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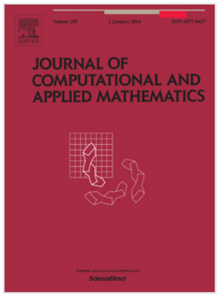

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A uniform monotone alternating direction scheme for nonlinear singularly perturbed parabolic problems

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Abstract

This paper deals with a monotone alternating direction (ADI) scheme for solving nonlinear singularly perturbed parabolic problems. Monotone sequences, based on the method of upper and lower solutions, are constructed for a nonlinear difference scheme which approximates the nonlinear parabolic problem. The monotone sequences possess quadratic convergence rate. An analysis of uniform convergence of the monotone ADI scheme to the solutions of the nonlinear difference scheme and to the continuous problem is given. Numerical experiments are presented.

Keywords: nonlinear parabolic problem, singular perturbation, monotone ADI scheme, quadratic convergence, uniform convergence

1. Introduction

In this paper we give a numerical treatment for the nonlinear singularly perturbed parabolic problem in the form

$$u_t - Lu + f(x,y,t,u) = 0, \quad Lu \equiv \mu^2(u_{xx} + u_{yy}), \tag{1}$$
$$(x,y,t) \in Q = \omega \times (0,T], \quad \omega = \{0 < x < 1\} \times \{0 < y < 1\},$$
$$u(x,y,t) = 0, \quad (x,y,t) \in \partial\omega \times (0,T],$$
$$u(x,y,0) = \psi(x,y), \quad (x,y) \in \overline{\omega},$$

where μ is a small positive parameter, $\partial\omega$ is the boundary of ω, the functions f and ψ are smooth in their respective domains, and f satisfies the constraint

$$f_u \geq \beta, \quad (x,y,t,u) \in \overline{\omega} \times [0,T] \times (-\infty,\infty), \tag{2}$$

where $\beta = \text{const} > 0$. This assumption can always be obtained via a change of variables. Indeed, introduce $z(x,y,t) = e^{-\lambda t}u(x,y,t)$, where λ is a constant. Now, $z(x,y,t)$ satisfies (1) with $\varphi = \lambda z + e^{-\lambda t} f(x,y,t,e^{\lambda t}z)$, instead of f, and we have $\varphi_z = \lambda + f_u$. Thus, if $\lambda \geq -\min f_u + \beta$, where minimum is taking over the domain from (2), we conclude $\varphi_z \geq \beta$.

For $\mu \ll 1$, the problem is singularly perturbed and characterized by boundary layers (regions with rapid change of solutions) near boundary $\partial\omega$ (see [2] for details). Various reaction-diffusion-type problems in chemical, physical and engineering sciences are described by problem (1).

In the study of numerical methods for nonlinear singularly perturbed problems, the two major points to be developed are: i) constructing robust difference schemes (this means that unlike classical schemes, the error does not increase to infinity, but rather remains bounded, as the small parameter approaches zero); ii) obtaining reliable and efficient computing algorithms for solving nonlinear discrete problems.

We shall employ a two-time level implicit scheme for approximating the semilinear problem (1). Alternating direction implicit (ADI) methods are very efficient methods for solving two or three dimensional parabolic problems. At each time-step, the ADI method reduces two or three dimensional problems to a succession of one dimensional problems, and, usually, one needs only to solve a sequence of tridiagonal systems. In the case of the nonlinear reaction function f in (1), the corresponding discrete problems become systems of nonlinear algebraic equations.

A fruitful method for solving the nonlinear difference scheme is the method of upper and lower solutions and its associated monotone iterations. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem. The above monotone iterative method is well known and has been widely used for continuous and discrete elliptic and parabolic boundary value problems. Most of publications on this topic involve monotone iterative schemes whose rate of convergence is linear. Accelerated monotone iterative methods for solving discrete parabolic problems are presented in [3, 9, 13]. An advantage of this accelerated approach is that it leads to sequences which converge quadratically.

In [4, 6], the ADI method based on the Douglas-Rachford ADI scheme [5] is applied to linear singularly perturbed reaction-diffusion problems of type (1). This ADI method is shown to be uniformly convergent (robust) with respect to the small parameter μ on special nonuniform meshes.

In this paper, we construct a nonlinear ADI scheme based on a modification of the Douglas-Rachford ADI scheme [5]. A monotone iterative method with quadratic convergence rate from [3] is in use for solving nonlinear discrete systems. We consider the case when on each time level a nonlinear difference scheme is solved inexactly, and give an analysis of convergence of a monotone ADI scheme on the whole interval of integration $[0, T]$.

The structure of the paper as follows. In Section 2, we introduce a nonlinear difference scheme for the numerical solution of (1), (2). In Section 3, we construct a nonlinear ADI scheme. The new monotone ADI scheme is presented in Section 4. Monotone properties of the ADI scheme are established. Based on these properties, existence and uniqueness of the solution to the nonlinear ADI scheme are proved. In Section 5, we show that on each time level the monotone iterative method possesses quadratic convergence rate. We analyze a convergence rate of the monotone ADI scheme on the whole interval of integration $[0, T]$. Section 6 deals with uniform convergence of the monotone ADI scheme to the nonlinear parabolic problem (1), (2). The final Section 7 presents results of numerical experiments.

2. The nonlinear difference scheme

On $\overline{Q}$ introduce a rectangular mesh $\overline{\omega}^h \times \overline{\omega}^\tau$, $\overline{\omega}^h = \overline{\omega}^{hx} \times \overline{\omega}^{hy}$:

$$\overline{\omega}^{hx} = \{x_i,\ 0 \le i \le N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\}, \tag{3}$$

$$\overline{\omega}^{hy} = \left\{y_j,\ 0 \le j \le N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\right\},$$

$$\overline{\omega}^{\tau} = \{t_k,\ 0 \le k \le N_\tau;\ t_0 = 0,\ t_{N_\tau} = T;\ \tau_k = t_k - t_{k-1}\}.$$

For solving (1), consider the nonlinear implicit difference scheme

$$\mathcal{L}U(p, t_k) + f(p, t_k, U) - \tau_k^{-1} U(p, t_{k-1}) = 0, \quad (p, t_k) \in \omega^h \times (\overline{\omega}^\tau \setminus \{0\}), \tag{4}$$

with the boundary and initial conditions

$$U(p, t_k) = 0, \quad (p, t_k) \in \partial\omega^h \times (\overline{\omega}^\tau \setminus \{0\}),$$

$$U(p, 0) = \psi(p), \quad p \in \overline{\omega}^h,$$

where $\partial\omega^h$ is the boundary of $\overline{\omega}^h$. When no confusion arises, we write $f(p, t_k, U(p, t_k)) = f(p, t_k, U)$. The difference operator $\mathcal{L}$ is defined by

$$\mathcal{L}U(p, t_k) = \mathcal{L}^h U(p, t_k) + \tau_k^{-1} U(p, t_k),$$

$$\mathcal{L}^h U = \mathcal{L}_x^h U + \mathcal{L}_y^h U, \quad \mathcal{L}_\nu^h U = -\mu^2 \mathcal{D}_\nu^2 U, \quad \nu = x, y,$$

where $\mathcal{D}_x^2 U$ and $\mathcal{D}_y^2 U$ are the central difference approximations to the second derivatives

$$\mathcal{D}_x^2 U_{ij}^k = (\hbar_{xi})^{-1} \left[\left(U_{i+1,j}^k - U_{ij}^k \right) (h_{xi})^{-1} - \left(U_{ij}^k - U_{i-1,j}^k \right) (h_{x,i-1})^{-1} \right],$$

$$\mathcal{D}_y^2 U_{ij}^k = (\hbar_{yj})^{-1} \left[\left(U_{i,j+1}^k - U_{ij}^k \right) (h_{yj})^{-1} - \left(U_{ij}^k - U_{i,j-1}^k \right) (h_{y,j-1})^{-1} \right],$$

$$\hbar_{xi} = 2^{-1} (h_{x,i-1} + h_{xi}), \quad \hbar_{yj} = 2^{-1} (h_{y,j-1} + h_{yj}), \quad U_{ij}^k \equiv U(x_i, y_j, t_k).$$

On each time level t_k, $k \geq 1$, introduce the linear difference problem

$$(\mathcal{L} + c(p, t_k) I) W(p, t_k) = \Phi(p, t_k), \quad p \in \omega^h, \tag{5}$$

$$W(p, t_k) = g(p, t_k), \quad p \in \partial\omega^h,$$

where I is the identity operator. We are concerned with maximal nodal errors, so we use the norm

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} = \max_{p \in \overline{\omega}^h} |W(p, t_k)|.$$

We now formulate the maximum principle and give an estimate to the solution of (5).

Lemma 1. *Let the assumption*

$$\tau_k^{-1} + \min_{p \in \overline{\omega}^h} c(p, t_k) > 0$$

hold true for $k \geq 1$.

(i) If a mesh function $W(p, t_k)$ satisfies the conditions

$$(\mathcal{L} + c(p, t_k) I) W(p, t_k) \geq 0 \ (\leq 0), \quad p \in \omega^h,$$

$$W(p, t_k) \geq 0 \ (\leq 0), \quad p \in \partial\omega^h,$$

then $W(p, t_k) \geq 0$ (≤ 0) in $\overline{\omega}^h$.

(ii) The following estimates of the solutions to (5) hold true

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} \leq \max_{p \in \omega^h} \frac{|\Phi(p, t_k)|}{\tau_k^{-1} + c(p, t_k)}. \tag{6}$$

The proof of the lemma can be found in [11].

Remark 1. *A difference scheme which satisfies the maximum principle from Lemma 1 is said to be monotone. The monotonicity condition guarantees that the systems of algebraic equations based on such methods are well-posed (see [11] for details).*

Remark 2. *The maximum principle from Lemma 1 holds true for the linear difference operators* $\mathcal{L}_\nu^h + (\tau_k^{-1} + c(p, t_k))I$, $\nu = x, y$.

3. The nonlinear ADI scheme

We use the following ADI scheme

$$\begin{gathered}\left(I + \tau_k \mathcal{L}_x^h\right) U^*(p, t_k) = U(p, t_{k-1}), \quad p \in \omega^h, \quad k \geq 1, \\ U^*(0, y_j, t_k) = U^*(1, y_j, t_k) = 0, \quad j = 1, \ldots, N_y - 1, \\ \left(I + \tau_k \mathcal{L}_y^h\right) U(p, t_k) = U^*(p, t_k) - \tau_k f(p, t_k, U), \quad p \in \omega^h, \quad k \geq 1, \\ U(x_i, 0, t_k) = U(x_i, 1, t_k) = 0, \quad i = 1, \ldots, N_x - 1, \\ U(p, 0) = \psi(p), \quad p \in \overline{\omega}^h.\end{gathered} \tag{7}$$

On each time level t_k, $k \geq 1$, for $U^*(p, t_k)$, $N_y - 1$ linear systems in the x-direction must be solved, and for $U(p, t_k)$, $N_x - 1$ nonlinear systems in the y-direction must be solved. The matrices corresponding to $I + \tau_k \mathcal{L}_\nu^h(p, t_k)$, $\nu = x, y$, are tridiagonal and can be inverted conveniently with the Thomas algorithm (see [8] for details).

Applying from the left the linear operator $I + \tau_k \mathcal{L}_x^h$ to the difference equation for $U(p, t_k)$ from (7), we get

$$\left(I + \tau_k \mathcal{L}_x^h\right) \left(I + \tau_k \mathcal{L}_y^h\right) U(p, t_k) = U(p, t_{k-1}) - \left(I + \tau_k \mathcal{L}_x^h\right) \tau_k f(p, t_k, U), \tag{8}$$

or

$$\begin{gathered}\left(I + \tau_k \mathcal{L}^h\right) U(p, t_k) + \tau_k f(p, t_k, U) - U(p, t_{k-1}) + \\ \tau_k^2 \mathcal{L}_x^h \left(f(p, t_k, U) + \mathcal{L}_y^h U(p, t_k)\right) = 0.\end{gathered} \tag{9}$$

The implicit scheme (4) and the nonlinear ADI scheme (7) differ by an order of τ_k^2. Taking error accumulation into account, both schemes will give the same first order accuracy in time.

4. The monotone ADI scheme

We say that mesh functions $\widetilde{U}(p,t_k)$, $\widetilde{U}^*(p,t_k)$ and $\widehat{U}(p,t_k)$, $\widehat{U}^*(p,t_k)$ are ordered upper and lower solutions of (7), if they satisfy $\widetilde{U}(p,t_k) \geq \widehat{U}(p,t_k)$, $\widetilde{U}^*(p,t_k) \geq \widehat{U}^*(p,t_k)$, $p \in \overline{\omega}^h$, $k \geq 1$, and

$$\begin{gathered}
\left(I+\tau_k \mathcal{L}_x^h\right) \widetilde{U}^*(p,t_k) \geq \widetilde{U}(p,t_{k-1}), \quad p \in \omega^h,\ k \geq 1, \\
\left(I+\tau_k \mathcal{L}_x^h\right) \widehat{U}^*(p,t_k) \leq \widehat{U}(p,t_{k-1}), \quad p \in \omega^h,\ k \geq 1, \\
\widehat{U}^*(0,y_j,t_k) \leq 0 \leq \widetilde{U}^*(1,y_j,t_k), \quad j=1,\ldots,N_y-1, \\
\left(I+\tau_k \mathcal{L}_y^h\right) \widetilde{U}(p,t_k) \geq \widetilde{U}^*(p,t_k) - \tau_k f(p,t_k,\widetilde{U}), \quad p \in \omega^h, \quad k \geq 1, \\
\left(I+\tau_k \mathcal{L}_y^h\right) \widehat{U}(p,t_k) \leq \widehat{U}^*(p,t_k) - \tau_k f(p,t_k,\widehat{U}), \quad p \in \omega^h, \quad k \geq 1, \\
\widehat{U}(x_i,0,t_k) \leq 0 \leq \widetilde{U}(x_i,1,t_k), \quad i=1,\ldots,N_x-1, \\
\widehat{U}(p,0) \leq \psi(p) \leq \widetilde{U}(p,0), \quad p \in \overline{\omega}^h.
\end{gathered} \tag{10}$$

We now construct an iterative method for solving (7) in the following way. Introduce the notation

$$\mathcal{L}_\nu = \mathcal{L}_\nu^h + \tau_k^{-1} I, \quad \nu = x, y.$$

On each time level t_k, $k \geq 1$, we calculate sequences of upper and lower solutions $\{V_\alpha^{(n)}(p,t_k)\}$ ($\alpha = 1$ and $\alpha = -1$ correspond to, respectively, the upper and lower cases) and define

$$V_1(p,t_k) = V_1^{(n_k)}(p,t_k), \quad k \geq 1, \quad V_1(p,0) = \psi(p), \quad p \in \overline{\omega}^h,$$

as an approximate solution of the nonlinear ADI scheme (7) on t_k, $k \geq 0$, where n_k is a number of iterative steps on time level t_k. Initial upper and lower solutions $V_\alpha^{(0)}(p,t_k)$, $\alpha = 1,-1$, are calculated by solving the linear problems

$$\begin{gathered}
\mathcal{L}_x V^*(p,t_k) = \tau_k^{-1} V_1(p,t_{k-1}), \quad p \in \omega^h, \\
V^*(0,y_j,t_k) = V^*(1,y_j,t_k) = 0, \quad j=1,\ldots,N_y-1, \\
\mathcal{L}_y Y_\alpha^{(0)}(p,t_k) = \alpha |\mathcal{R}(p,t_k,S)|, \quad p \in \omega^h, \\
Y_\alpha^{(0)}(p,t_k) = 0, \quad p \in \partial\omega^h, \\
\mathcal{R}(p,t_k,S) = \mathcal{L}_y S(p,t_k) + f(p,t_k,S) - \tau_k^{-1} V^*(p,t_k),
\end{gathered} \tag{11}$$

$$V_\alpha^{(0)}(p,t_k) = S(p,t_k) + Y_\alpha^{(0)}(p,t_k), \quad p \in \overline{\omega}^h,$$

where $S(p,t_k)$ is an arbitrary mesh function, defined on $\overline{\omega}^h$, which satisfies $S(p,t_k) = 0$ on $\partial\omega^h$. For $n \geq 1$, we calculate upper and lower solutions by using the recurrence formulae

$$\left(\mathcal{L}_y + c^{(n-1)}(p,t_k)I\right) Z_\alpha^{(n)}(p,t_k) = -\mathcal{R}(p,t_k,V_\alpha^{(n-1)}), \quad p \in \omega^h, \tag{12}$$

$$\begin{aligned}
\mathcal{R}(p,t_k,V_\alpha^{(n-1)}) \quad = \quad & \mathcal{L}_y V_\alpha^{(n-1)}(p,t_k) + f(p,t_k,V_\alpha^{(n-1)}) - \\
& \tau_k^{-1} V^*(p,t_k),
\end{aligned}$$

$$\begin{gathered}
Z_\alpha^{(1)}(x_i,0,t_k) = -V_\alpha^{(0)}(x_i,0,t_k), \quad i = 1,\ldots,N_x - 1, \\
Z_\alpha^{(1)}(x_i,1,t_k) = -V_\alpha^{(0)}(x_i,1,t_k), \quad i = 1,\ldots,N_x - 1, \\
Z_\alpha^{(n)}(x_i,0,t_k) = Z_\alpha^{(n)}(x_i,1,t_k) = 0, \quad i = 1,\ldots,N_x - 1, \quad n \geq 2, \\
V_\alpha^{(n)}(p,t_k) = V_\alpha^{(n-1)}(p,t_k) + Z_\alpha^{(n)}(p,t_k), \quad p \in \overline{\omega}^h,
\end{gathered}$$

The mesh function $c^{(n-1)}(p,t_k)$ is given by

$$c^{(n-1)}(p,t_k) = \max_V \{f_u(p,t_k,V),\ V_{-1}^{(n-1)}(p,t_k) \leq V \leq V_1^{(n-1)}(p,t_k)\}, \tag{13}$$

where below in Theorem 1, we prove that $V_{-1}^{(n-1)}(p,t_k) \leq V_1^{(n-1)}(p,t_k)$, $p \in \overline{\omega}^h$.

In the following theorem we prove the monotone property of the ADI scheme (11)–(13).

Theorem 1. *Let (2) hold. The sequences $\{V_1^{(n)}\}$, $\{V_{-1}^{(n)}\}$, generated by (11)–(13) are ordered upper and lower solutions to (7) and converge monotonically*

$$V_{-1}^{(n-1)}(p,t_k) \leq V_{-1}^{(n)}(p,t_k) \leq V_1^{(n)}(p,t_k) \leq V_1^{(n-1)}(p,t_k), \quad p \in \overline{\omega}^h, \tag{14}$$

where $k \geq 1$ and $n \geq 1$.

Proof. We show that $V_1^{(0)}(p,t_k)$, $k \geq 1$, defined by (11) is an upper solution. From (11), by the maximum principle in Lemma 1, it follows that

$$Y_1^{(0)}(p,t_k) \geq 0, \quad p \in \overline{\omega}^h.$$

Using the difference equation and the mean-value theorem, we have

$$\mathcal{L}_y(S(p,t_k)+Y_1^{(0)}(p,t_k))+f(p,t_k,S+Y_1^{(0)})-\tau_k^{-1}V_1(p,t_{k-1})=$$
$$\mathcal{R}(p,t_k,S)+|\mathcal{R}(p,t_k,S)|+f_u(p,t_k,R)Y_1^{(0)}(p,t_k),$$

where $S(p,t_k) \le R(p,t_k) \le S(p,t_k)+Y^{(0)}(p,t_k)$. From (2) and $Y_1^{(0)}$ is nonnegative, we conclude that $V_1^{(0)}(p,t_k)=S(p,t_k)+Y_1^{(0)}(p,t_k)$ is an upper solution. Similarly, we can prove that $V_{-1}^{(0)}(p,t_k)=S(p,t_k)+Y_{-1}^{(0)}(p,t_k)$ is a lower solution.

Since $V_1^{(0)}$ is an upper solution, then from (12), we have

$$\left(\mathcal{L}_y+c^{(0)}(p,t_1)I\right)Z_1^{(1)}(p,t_1)\le 0, \quad p\in\omega^h,$$

$$Z_1^{(1)}(x_i,0,t_1)\le 0, \quad Z_1^{(1)}(x_i,1,t_1)\le 0, \quad i=1,\ldots,N_x-1.$$

From (2) and (13), by Lemma 1, it follows that

$$Z_1^{(1)}(p,t_1)\le 0, \quad p\in\overline{\omega}^h. \tag{15}$$

Similarly, for a lower solution $V_{-1}^{(0)}$, we conclude that

$$Z_{-1}^{(1)}(p,t_1)\ge 0, \quad p\in\overline{\omega}^h. \tag{16}$$

We now prove that

$$V_{-1}^{(1)}(p,t_1)\le V_1^{(1)}(p,t_1), \quad p\in\overline{\omega}^h. \tag{17}$$

Letting $W^{(n)}=V_1^{(n)}-V_{-1}^{(n)}$, $n\ge 0$, from (12) and the mean-value theorem, we have

$$\left(\mathcal{L}_y+c^{(0)}(p,t_1)I\right)W^{(1)}(p,t_1)=(c^{(0)}(p,t_1)-f_u(p,t_1,E))W^{(0)}(p,t_1),$$

$$p\in\omega^h, \quad W^{(1)}(x_i,0,t_1)\ge 0, \quad W^{(1)}(x_i,1,t_1)\ge 0, \quad i=1,\ldots,N_x-1,$$

where $V_{-1}^{(0)}(p,t_1)\le E(p,t_1)\le V_1^{(0)}(p,t_1)$. From $W^{(0)}(p,t_1)\ge 0$, $p\in\overline{\omega}^h$, (2) and (13), we conclude that the right hand side in the difference equation is nonnegative. Taking into account (2) and (13), the positivity property in Lemma 1 implies $W^{(1)}(p,t_1)\ge 0$, $p\in\overline{\omega}^h$, and this leads to (17).

We now prove that $V_1^{(1)}(p,t_1)$ and $V_{-1}^{(1)}(p,t_1)$ are upper and lower solutions (10), respectively. Using the mean-value theorem, from (12) we obtain

$$\mathcal{R}(p,t_1,V_1^{(1)})=-\left(c^{(0)}(p,t_1)-f_u(p,t_1,Q)\right)Z_1^{(1)}(p,t_1), \tag{18}$$

where $V_1^{(1)}(p,t_1) \le Q(p,t_1) \le V_1^{(0)}(p,t_1)$. From here, (13), (15), (16) and (17), it follows that

$$c^{(0)}(p,t_1) \ge f_u(p,t_1,Q), \quad p \in \omega^h.$$

From here and (15), we conclude that

$$\mathcal{R}(p,t_1,V_1^{(1)}) \ge 0, \quad p \in \omega^h,$$

$$V_1^{(1)}(x_i,0,t_1) = V_1^{(1)}(x_i,1,t_1) = 0, \quad i = 1,\dots,N_x - 1.$$

Thus, $V_1^{(1)}(p,t_1)$ is an upper solution. Similarly, we can prove that $V_{-1}^{(1)}(p,t_1)$ is a lower solution, that is,

$$\mathcal{R}(p,t_1,V_{-1}^{(1)}) \le 0, \quad p \in \omega^h,$$

$$V_{-1}^{(1)}(x_i,0,t_1) = V_{-1}^{(1)}(x_i,1,t_1) = 0, \quad i = 1,\dots,N_x - 1.$$

By induction on n, we can prove that $\{V_1^{(n)}(p,t_1)\}$ is a monotonically decreasing sequence of upper solutions and $\{V_{-1}^{(n)}(p,t_1)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (14) for t_1.

By induction on k, $k \ge 1$, we can prove that $\{V_1^{(n)}(p,t_k)\}$ is a monotonically decreasing sequence of upper solutions and $\{V_{-1}^{(n)}(p,t_k)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (14). Thus, we prove the theorem. □

Applying Theorem 1, we investigate existence and uniqueness of a solution to the nonlinear ADI scheme (7).

Theorem 2. *Let (2) hold. Then the nonlinear ADI scheme (7) has a unique solution.*

Proof. Let $U_1(p,t_1) = \lim_{n\to\infty} V_1^{(n)}(p,t_1)$, $p \in \overline{\omega}^h$. It follows from (14) that the limit exists and

$$U_1(p,t_1) \le V_1^{(n)}(p,t_1), \quad \lim_{n\to\infty} Z_1^{(n)}(p,t_1) = 0, \quad p \in \overline{\omega}^h. \tag{19}$$

Similar to (18), we can prove that

$$\mathcal{R}(p,t_1,V_1^{(n)}) = -(c^{(n-1)}(p,t_1) - f_u(p,t_1,Q^{(n)})Z_1^{(n)}(p,t_1), \quad n \ge 1, \tag{20}$$

where $V_1^{(n)}(p,t_1) \le Q^{(n)}(p,t_1) \le V_1^{(n-1)}(p,t_1)$. From here and (19), we conclude that $U_1(p,t_1)$ solves (7) at t_1. By induction on k, $k \ge 1$, we can prove that

$$U_1(p,t_k) = \lim_{n\to\infty} V_1^{(n)}(p,t_k), \quad p \in \overline{\omega}^h, \quad k \ge 1,$$

is a solution of the nonlinear ADI scheme (7). Similarly, we can prove that

$$U_{-1}(p,t_k) = \lim_{n\to\infty} V_{-1}^{(n)}(p,t_k), \quad p \in \overline{\omega}^h, \quad k \ge 1,$$

is a solution of the nonlinear ADI scheme (7).

We now show that

$$U_1(p,t_k) = U_{-1}(p,t_k), \quad p \in \overline{\omega}^h, \quad k \ge 1,$$

where $U_1(p,t_k)$ and $U_{-1}(p,t_k)$ are solutions to the nonlinear ADI scheme (7), which are defined above. Let $W(p,t_k) = U_1(p,t_k) - U_{-1}(p,t_k)$. From (7), by Lemma 1, it follows that $U^*(p,t_1) = 0$, $p \in \overline{\omega}^h$. From here and (7), we have

$$\mathcal{L}_y W(p,t_1) + f(p,t_1,U_1) - f(p,t_1,U_{-1}) = 0, \quad p \in \omega^h,$$
$$W(p,t_1) = 0, \quad p \in \partial\omega^h.$$

From (14), it follows that

$$V_{-1}^{(n)}(p,t_1) \le U_{-1}(p,t_1) \le U_1(p,t_1) \le V_1^{(n)}(p,t_1), \quad p \in \overline{\omega}^h. \tag{21}$$

Using the mean-value theorem, we obtain

$$(\mathcal{L}_y + f_u(p,t_1,E)I)W(p,t_1) = 0, \quad p \in \omega^h, \quad W(p,t_1) = 0, \quad p \in \partial\omega^h,$$

where $U_{-1}(p,t_1) \le E(p,t_1) \le U_1(p,t_1)$. Using (2), by Lemma 1, we conclude that $W(p,t_1) = 0$, $p \in \overline{\omega}^h$. By induction on k, $k \ge 1$, we can prove that $W(p,t_k) = 0$, $p \in \overline{\omega}^h$, $k \ge 1$, and prove the theorem. □

5. Convergence properties of the monotone ADI scheme

5.1. Quadratic convergence of the monotone iterative method

Introduce the notation

$$r_k = \max_{p\in\overline{\omega}^h}[\max_V\{|f_{uu}(p,t_k,V)|, \widehat{U}(p,t_k) \le V \le \widetilde{U}(p,t_k)\}]. \tag{22}$$

The following theorem gives the quadratic convergence of the monotone iterative method (11)–(13).

Theorem 3. *Let (2) hold. On each time level, for the sequences* $\{V_\alpha^{(n)}\}$, $\alpha = 1, -1$, *generated by (11)–(13), the following estimate holds:*

$$\|W^{(n+1)}(\cdot, t_k)\|_{\overline{\omega}^h} \le \rho_k \|W^{(n)}(\cdot, t_k)\|^2_{\overline{\omega}^h}, \quad \rho_k = \tau_k r_k, \tag{23}$$

where $W^{(n)}(p, t_k) = V_1^{(n)}(p, t_k) - V_{-1}^{(n)}(p, t_k)$.

Proof. From (12), we have

$$(\mathcal{L}_y + c^{(n)}(p, t_k) I) W^{(n+1)}(p, t_k) = H^{(n)}(p, t_k), \quad p \in \omega^h, \tag{24}$$

$$H^{(n)}(p, t_k) = c^{(n)}(p, t_k) W^{(n)}(p, t_k) - (f(p, t_k, V_1^{(n)}) - f(p, t_k, V_{-1}^{(n)})),$$

$$W^{(n+1)}(x_i, 0, t_k) = W^{(n+1)}(x_i, 1, t_k) = 0, \quad i = 1, \dots, N_x - 1.$$

By the mean-value theorem,

$$f(p, t_k, V_1^{(n)}) - f(p, t_k, V_{-1}^{(n)}) = f_u(p, t_k, E^{(n)}) W^{(n)}(p, t_k),$$

where $V_{-1}^{(n)}(p, t_k) \le E^{(n)}(p, t_k) \le V_1^{(n)}(p, t_k)$. From (13), it follows that

$$c^{(n)}(p, t_k) = f_u(p, t_k, Q^{(n)}), \quad V_{-1}^{(n)}(p, t_k) \le Q^{(n)}(p, t_k) \le V_1^{(n)}(p, t_k).$$

Thus, we represent the right hand side $H^{(n)}$ from (24) in the form

$$(f_u(p, t_k, Q^{(n)}) - f_u(p, t_k, E^{(n)})) W^{(n)}(p, t_k).$$

Applying again the mean-value theorem, we get

$$f_u(p, t_k, Q^{(n)}) - f_u(p, t_k, E^{(n)}) = f_{uu}(p, t_k, G^{(n)})(Q^{(n)}(p, t_k) - E^{(n)}(p, t_k)),$$

where $G^{(n)}$ lies between $Q^{(n)}$ and $E^{(n)}$. Taking into account that

$$|Q^{(n)}(p, t_k) - E^{(n)}(p, t_k)| \le V_1^{(n)}(p, t_k) - V_{-1}^{(n)}(p, t_k),$$

in the notation (22), we estimate $H^{(n)}$ from (24) as follows

$$\|H^{(n)}(\cdot, t_k)\|_{\omega^h} \le r_k \|W^{(n)}(\cdot, t_k)\|^2_{\overline{\omega}^h}.$$

From here, using (6), we prove the estimate (23). $\square$

Remark 3. *If on each time level t_k, $k \geq 1$, the nonlinear function f satisfies the constraint*

$$\min_{p \in \overline{\omega}^h} [\min_V \{f_{uu}(p, t_k, V), \widehat{U}(p, t_k) \leq V \leq \widetilde{U}(p, t_k)\}] \geq 0, \tag{25}$$

then for the upper sequence $\{V_1^{(n)}\}$ in Theorem 3, we have the estimate

$$\|V_1^{(n+1)}(\cdot, t_k) - U(\cdot, t_k)\|_{\overline{\omega}^h} \leq \rho_k \|V_1^{(n)}(\cdot, t_k) - U(\cdot, t_k)\|^2_{\overline{\omega}^h},$$

where U is the exact solution of the nonlinear ADI scheme (7). From assumption (25) on f_{uu} and (13), it follows that

$$c^{(n)}(p, t_k) = f_u(p, t_k, V_1^{(n)}). \tag{26}$$

If we take into account that $W^{(n)}(p, t_k) = V_1^{(n)}(p, t_k) - U(p, t_k)$, $U^*(p, t_k) \leq E^{(n)}(p, t_k) \leq V_1^{(n)}(p, t_k)$ and $Q^{(n)} = V_1^{(n)}$, the proof of the estimate repeats the proof of Theorem 3.

In the case of the constraint (25), from (26), it follows that on each time level t_k, $k \geq 1$, one can calculate only the sequence of upper solutions $\{V_1^{(n)}(p, t_k)\}$ in the monotone ADI scheme (11)–(13). Thus, calculation of the mesh function $c^{(n-1)}(p, t_k)$ is simplified to compare (26) to (13).

If on each time level t_k, $k \geq 1$, the nonlinear function f satisfies the constraint

$$\max_{p \in \overline{\omega}^h} [\max_V \{f_{uu}(p, t_k, V), \widehat{U}(p, t_k) \leq V \leq \widetilde{U}(p, t_k)\}] \leq 0, \tag{27}$$

then for the lower sequence $\{V_{-1}^{(n)}\}$ in Theorem 3, we have the estimate

$$\|V_{-1}^{(n+1)}(\cdot, t_k) - U(\cdot, t_k)\|_{\overline{\omega}^h} \leq \rho_k \|V_{-1}^{(n)}(\cdot, t_k) - U(\cdot, t_k)\|^2_{\overline{\omega}^h}.$$

From assumption (27) on f_{uu} and (13), it follows that

$$c^{(n)}(p, t_k) = f_u(p, t_k, V_{-1}^{(n)}). \tag{28}$$

If we take into account that $W^{(n)}(p, t_k) = V_{-1}^{(n)}(p, t_k) - U(p, t_k)$, $V_{-1}^{(n)}(p, t_k) \leq E^{(n)}(t_k) \leq U^*(p, t_k)$ and $Q^{(n)} = V_{-1}^{(n)}$, the proof of the estimate repeats the proof of Theorem 3.

In the case of the constraint (27), from (28), it follows that on each time level t_k, $k \geq 1$, one can calculate only the sequence of lower solutions $\{V_{-1}^{(n)}(p, t_k)\}$ in the monotone ADI scheme (11)–(13). Thus, calculation of the mesh function $c^{(n-1)}(p, t_k)$ is simplified to compare (28) to (13).

Introduce the notation

$$q_n(t_k) = \|V_1^{(n)}(\cdot,t_k) - V_{-1}^{(n)}(\cdot,t_k)\|_{\overline{\omega}^h}.$$

In the following theorem we estimate the quadratic convergence rate in (23).

Theorem 4. *Let (2) hold. Then on each time level, for the sequences* $\{V_\alpha^{(n)}\}$*, $\alpha = 1,2$, generated by (11)–(13), there exists n_k, such that $\rho_k q_{n_k} < 1$, and the following estimate holds:*

$$q_n(t_k) \leq \frac{1}{\rho_k}\left[\rho_k q_{n_k}(t_k)\right]^{2^{n-n_k}}, \quad n \geq n_k, \tag{29}$$

where ρ_k is defined in (23).

Proof. Let $\kappa_n(t_k) = \rho_k q_n(t_k)$. Multiplying (23) by ρ_k, we have

$$\kappa_{n+1}(t_k) \leq \left[\kappa_n(t_k)\right]^2, \quad n \geq 0.$$

Since the sequences $\{V_\alpha^{(n)}(p,t_k)\}$, $\alpha = 1,-1$, converge to the exact solution $U(p,t_k)$ of the nonlinear scheme (7), then for some n_k the inequality $\kappa_{n_k} < 1$ holds. By induction, we show that

$$\kappa_n(t_k) \leq \left[\kappa_{n_k}(t_k)\right]^{2^{n-n_k}}, \quad n \geq n_k. \tag{30}$$

It is true for $n = n_k$. Assuming that it holds true for $n = l$, we have

$$\kappa_{l+1}(t_k) \leq \left[\kappa_l(t_k)\right]^2 \leq \left(\left[\kappa_{n_k}(t_k)\right]^{2^{l-n_k}}\right)^2 = \left[\kappa_{n_k}(t_k)\right]^{2^{l+1-n_k}},$$

and prove (30). From (30) and $\kappa_n(t_k) = \rho_k q_n(t_k)$, we conclude (29). □

5.2. Convergence of the monotone ADI scheme on $[0,T]$

In Theorems 1, 3 and 4, we have investigated convergence properties of the monotone ADI scheme (11)–(13) on each time-level t_k, $k \geq 1$. We now investigate convergence of the monotone ADI scheme (11)–(13) on the whole interval of integration $[0,T]$. In (12), we assume that on each time-level t_k, $k \geq 1$, $V_1(p,t_k)$ is the approximation of the exact solution, where $V_1(p,t_k) = V_1^{(n_k)}(p,t_k)$. Thus, on the whole interval of integration, we estimate $\max_{t_k \in \overline{\omega}^\tau} \|V_1(\cdot,t_k) - U(\cdot,t_k)\|_{\overline{\omega}^h}$, where U is the exact solution to the nonlinear ADI scheme (7).

We now choose the stopping criterion of the monotone ADI scheme (11)–(13) in the form

$$\|\mathcal{R}(\cdot, t_k, V_1^{(n)})\|_{\omega^h} \leq \delta, \tag{31}$$

where δ is a prescribed accuracy, and set up $V_1(p, t_k) = V_1^{(n_k)}(p, t_k)$, $p \in \overline{\omega}^h$, such that n_k is minimal subject to (31).

We prove the following convergence result for the monotone ADI scheme (11)–(13), (31).

Theorem 5. *Let (2) hold true. The sequence $\{V_1^{(n)}\}$, generated by (11)–(13), (31), converges uniformly in the perturbation parameter μ:*

$$\max_{t_k \in \overline{\omega}^\tau} \|V_1(\cdot, t_k) - U(\cdot, t_k)\|_{\overline{\omega}^h} \leq T\delta, \tag{32}$$

where $U(p, t_k)$ is the unique solution to (7).

Proof. The difference problem for $V_1(p, t_k) = V_1^{(n_k)}(p, t_k)$, $k \geq 1$, can be represented in the form

$$\mathcal{L}_y V_1(p, t_k) + f(p, t_k, V_1) - \tau_k^{-1} V^*(p, t_k) = \mathcal{R}(p, t_k, V_1^{(n_k)}), \quad p \in \omega^h,$$

$$V_1(x_i, 0, t_k) = V_1(x_i, 1, t_k) = 0, \quad i = 1, \ldots, N_x - 1.$$

From here, (7) and using the mean-value theorem, we get the following difference problems for $W^*(p, t_k) = V^*(p, t_k) - U^*(p, t_k)$ and $W(p, t_k) = V_1(p, t_k) - U(p, t_k)$:

$$\mathcal{L}_x W^*(p, t_k) = \tau_k^{-1} W(p, t_{k-1}), \quad p \in \omega^h, \tag{33}$$

$$W^*(0, y_j, t_k) = W^*(1, y_j, t_k) = 0, \quad j = 1, \ldots, N_y - 1,$$

$$(\mathcal{L}_y + f_u(p, t_k, E)I) W(p, t_k) = \mathcal{R}(p, t_k, V_1) + \tau_k^{-1} W^*(p, t_k), \quad p \in \omega^h,$$

$$W(x_i, 0, t_k) = W(x_i, 1, t_k) = 0, \quad i = 1, \ldots, N_x - 1,$$

where $E(p, t_k)$ lies between $V_1(p, t_k)$ and $U(p, t_k)$. From here and (2), by using (6), we have

$$\|W^*(\cdot, t_k)\|_{\overline{\omega}^h} \leq \|W(\cdot, t_{k-1})\|_{\overline{\omega}^h}, \tag{34}$$

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} \leq \tau_k \|\mathcal{R}(\cdot, t_k, V_1)\|_{\omega^h} + \|W^*(\cdot, t_k)\|_{\overline{\omega}^h}.$$

Since $W(p,0)=0$, from here, by using (6), we conclude that $W^*(p,t_1)=0$, $p\in\overline{\omega}^h$. From here, (34) and taking into account that according to Theorem 1 the stopping criterion (31) can always be satisfied, it follows that

$$\|W(\cdot,t_1)\|_{\overline{\omega}^h}\le\delta\tau_1.$$

Similarly, from here and (34), we obtain

$$\|W^*(\cdot,t_2)\|_{\overline{\omega}^h}\le\delta\tau_1,\quad \|W(\cdot,t_2)\|_{\overline{\omega}^h}\le\delta\tau_1+\delta\tau_2.$$

Now, by induction on k, we conclude that

$$\|W(\cdot,t_k)\|_{\overline{\omega}^h}\le\delta\sum_{l=1}^{k}\tau_l\le\delta T,\quad k\ge 1.$$

Thus, we prove the theorem. □

6. Uniform convergence of the monotone ADI scheme to the nonlinear parabolic problem

We suppose sufficient smoothness of functions f and ψ in (1) and also sufficient compatibility conditions between the initial and boundary data, in such a way that for l sufficiently large integer and $0<\epsilon<1$, the solution of (1) satisfies

$$u(x,y,t)\in C^{l+\epsilon,l+\epsilon,(l+\epsilon)/2}(\overline{Q}).$$

Using the mean-value theorem, the reaction function f in (1) can be written in the form $f(x,y,t,u)=f(x,y,t,0)+f_u u$. Now, we may consider (1), (2) as a linear parabolic problem with the smooth coefficient f_u and use the bounds of the exact solution and its derivatives obtained in [12] for a linear problem. According to [12], the solution can be decomposed into two parts $u=S+E$, where S and E are the regular and singular parts of u, respectively. In turn, the singular part can be decomposed in the form

$$E=\Phi+\Psi+(\Upsilon_{00}+\Upsilon_{10}+\Upsilon_{01}+\Upsilon_{11}),$$

where Φ and Ψ are essentially one-dimensional boundary layer functions in some neighborhoods of sides $x=0$, $x=1$ and $y=0$, $y=1$, respectively, and

$\Upsilon_{mn}, m, n = 0, 1$ are corner layers in the neighborhood of (m, n). According to the results from [12], the following bounds hold true:

$$\left|\frac{\partial^k S(x,y,t)}{\partial x^{k_x}\partial y^{k_y}\partial t^{k_t}}\right| \le C, \tag{35}$$

$$\begin{aligned}
&\left|\frac{\partial^k \Phi(x,y,t)}{\partial x^{k_x}\partial y^{k_y}\partial t^{k_t}}\right| \le C\mu^{-k_x}\Pi(x), \quad \Pi(x) = \Pi_0(x) + \Pi_1(x),\\
&\left|\frac{\partial^k \Psi(x,y,t)}{\partial x^{k_x}\partial y^{k_y}\partial t^{k_t}}\right| \le C\mu^{-k_y}\hat{\Pi}(y), \quad \hat{\Pi}(y) = \hat{\Pi}_0(y) + \hat{\Pi}_1(y),\\
&\left|\frac{\partial^k \Upsilon_{mn}(x,y,t)}{\partial x^{k_x}\partial y^{k_y}\partial t^{k_t}}\right| \le C\mu^{-(k_y+k_y)}\Pi_m(x)\hat{\Pi}_n(y), \; m,n = 0,1,\\
&\Pi_0(x) = \exp\left(-\sqrt{\beta}x/\mu\right), \quad \Pi_1(x) = \exp\left(-\sqrt{\beta}(1-x)/\mu\right),\\
&\hat{\Pi}_0(y) = \exp\left(-\sqrt{\beta}y/\mu\right), \quad \hat{\Pi}_1(y) = \exp\left(-\sqrt{\beta}(1-y)/\mu\right),
\end{aligned}$$

where $k = (k_x, k_y, k_t)$, $k_x + k_y + 2k_t \le l$, and here and throughout C denotes a generic positive constant which is independent of μ and the mesh parameters.

6.1. Layer-adapted meshes

We employ a layer-adapted mesh from [10] which is formed in the following manner. We divide each of the intervals $\overline{\omega}^x = [0, 1]$ and $\overline{\omega}^y = [0, 1]$ into three parts $[0, \varsigma_x]$, $[\varsigma_x, 1 - \varsigma_x]$, $[1 - \varsigma_x, 1]$, and $[0, \varsigma_y]$, $[\varsigma_y, 1 - \varsigma_y]$, $[1 - \varsigma_y, 1]$, respectively. Assuming that N_x, N_y are divisible by 4, in the parts $[0, \varsigma_x]$, $[1 - \varsigma_x, 1]$ and $[0, \varsigma_y]$, $[1 - \varsigma_y, 1]$ we allocate $N_x/4+1$ and $N_y/4+1$ mesh points, respectively, and in the parts $[\varsigma_x, 1 - \varsigma_x]$ and $[\varsigma_y, 1 - \varsigma_y]$ we allocate $N_x/2 + 1$ and $N_y/2 + 1$ mesh points, respectively. Points ς_x, $(1 - \varsigma_x)$ and ς_y, $(1 - \varsigma_y)$ correspond to transition to the boundary layers. We consider meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ which are equidistant in $\left[x_{N_x/4}, x_{3N_x/4}\right]$ and $\left[y_{N_y/4}, y_{3N_y/4}\right]$ but graded in $\left[0, x_{N_x/4}\right]$, $\left[x_{3N_x/4}, 1\right]$ and $\left[0, y_{N_y/4}\right]$, $\left[y_{3N_y/4}, 1\right]$. On $\left[0, x_{N_x/4}\right]$, $\left[x_{3N_x/4}, 1\right]$ and $\left[0, y_{N_y/4}\right]$, $\left[y_{3N_y/4}, 1\right]$ let our mesh be given by a mesh generating function ϕ with $\phi(0) = 0$ and $\phi(1/4) = 1$ which is supposed to be continuous, monotonically increasing, and piecewise continuously differentiable. Then our mesh is defined by

$$x_i = \begin{cases} \varsigma_x\phi\left(\xi_i\right), & \xi_i = \frac{i}{N_x},\; i = 0, \dots, \frac{N_x}{4};\\ ih_x, & i = \frac{N_x}{4} + 1, \dots, \frac{3N_x}{4} - 1;\\ 1 - \varsigma_x\left(1 - \phi\left(\xi_i\right)\right), & \xi_i = (i - \frac{3N_x}{4})N_x^{-1},\; i = \frac{3N_x}{4} + 1, \dots, N_x, \end{cases}$$

$$y_j = \begin{cases} \varsigma_y \phi\left(\xi_j\right), & \xi_j = \frac{j}{N_y},\ j = 0, \ldots, \frac{N_y}{4}; \\ j h_y, & j = \frac{N_y}{4} + 1, \ldots, \frac{3N_y}{4} - 1; \\ 1 - \varsigma_y\left(1 - \phi\left(\xi_j\right)\right), & \xi_j = (j - \frac{3N_y}{4}) N_y^{-1},\ j = \frac{3N_y}{4} + 1, \ldots, N_y, \end{cases}$$

$$h_x = 2\left(1 - 2\varsigma_x\right) N_x^{-1}, \quad h_y = 2\left(1 - 2\varsigma_y\right) N_y^{-1}.$$

We also assume that ϕ' does not decrease. This condition implies that

$$h_{xi} \le h_{x,i+1},\ i = 1, \ldots, \frac{N_x}{4} - 1, \quad h_{xi} \ge h_{x,i+1},\ i = \frac{3N_x}{4} + 1, \ldots, N_x - 1,$$

$$h_{yj} \le h_{y,j+1},\ j = 1, \ldots, \frac{N_y}{4} - 1, \quad h_{yj} \ge h_{y,j+1},\ j = \frac{3N_y}{4} + 1, \ldots, N_y - 1.$$

Piecewise uniform meshes of Shishkin-type. The piecewise uniform meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are defined in the manner of [7] and are referred to as Shishkin meshes. The boundary layer thicknesses ς_x and ς_y are chosen as

$$\varsigma_x = \min\left\{1/4, m_1 \mu \ln N_x\right\}, \quad \varsigma_y = \min\left\{1/4, m_2 \mu \ln N_y\right\},$$

where m_1 and m_2 are positive constants independent of μ, N_x and N_y. If $\varsigma_{x,y} = 1/4$, then $N_{x,y}^{-1}$ are very small relative to μ, and in this case, the difference scheme can be analyzed using standard techniques. We therefore assume that

$$\varsigma_x = m_1 \mu \ln N_x, \quad \varsigma_y = m_2 \mu \ln N_y. \tag{36}$$

Consider the mesh generating function ϕ in the form

$$\phi(\xi) = 4\xi. \tag{37}$$

In this case the meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are piecewise equidistant with the step sizes

$$N_x^{-1} < h_x < 2N_x^{-1}, \quad h_{x\mu} = m_1 \mu N_x^{-1} \ln N_x,$$
$$N_y^{-1} < h_y < 2N_y^{-1}, \quad h_{y\mu} = m_2 \mu N_y^{-1} \ln N_y.$$

The mesh $\overline{\omega}^{hy}$ is defined similarly.

Log-meshes of Bakhvalov-type. We choose the transition points ς_x, $(1 - \varsigma_x)$ and ς_y, $(1 - \varsigma_y)$ in Bakhvalov's sense (see [1] for details), i.e.

$$\varsigma_x = m_1 \mu \ln\left(1/\mu\right), \quad \varsigma_y = m_2 \mu \ln\left(1/\mu\right), \tag{38}$$

and the mesh generating function ϕ is given in the form

$$\phi(\xi) = \frac{\ln[1 - 4(1 - \mu)\xi]}{\ln \mu}, \tag{39}$$

where m_1 and m_2 are positive constants independent of μ, N_x and N_y.

6.2. Error analysis

Firstly, we analyze a local truncation error of the exact solution $u(x, y, t)$ to the nonlinear problem (1), (2) on the nonlinear ADI scheme (7). Since the exact solution $U(p, t_k)$ of (7) satisfies (9), then on each time level t_k the local truncation error $\sigma(p, t_k)$ is defined by the left hand side of (9), where $u(p, t_k)$ is in use instead of $U(p, t_k)$.

Lemma 2. *The following error bounds hold true:*

$$\|\sigma(\cdot, t_k)\|_{\overline{\omega}^h} \leq \begin{cases} C\tau_k \left(N^{-1} \ln N + \tau_k\right), & \text{on mesh } (36), (37), \\ C\tau_k \left(N^{-1} + \tau_k\right), & \text{on mesh } (38), (39), \end{cases} \tag{40}$$

where $N = \min\{N_x, N_y\}$, and constant C is independent of μ, N and τ_k.

Proof. We split $\sigma(p, t_k)$ into two parts $\sigma_1(p, t_k)$ and $\sigma_2(p, t_k)$, where

$$\sigma_1(p, t_k) = (I + \tau_k \mathcal{L}^h) u(p, t_k) + \tau_k f(p, t_k, u) - u(p, t_{k-1}),$$

$$\sigma_2(p, t_k) = \tau_k^2 \mathcal{L}_x^h (f(p, t_k, u) + \mathcal{L}_y^h u(p, t_k)).$$

The part $\sigma_1(p, t_k)$ corresponds to the nonlinear implicit difference scheme for solving (1), (2), and for $\sigma_1(p, t_k)$, we proved the error bounds (40) in [2].

We now estimate $\sigma_2(p, t_k)$. The representation

$$\begin{aligned} \mathcal{D}_x^2 g_{ij} &= (\hbar_{xi} h_{xi})^{-1} \int_{x_i}^{x_{i+1}} \int_{x_i}^{s} \frac{\partial^2 g(z, y_j)}{\partial x^2} dz ds + \\ &\quad (\hbar_{xi} h_{x,i-1})^{-1} \int_{x_i}^{x_{i-1}} \int_{x_i}^{s} \frac{\partial^2 g(z, y_j)}{\partial x^2} dz ds \end{aligned}$$

for any smooth g yields

$$\left|\mathcal{D}_x^2 g_{ij}\right| \leq \left\|\partial^2 g / \partial x^2\right\|_{[x_{i-1}, x_{i+1}]}.$$

From here, (35) and using the mean-value theorem, we obtain

$$\|\mathcal{L}_x^h f(\cdot, t_k, u)\|_{\omega^h} \leq \|\mathcal{L}_x^h f(\cdot, t_k, 0)\|_{\omega^h} + \|\mathcal{L}_x^h f_u u(\cdot, t_k)\|_{\omega^h} \leq C.$$

For any smooth g, we have

$$\begin{aligned} \mathcal{D}_x^2 \mathcal{D}_y^2 g_{ij} &= (\hbar_{yj} h_{yj})^{-1} \int_{y_j}^{y_{j+1}} \int_{y_j}^{s} \mathcal{D}_x^2 \frac{\partial^2 g(x_i, z)}{\partial y^2} dz ds + \\ &\quad (\hbar_{yj} h_{y,j-1})^{-1} \int_{y_j}^{y_{j-1}} \int_{y_j}^{s} \mathcal{D}_x^2 \frac{\partial^2 g(x_i, z)}{\partial y^2} dz ds. \end{aligned}$$

Using the above integral representation for $\mathcal{D}_x^2$, we get

$$\left|\mathcal{D}_x^2\mathcal{D}_y^2 g_{ij}\right| \le \left\|\partial^4 g/(\partial x^2 \partial y^2)\right\|_{[x_{i-1},x_{i+1}]\times[y_{j-1},y_{j+1}]}.$$

From here and (35), we obtain

$$\|\mathcal{L}_x^h\mathcal{L}_y^h u(\cdot,t_k)\|_{\omega^h} \le C.$$

Collecting the above bounds, we estimate $\sigma_2(p,t_k)$ in the form

$$\|\sigma_2(\cdot,t_k)\|_{\omega^h} \le C\tau_k^2.$$

From here, we prove the error bounds (40) for $\sigma(p,t_k)$. □

We now investigate μ-uniform convergence of the nonlinear ADI scheme (7) on layer-adapted meshes to the nonlinear parabolic problem (1), (2).

Lemma 3. *On each time level t_k, $k \ge 1$, the nonlinear ADI scheme (7) converges μ-uniformly to the nonlinear parabolic problem (1), (2):*

$$\|U(\cdot,t_k) - u(\cdot,t_k)\|_{\overline{\omega}^h} \le \begin{cases} C\left(N^{-1}\ln N + \tau\right), & \text{on mesh } (36), (37), \\ C\left(N^{-1} + \tau\right), & \text{on mesh } (38), (39), \end{cases} \tag{41}$$

where $N = \min\{N_x, N_y\}$, $\tau = \max_k \tau_k$, and constant C is independent of μ, N and τ.

Proof. Let $e(p,t_k) = U(p,t_k) - u(p,t_k)$. We use (8) and the mean-value theorem to represent the difference problem for $e(p,t_k)$ in the form

$$\left(I + \tau_k\mathcal{L}_x^h\right)\left((1+\tau_k f_u)I + \tau_k\mathcal{L}_y^h\right)e(p,t_k) = e(p,t_{k-1}) - \sigma(p,t_k), \quad p \in \omega^h,$$

$$e(p,t_k) = 0, \quad p \in \partial\omega^h, \quad e(p,0) = 0, \quad p \in \overline{\omega}^h.$$

From here and taking into account that

$$\left\|\left(I + \tau_k\mathcal{L}_x^h\right)^{-1}\right\| \le 1, \quad \left\|\left((1+\tau_k f_u)I + \tau_k\mathcal{L}_y^h\right)^{-1}\right\| \le 1,$$

we obtain the following estimate for $e(p,t_k)$:

$$\|e(\cdot,t_k)\|_{\overline{\omega}^h} \le \|e(\cdot,t_{k-1})\|_{\overline{\omega}^h} + \|\sigma(\cdot,t_k)\|_{\overline{\omega}^h}.$$

From here and $e(p,0) = 0$, $p \in \overline{\omega}^h$, we conclude that

$$\|e(\cdot,t_k)\|_{\overline{\omega}^h} \le \left(\sum_{l=1}^{k}\tau_l\right)\|\tau_l^{-1}\sigma(\cdot,t_l)\|_{\overline{\omega}^h},$$

From here, $\sum_{l=1}^{k}\tau_k \le T$ and (40), we prove (41). □

We now arrive at our main theoretical result.

Theorem 6. *The sequence $\{V_1^{(n)}\}$, generated by the monotone ADI scheme (11)–(13), (31), converges μ-uniformly to the solution u of the nonlinear parabolic problem (1), (2):*

$$\|V_1(\cdot,t_k)-u(\cdot,t_k)\|_{\overline{\omega}^h} \le \begin{cases} C\,(\delta+N^{-1}\ln N+\tau)\,, & \textit{on mesh } (36),(37), \\ C\,(\delta+N^{-1}+\tau)\,, & \textit{on mesh } (38),(39), \end{cases}$$

where constant C is independent of μ, N and τ.

Proof. The proof follows from (32) and (41). □

7. Numerical experiments

In this section, we present some numerical experiments for the monotone ADI scheme. Our test problem arises from the enzyme reaction model where the reaction function is based on the Michaelis-Menton hypothesis. In the test problem, the true continuous solution is explicitly known and is used to compare with the numerical solution from the monotone iterations by the monotone ADI scheme.

We choose the stopping criterion in the form (31) with $\delta = 10^{-5}$. In all numerical experiments, the monotone property of upper and lower solutions is observed at every mesh point of the computational domain.

We consider the enzyme reaction model with an internal source $q(x,y,t)$ in $\omega = \{0<x<1,\ 0<y<1\}$. This is given by

$$u_t-\mu^2(u_{xx}+u_{yy})-\frac{a+u}{b+u}=q(x,y,t), \quad (x,y,t)\in\omega\times(0,T],$$

$$u(x,y,t)=0, \quad (x,y,t)\in\partial\omega\times(0,T],$$

$$u(x,y,0)=0, \quad (x,y)\in\overline{\omega},$$

where a and b are positive constants, $a>b>0$. We choose $q(x,y,t)$ such that the exact solution is

$$u(x,y,t)=(1-e^{-t})\phi(x)\phi(y), \quad \phi(z)=1-\frac{e^{-z/\mu}+e^{-(1-z)/\mu}}{1+e^{-1/\mu}}. \tag{42}$$

For $f(x,y,t,u)=-(a+u)/(b+u)-q(x,y,t)$, we have

$$f_u=\frac{a-b}{(b+u)^2}, \quad \frac{a-b}{(1+b)^2}\le f_u\le\frac{a-b}{b^2}, \quad u\ge 0.$$

Thus, $\beta = (a-b)/(1+b)^2$ in (2). Since $f_{uu} = -2(a-b)(b+u)^{-3} \leq 0$, $u \geq 0$, then $c^{(n-1)}$ in (13) is given in the form of (28), and we can calculate only the sequence of lower solutions $\{V_{-1}^{(n)}(p, t_k)\}$, where $V_{-1}(p, t_{k-1}) = V_{-1}^{(n_{k-1})}(p, t_{k-1})$ is in use in (12), instead of $V_1(p, t_{k-1})$.

Piecewise uniform mesh of Shishkin-type. Here we consider the monotone ADI scheme on the piecewise uniform mesh (36), (37). In Table 1, for $a = 50$, $b = 1$ and $T = 0.5$ and different values of μ, N and $\tau = T/N$, we present the maximum numerical error

$$\text{error}(N) = \max_{0 \leq k \leq N_\tau} \|V_{-1}(\cdot, t_k) - u(\cdot, t_k)\|_{\overline{\omega}^h},$$

where $u(x, t)$ is the exact solution (42), and number of monotone iterations on each time level is given in parentheses. The order of maximum numerical error, corresponding to the data from Table 1,

$$\text{order}(N) = \log_2 \left(\frac{\text{error}(N)}{\text{error}(2N)} \right),$$

is reported in Table 2. The data in Tables 1 and 2 show that the monotone ADI scheme converges μ-uniformly, the numerical solution has the first-order accuracy in the time variable, and the monotone lower sequences converge in a few iterations to the exact solution.

μ/N	64	128	256	512	1024
1	2.21e-4(2)	1.34e-4(2)	7.60e-5(2)	4.06e-5(2)	2.10e-5(2)
10^{-1}	1.54e-3(4)	9.75e-4(4)	5.89e-4(3)	3.29e-4(3)	1.76e-4(3)
10^{-2}	2.10e-3(4)	1.27e-3(4)	5.97e-4(3)	3.34e-4(3)	1.78e-4 (3)
10^{-3}	1.05e-2(4)	6.23e-3(4)	3.52e-3(3)	1.93e-3(3)	1.01e-3(3)
10^{-4}	7.31e-3(4)	4.19e-3(4)	2.30e-3(3)	1.21e-3(3)	6.26e-4(3)
$\leq 10^{-5}$	7.66e-3(4)	4.21e-3(4)	2.32e-3(3)	1.22e-3(3)	6.28e-4(3)

Table 1: Errors in the monotone ADI scheme on mesh (36), (37) for $\tau = T/N$.

For the same set of parameters as in Table 1 but $\tau = (T/N)^2$, we present the maximum numerical error and the order of maximum numerical error in Tables 3 and 4, respectively. The data in Tables 3 and 4 show that the monotone ADI scheme converges μ-uniformly, for $\mu \leq 10^{-3}$, the numerical solution has the first-order accuracy in the space variables, and the monotone lower sequences converge in a few iterations to the exact solution.

μ/N	64	128	256	512
1	0.72	0.82	0.91	0.95
10^{-1}	0.66	0.73	0.84	0.90
10^{-2}	0.73	1.09	0.84	0.91
10^{-3}	0.76	0.82	0.87	0.93
10^{-4}	0.80	0.84	0.90	0.95
$\leq 10^{-5}$	0.86	0.90	0.93	0.97

Table 2: Order of convergence in the monotone ADI scheme on mesh (36), (37) for $\tau = T/N$.

μ/N	64	128	256	512	1024
1	4.81e-6(2)	1.28e-6(2)	3.30e-7(2)	8.37e-8(2)	2.08e-8(2)
10^{-1}	9.17e-5(3)	1.94e-5(3)	3.54e-6(3)	6.62e-7(2)	1.36e-7(2)
10^{-2}	2.21e-3(3)	1.23e-3(3)	5.08e-4(3)	1.64e-4(3)	4.30e-5(2)
10^{-3}	1.05e-2(3)	6.03e-3(3)	3.38e-3(3)	1.86e-3(3)	9.81e-4(2)
10^{-4}	7.26e-3(3)	4.11e-3(3)	2.27e-3(3)	1.18e-3(3)	6.11e-4(2)
$\leq 10^{-5}$	7.66e-3(3)	4.19e-3(3)	2.28e-3(3)	1.18e-3(3)	5.98e-4(2)

Table 3: Errors in the monotone ADI scheme on mesh (36), (37) for $\tau = (T/N)^2$.

μ/N	64	128	256	512
1	1.91	1.96	1.98	2.01
10^{-1}	2.24	2.45	2.42	2.28
10^{-2}	0.85	1.29	1.62	1.93
10^{-3}	0.80	0.84	0.86	0.90
10^{-4}	0.83	0.86	0.92	0.95
$\leq 10^{-5}$	0.87	0.92	0.95	0.98

Table 4: Order of convergence in the monotone ADI scheme on mesh (36), (37) for $\tau = (T/N)^2$.

Log-meshes of Bakhvalov-type. Here we consider the monotone ADI scheme on log-mesh of Bakhvalov-type (38), (39). For the same set of parameters as in Table 1, we present the maximum numerical error and the order of maximum numerical error in Tables 5 and 6, respectively. The data in Tables 5 and 6 indicate that the monotone ADI scheme converges μ-uniformly, the numerical solution has the first-order accuracy in the time variable, and the monotone lower sequences converge in a few iterations to the exact solution.

μ/N	64	128	256	512	1024
1	2.18e-4(2)	1.35e-4(2)	7.61e-5(2)	4.06e-5(2)	2.10e-5(2)
10^{-1}	2.04e-3(4)	1.21e-3(4)	6.74e-4(3)	3.57e-4(3)	1.84e-4(3)
10^{-2}	2.06e-3(4)	1.23e-3(4)	6.81e-4(3)	3.61e-4(3)	1.86e-4 (3)
10^{-3}	8.50e-3(4)	4.96e-3(4)	2.73e-3(3)	1.48e-3(3)	7.77e-4(3)
10^{-4}	5.94e-3(4)	3.37e-3(4)	1.85e-3(3)	9.70e-4(3)	5.02e-4(3)
$\leq 10^{-5}$	6.63e-3(4)	3.62e-3(4)	1.91e-3(3)	9.80e-4(3)	4.97e-4(3)

Table 5: Errors in the monotone ADI scheme on mesh (38), (39) for $\tau = T/N$.

μ/N	64	128	256	512
1	0.70	0.83	0.91	0.95
10^{-1}	0.75	0.84	0.92	0.96
10^{-2}	0.74	0.85	0.92	0.96
10^{-3}	0.78	0.86	0.88	0.93
10^{-4}	0.82	0.87	0.93	0.96
$\leq 10^{-5}$	0.87	0.92	0.96	0.98

Table 6: Order of convergence in the monotone ADI scheme on mesh (38), (39) for $\tau = T/N$.

For the same set of parameters as in Table 5 but $\tau = (T/N)^2$, we present the maximum numerical error and the order of maximum numerical error in Tables 7 and 8, respectively. From the data in Tables 7 and 8, we can conclude that the monotone ADI scheme converges μ-uniformly, for $\mu \leq 10^{-3}$, the numerical solution has the first-order accuracy in the space variables, and the monotone lower sequences converge in a few iterations to the exact solution.

Our numerical experiments confirm the theoretical results proved in Theorem 6 that on meshes (36), (37) and (38), (39), the monotone ADI scheme (11)–(13), (31), converges μ-uniformly to the solution of the nonlinear parabolic

μ/N	64	128	256	512	1024
1	4.80e-6(2)	1.28e-6(2)	3.30e-7(2)	8.37e-8(2)	2.08e-8(2)
10^{-1}	1.30e-4(3)	3.66e-5(3)	9.57e-6(3)	2.42e-6(3)	6.22e-7(2)
10^{-2}	1.82e-3(3)	1.05e-3(3)	4.80e-4(3)	1.81e-4(3)	5.69e-5 (2)
10^{-3}	2.62e-3(3)	1.66e-3(3)	9.53e-4(3)	5.32e-4(3)	2.89e-4(2)
10^{-4}	1.94e-3(3)	1.12e-3(3)	6.25e-4(3)	3.39e-4(3)	1.78e-4(2)
$\leq 10^{-5}$	2.23e-3(3)	1.26e-3(3)	6.85e-4(3)	3.62e-4(3)	1.86e-4(2)

Table 7: Errors in the monotone ADI scheme on mesh (38), (39) for $\tau = (T/N)^2$.

μ/N	64	128	256	512
1	1.91	1.96	1.98	2.01
10^{-1}	1.83	1.94	1.98	1.96
10^{-2}	0.79	1.13	1.41	1.67
10^{-3}	0.75	0.80	0.84	0.88
10^{-4}	0.79	0.84	0.88	0.93
$\leq 10^{-5}$	0.83	0.88	0.92	0.96

Table 8: Order of convergence in the monotone ADI scheme on mesh (38), (39) for $\tau = (T/N)^2$.

problem (1), (2). The numerical solutions have the first-order accuracy in the time variable and for $\mu \leq 10^{-3}$, the first order accuracy in the space variables.

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