) \in W^1[-K,K]$; hence we have

$$\begin{aligned}\overline{Y}(s) &= \sum_{i=1}^{\infty}(\overline{Y}(s), \overline{\psi}_i(s))\overline{\psi}_i(s) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(\overline{Y}(s), L^*\varphi_k(s))\overline{\psi}_i(s) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(L\overline{Y}(s), \varphi_k(s))\overline{\psi}_i(s) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(\overline{F}(s), \varphi_k(s))\overline{\psi}_i(s) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\overline{F}(s_k)\overline{\psi}_i(s)\end{aligned} \tag{2.13}$$

and the proof of the theorem is complete. □

The approximate solution $\overline{Y}_N(s)$ can be obtained by taking finitely many terms in the series representation of $\overline{Y}(s)$ and

$$\overline{Y}_N(s) = \sum_{i=1}^{N}\sum_{k=1}^{i} \beta_{ik}\overline{F}(s_k)\overline{\psi}_i(s). \tag{2.14}$$

Theorem 2.5. *The approximate solution $\overline{Y}_N(s)$ and its derivatives $\overline{Y}_N^{(i)}(x)$, $i = 1, 2$ are all convergent.*

Proof. Noting that $W^4[-K,K]$ is a Hilbert space, we have

$$\|\overline{Y}_N(s) - \overline{Y}(s)\|_4 \to 0, \qquad N \to \infty.$$

Since

$$\begin{aligned}|\overline{Y}_N^{(i)}(s) - \overline{Y}^{(i)}(s)| &= \left|\left(\overline{Y}_N(t) - \overline{Y}(t), \frac{\partial^i k(s,t)}{\partial s^i}\right)_4\right| \\ &\le \|\overline{Y}_N - \overline{Y}\|_4 \left\|\frac{\partial^i k(s,t)}{\partial s^i}\right\|_4, \quad (i = 0, 1, 2),\end{aligned}$$

we must have

$$|\overline{Y}_N^{(i)}(s) - \overline{Y}^{(i)}(s)| \to 0, \qquad N \to \infty \ (i = 0, 1, 2).$$

Therefore, the approximate solution $\overline{Y}_N(x)$ and its derivatives $\overline{Y}_N^{(i)}(x)$, $i = 1, 2$ are all convergent, and the proof is complete. □

Combining (2.8) and (2.14), leads to the approximate solution of (2.6)

$$\overline{y}_N(s) = \overline{Y}_N(s) + \phi(s) = \sum_{i=1}^{N}\sum_{k=1}^{i} \beta_{ik}\overline{F}(s_k)\overline{\psi}_i(s) + \phi(s). \tag{2.15}$$

Furthermore, the approximation of the solution of interior layer region problem (2.5) can be obtained by

$$U_{I,N}(x) = \overline{y}_N\left(\frac{x}{\varepsilon^\rho}\right). \tag{2.16}$$

By (2.1), (2.3) and (2.16), the approximate solution of (1.1) on the entire region $[-1, 1]$ is immediately obtained

$$U_{M,N}(x) = \begin{cases} U_{L,M}(x), & -1 \le x < -K\varepsilon^\rho, \\ U_{I,N}(x), & -K\varepsilon^\rho \le x \le K\varepsilon^\rho, \\ U_{R,M}(x), & K\varepsilon^\rho < x \le 1. \end{cases} \tag{2.17}$$

3. Error analysis

Firstly, we discuss the errors of solution in the regular domain.

Theorem 3.1. *The approximate solutions $U_{R,M}(x)$ and $U_{L,M}(x)$ of regular regions satisfy*

$$\|U_{R,M}(x) - u(x)\|_\infty = \max_{s\in[K\varepsilon^\rho,1]} |U_{R,M}(x) - u(x)| \le d_1\varepsilon^{M+1},$$

and

$$\|U_{L,M}(x) - u(x)\|_\infty = \max_{s\in[-1,-K\varepsilon^\rho]} |U_{L,M}(x) - u(x)| \le d_2\varepsilon^{M+1},$$

where d_1, d_2 are positive constants.

Please refer to [28] for the proof of Theorem 3.1.

Theorem 3.1 shows that it is enough to take a small value of M.

Secondly, we discuss the errors of solution in the interior layer region.

Suppose that the exact values of $u(-K\varepsilon^\rho)$ and $u(K\varepsilon^\rho)$ are δ_0 and δ_1. From Theorem 3.1, one obtains

$$|\delta_0 - \overline{\delta}_0| \le d_2\varepsilon^{M+1}, \qquad |\delta_1 - \overline{\delta}_1| \le d_1\varepsilon^{M+1}. \tag{3.1}$$

Replacing $\overline{\delta}_0$ and $\overline{\delta}_1$ with δ_0 and δ_1 in (2.5) and (2.9), we have

$$\begin{cases} \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), & -K\varepsilon^\rho < x < K\varepsilon^\rho, \\ u(-K\varepsilon^\rho) = \delta_0, \qquad u(K\varepsilon^\rho) = \delta_1. \end{cases} \tag{3.2}$$

and

$$\begin{cases} LY(s) = F(s), & -K < s < K, \\ Y(-K) = 0, \qquad Y(K) = 0. \end{cases} \tag{3.3}$$

By using the RKM, we can obtain the approximate solutions $Y_N(s)$ of (3.3) in the space $W^4[-K, K]$. Let $u(x)$ and $Y(s)$ be the exact solutions of (3.2) and (3.3), respectively.

From [10], we have the following lemmas.

Lemma 3.1 (*Minimum Principle*)**.** *Consider (2.9). If $Y(-K) \ge 0, Y(K) \ge 0$ and $LY(s) \ge 0$, $\forall\, s \in (-K, K)$, then $Y(s) \ge 0$, $\forall\, s \in (-K, K)$.*

Lemma 3.2. *If $\overline{Y}(s)$ is the solution of (2.9), then there exists a positive C such that*

$$\|\overline{Y}(s)\|_\infty \le C\|\overline{F}\|_\infty.$$

Lemma 3.3. *If $-K = s_1 < s_2 < \cdots < s_N = K$, and if $a(\varepsilon^\rho s), b(\varepsilon^\rho s), \overline{F}(s) \in C^2[-K, K]$, then the approximate solution $\overline{Y}_N(s)$ of (2.9) satisfies*

$$\|L\overline{Y}_N - \overline{F}\|_\infty = \max_{s\in[-K,K]} |L\overline{Y}_N - \overline{F}| \le d_3h^2,$$

where d_3 is a positive constant, $h = \max_{1\le i\le N-1} |s_{i+1} - s_i|$.

Proof. For the details of the proof, one may refer to [15]. □

Theorem 3.2. *The approximate solution* $\overline{Y}_N(s)$ *of* (2.9) *satisfies*

$$\|\overline{Y}_N(s) - Y(s)\|_\infty \le d_5\, h^2 + d_6\varepsilon^{M+1},$$

where d_5, d_6 *are positive constants.*

Proof. By Theorem 3.1, it follows that

$$\|F - \overline{F}\|_\infty \le d_4\varepsilon^{M+1}. \tag{3.4}$$

From (3.4) and Lemma 3.3, one obtains

$$\begin{aligned} \|L\overline{Y}_N - F\|_\infty &= \|L\overline{Y}_N - \overline{F} + \overline{F} - F\|_\infty \\ &\le \|L\overline{Y}_N - \overline{F}\|_\infty + \|\overline{F} - F\|_\infty \\ &\le d_3h^2 + d_4\varepsilon^{M+1}. \end{aligned} \tag{3.5}$$

Note that $\overline{Y}_N(s) - Y(s)$ is the solution of

$$\begin{cases} LV = L\overline{Y}_N - F, & -K < s < K, \\ V(-K) = 0, & V(K) = 0. \end{cases}$$

Applying Lemma 3.2, we get

$$\|\overline{Y}_N(s) - Y(s)\|_\infty \le d_5\, h^2 + d_6\varepsilon^{M+1},$$

where d_5, d_6 are positive constants. □

Theorem 3.3. *The approximate solution* $U_{I,N}(x)$ *of the interior layer region problem satisfies*

$$\|U_{I,N}(x) - u(x)\|_\infty = \max_{s\in[-K\varepsilon^\rho, K\varepsilon^\rho]} |U_{I,N}(x) - u(x)| \le d_7\, h^2 + d_8\varepsilon^{M+1}.$$

Proof. Combining (2.8) and Theorem 3.2, we have

$$\|\overline{y}_N(s) - y(s)\|_\infty = \max_{s\in[-K,K]} |\overline{y}_N(s) - y(s)| \le d_7\, h^2 + d_8\varepsilon^{M+1},$$

where d_7, d_8 are positive constants.

Note that

$$\max_{x\in[-K\varepsilon^\rho, K\varepsilon^\rho]} |U_{I,N}(x) - u(x)| = \max_{s\in[-K,K]} |\overline{y}_N(s) - y(s)|.$$

Hence,

$$\|U_{I,N}(x) - u(x)\|_\infty \le d_7\, h^2 + d_8\varepsilon^{M+1}. \quad \square$$

From Theorems 3.1 and 3.3, the following theorem can be obtained.

Theorem 3.4. *The function* $U_{M,N}(x)$ *defined in* (2.17) *is an approximate solution in the sense that*

$$\|U_{M,N}(x) - u(x)\|_\infty = \max_{x\in[-1,1]} |U_{M,N}(x) - u(x)| \le d\, h^2 + dd\, \varepsilon^{M+1},$$

where d, dd *are positive constants,* $h = \max_{1\le i\le N-1} |s_{i+1} - s_i|$.

4. Numerical examples

Example 4.1. Consider the following singular perturbation problem [1]

$$\begin{cases} \varepsilon u''(x) + 2xu'(x) = 0, & -1 < x < 1, \\ u(-1) = -1, & u(1) = 1. \end{cases}$$

This problem has an internal layer of width $O(\sqrt{\varepsilon})$. The exact solution is $u(x) = \operatorname{erf}(\frac{x}{\sqrt{\varepsilon}})$. Using the present method, taking $\rho = \frac{1}{2}, M = 1, K = 10, s_i = -K + (i-1)h, h = \frac{2K}{N-1},\ i = 1, 2, \ldots, N$. All computations are performed by using Mathematica 7.0. It is easy to obtain $U_{L,1} = -1$, $U_{R,1} = 1$. The relative errors using the present method (PM) are compared with [1] in Table 1 for $\varepsilon = 2^{-5}, 2^{-10}, 2^{-30}$. Taking $\varepsilon = 10^{-5}$, $N = 200, 400$, the absolute errors are shown in Fig. 1.

Table 1
Comparison of relative errors for Example 4.1.

ε	$h = 2^{-5}$ ([1])	$h = 2^{-5}$ (PM)	$h = 2^{-6}$ ([1])	$h = 2^{-6}$ (PM)	$h = 2^{-7}$ ([1])	$h = 2^{-7}$ (PM)
2^{-5}	1.53×10^{-2}	8.40×10^{-5}	1.45×10^{-2}	5.20×10^{-6}	1.20×10^{-2}	3.20×10^{-7}
2^{-10}	2.63×10^{-3}	8.33×10^{-5}	1.20×10^{-3}	5.15×10^{-6}	2.40×10^{-3}	3.30×10^{-7}
2^{-30}	1.07×10^{-5}	8.40×10^{-5}	1.07×10^{-5}	5.22×10^{-6}	1.07×10^{-5}	3.33×10^{-7}

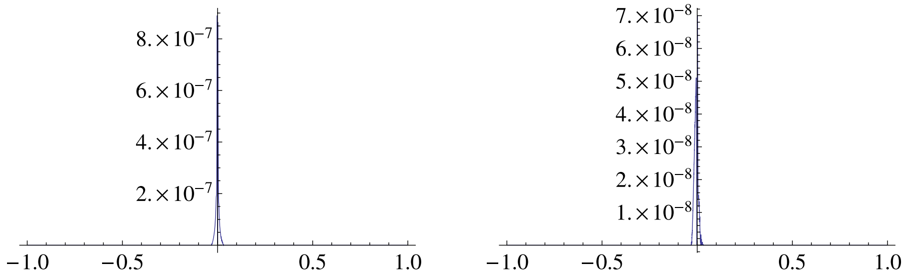

Fig. 1. Absolute errors between the approximate solution and exact solution of Example 4.1 for $\varepsilon = 10^{-5}$ (the left: $N = 200$; the right: $N = 400$).

Example 4.2. Consider the following singular perturbation problem [1]

$$\begin{cases} \varepsilon u''(x) + xu'(x) - u(x) = 0, & -1 < x < 1, \\ u(-1) = 1, \quad u(1) = 2. \end{cases}$$

This problem has an internal layer of width $O(\sqrt{\varepsilon})$. The exact solution is

$$u(x) = \frac{2\sqrt{\varepsilon}\left(x + 3e^{-\frac{x^2-1}{2\varepsilon}}\right) + e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}x\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) + 3e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}x\operatorname{erf}\left(\frac{x}{\sqrt{2}\sqrt{\varepsilon}}\right)}{2e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) + 4\sqrt{\varepsilon}}.$$

Using the present method, taking $\rho = \frac{1}{2}, M = 1, K = 10, s_i = -K + (i-1)h, h = \frac{2K}{N-1}, \ i = 1, 2, \ldots, N$. It is easy to obtain $U_{L,1} = -x, U_{R,1} = 2x$. The numerical results compared with [1] are given in Table 2 for $\varepsilon = 2^{-5}, 2^{-10}$. Taking $\varepsilon = 10^{-5}, N = 100, 200$, the absolute errors are shown in Fig. 2.

Example 4.3. Consider the following singular perturbation problem [2]

$$\begin{cases} \varepsilon u''(x) + 2(2x-1)u'(x) - 4u(x) = 0, & 0 < x < 1, \\ u(0) = 1, \quad u(1) = 1. \end{cases}$$

This problem has an internal layer of width $O(\sqrt{\varepsilon})$. The exact solution is

$$u(x) = -\frac{e^{\frac{1}{2\varepsilon} - \frac{(1-2x)^2}{2\varepsilon}}\left(2e^{\frac{(1-2x)^2}{2\varepsilon}}\sqrt{2\pi}x\operatorname{erf}\left(\frac{1-2x}{\sqrt{2}\sqrt{\varepsilon}}\right) - e^{\frac{(1-2x)^2}{2\varepsilon}}\sqrt{2\pi}\operatorname{erf}\left(\frac{1-2x}{\sqrt{2}\sqrt{\varepsilon}}\right) - 2\sqrt{\varepsilon}\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right) + 2\sqrt{\varepsilon}}.$$

Using the present method, taking $\rho = \frac{1}{2}, M = 1, K = 10, s_i = -K + (i-1)h, h = \frac{2K}{N-1}, \ i = 1, 2, \ldots, N$. It is easy to obtain $U_{L,1} = 1 - 2x, U_{R,1} = 2x - 1$. The numerical results compared with [2] are given in Table 3. Taking $\varepsilon = 10^{-6}, N = 100, 200$, the absolute errors are shown in Fig. 3.

Example 4.4. Consider the following singular perturbation problem [7]

$$\begin{cases} \varepsilon u''(x) + u'(x) = -(1 + 2x), & 0 < x < 1, \\ u'(0) = -1, \quad u(1) = 0, \end{cases}$$

whose exact solution is given by

$$u(x) = 2\varepsilon^2\left(e^{-\frac{x}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}\right) - x(x - 2\varepsilon + 1) + 2(1 - \varepsilon).$$

Using the present method, taking $\rho = \frac{1}{2}, M = 1, K = 10, s_i = (i-1)h, h = \frac{K}{N-1}, \ i = 1, 2, \ldots, N = 100$. It is easy to obtain $U_{R,1} = 2 - x - x^2 + 2\varepsilon(x - 1)$. The numerical results compared with [7] are given in Table 4.

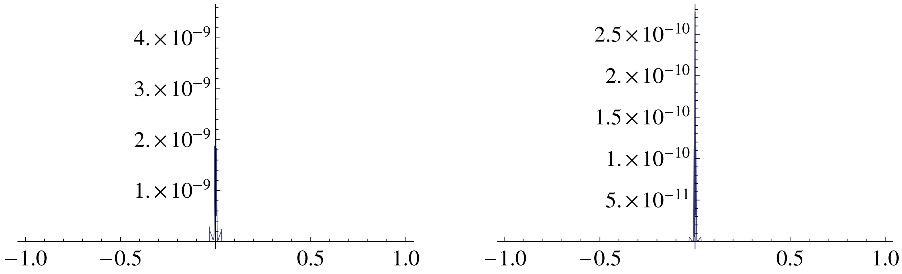

Fig. 2. Absolute errors between the approximate solution and exact solution of Example 4.2 for $\varepsilon = 10^{-5}$ (the left: $N = 100$; the right: $N = 200$).

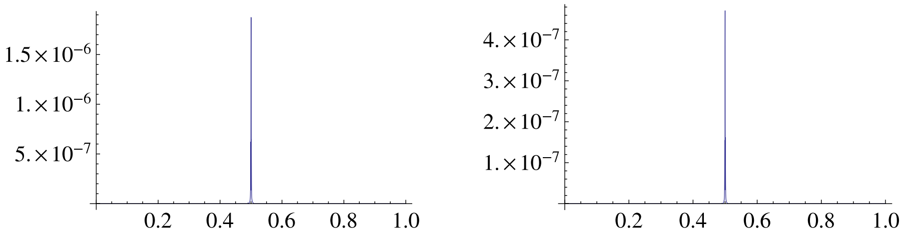

Fig. 3. Absolute errors between the approximate solution and exact solution of Example 4.3 for $\varepsilon = 10^{-6}$ (the left: $N = 100$; the right: $N = 200$).

Table 2
Comparison of numerical results for Example 4.2 with $N = 100$.

x	$\varepsilon = 2^{-5}$ (Exact solution)	[1]	PM	$\varepsilon = 2^{-10}$ (Exact solution)	[1]	PM
−1.000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
−0.875	0.875000	0.875005	0.875000	0.875000	0.875000	0.875000
−0.500	0.500367	0.501224	0.500367	0.500000	0.500001	0.500000
−0.125	0.199865	0.200683	0.199865	0.125001	0.127626	0.125001
−0.000	0.211571	0.210547	0.211571	0.0374008	0.021125	0.0374009
0.125	0.324865	0.325683	0.324865	0.250001	0.252626	0.250001
0.500	1.000370	1.001224	1.000370	1.000000	1.000001	1.000000
0.875	1.750000	1.750005	1.750000	1.750000	1.750000	1.750000
1.000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000

Table 3
Comparison of maximum absolute errors for Example 4.3.

ε	$N = 128$ ([2])	$N = 128$ (PM)	$N = 256$ ([2])	$N = 256$ (PM)
2^{-6}	7.40×10^{-4}	1.41×10^{-4}	2.53×10^{-4}	3.53×10^{-5}
2^{-10}	9.86×10^{-4}	3.52×10^{-5}	4.89×10^{-4}	8.10×10^{-6}
2^{-14}	1.79×10^{-3}	8.05×10^{-6}	4.92×10^{-4}	2.60×10^{-6}
2^{-20}	9.92×10^{-4}	2.61×10^{-7}	4.92×10^{-4}	6.00×10^{-8}

Table 4
Comparison of absolute errors for Example 4.4 with $\varepsilon = 10^{-3}$.

x	Method in [7]	Present method
0.00000	3.50×10^{-8}	2.43×10^{-10}
0.00010	3.47×10^{-8}	2.39×10^{-10}
0.00050	3.35×10^{-8}	2.22×10^{-10}
0.00090	3.25×10^{-8}	2.10×10^{-10}
0.00300	2.84×10^{-8}	1.71×10^{-10}
0.00700	2.16×10^{-8}	1.22×10^{-10}
0.01000	1.66×10^{-8}	9.08×10^{-11}
0.02000	8.27×10^{-15}	4.12×10^{-15}
0.10000	1.80×10^{-3}	5.85×10^{-18}
0.30000	7.00×10^{-3}	1.06×10^{-17}
0.70000	3.00×10^{-3}	6.78×10^{-17}
0.90000	5.00×10^{-3}	1.98×10^{-16}
1.00000	0	0

5. Conclusion

In this paper, a new method is proposed for solving singularly perturbed turning point problems with an interior layer. The present method is based on the RKM, the asymptotic expansion technique and the method of stretching variable. The major advantage of the method is that it can produce good globally continuous approximate solutions. The results from the numerical example show that the present method is an accurate and reliable analytical technique for treating singularly perturbed turning point problems with an interior layer.

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