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Discrete multiple delay advection–reaction operators

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ABSTRACT

In this work we introduce a multiple delay term to the family of discrete operators known as advection–reaction operators. We present detailed quantitative information of the delay inclusion on the dynamics of the discrete dynamical systems with iteration functions given by the advection–reaction operators. Those dynamical systems are infinite dimensional linear discrete dynamical systems and by using a matrix representation on the system we compute the iterates of such operators which are given by the powers of such matrices. Those matrices are sparse lower triangular matrices where only two subdiagonals are different than zero. The entries of the power matrices are calculated explicitly and with the aid of some particular cases we can show their contained dynamical information.

1. Introduction

In our days, delay modeling is an active and important research in Science and Engineering [1–8], thus it is very common to find delay models in diverse areas such as Biology (e.g., population dynamics and epidemiology), Physics (e.g. Quantum Dot Lasers and Solar physics), Chemistry (e.g., Chemical-kinetics and Chemical reactions) and Economics (e.g., Finance and Cost) to name a few. Usually a delay or a lag term is used in a model when usual assumptions are replaced with more realistic temporary assumptions or when the result of some phase or stage structure in the system is under study. Also, the differences between systems of delay models and those without delay have been and remain an active area of research, see for example [9,10].

In this paper, we consider the family of advection–reaction operators introduced in [11] for which a multiple delay term is included. It has been shown that these operators appear in a natural way in partial differential equations and that their iterates using a one delay system can be interpreted as an interpolation problem [12,13]. Moreover, it has been shown how easily the pointwise convergence or non convergence of the system orbits can be achieved. Our goal is to present detailed quantitative information of the multiple delay inclusion on the dynamics of discrete systems where the iteration functions are given by the advection–reaction operators. To do this, we present in Section 2 an algorithm to show the equivalence of the multiple delay system to several one delay systems and with a matrix representation of the advection–reaction operators we explicitly find their iterates. Finally in Section 3 we will give sufficient conditions on the uniform convergence of the iterates for some particular cases.

2. Family of discrete delay advection–reaction operators

Let us start by defining the class of dynamical systems of our interest:

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Definition. Given two real nonzero sequences $\lambda = \{\lambda_i\}_{i=1}^{\infty}$ and $\mathbf{c} = \{c_i\}_{i=1}^{\infty}$ and a fixed natural number $k \geq 1$, we define the following bidimensional discrete k-delay dynamical system:

$$u_{m,n+1} = \begin{cases} \lambda_m u_{m,n} & \text{if } n \in \mathbb{N} \text{ and } m = 1, 2, 3, \ldots, k \\ \lambda_m u_{m,n} + c_{m-k} u_{m-k,n} & \text{if } m, n \in \mathbb{N} \text{ and } m \geq k+1. \end{cases} \tag{1}$$

Our interest in this family of systems has several reasons. First, notice that system (1) is basically a linear system and convergence of finite dimension linear systems is determined by their associated eigenvalues. In our case, the convergence of (1) will be determined by the sequence $\{c_i\}_{i=1}^{\infty}$, which introduces the delay and by its eigenvalues, which are given by sequence $\{\lambda_i\}_{i=1}^{\infty}$. Second, a sufficient condition for convergence of finite dimensional systems is that their eigenvalues belong to the unit ball; for system (1) this is not the case, we have to impose extra conditions on the eigenvalues and also on the sequence $\{c_i\}_{i=1}^{\infty}$, which will play an important role as we will discover later on. Third, besides its intrinsic value, it is important for their role in several applications for example by approximating the equation

$$\frac{\partial u(x,t)}{\partial t} + \alpha_1(x)\frac{\partial u(x,t)}{\partial x} + \alpha_2(x)\frac{\partial^k u(x,t)}{\partial x^k} = \beta(x)u(x,t),$$

via finite differences equations. For the higher order derivative one can construct approximations based on k-delay systems by sampling a number of points to the left or to the right of the center point. In particular if we set $k = 2$, then the system (1) can be viewed as a consistent discretization via finite differences on an infinite rectangular grid of the one dimensional *advection–diffusion–reaction equation* with weak diffusion by setting $u_{m,n} = u(mh, nk)$, $\epsilon = h$, $c_{m-2} = -\alpha_2(mh)\frac{k}{h^2}$ and $\lambda_m = -\alpha_1(mh)\frac{k}{h} - \alpha_2(mh)\frac{k}{h^2} + \beta(mh)k$ with h and k being the step sizes and the indexes m and n correspond to the space variable x and to the time variable t, respectively, see [11].

2.1. Decoupling

In many cases, delay ordinary differential equations are equivalent to systems of non-delay ordinary differential equations. In a previous work, [13], a discrete dynamical system with a bidiagonal structure was analyzed, which corresponds to the problem studied here, in the special case of $k = 1$. Our next goal here will be to decouple system (1) obtaining the equivalent representation of k discrete 1-delay dynamical systems as follows: Let ℓ be a fixed natural number satisfying $1 \leq \ell \leq k$, then for all $m, n \in \mathbb{N}$, we have the system:

$$v_{m,n+1}(\ell) = \begin{cases} \mu_1(\ell) v_{1,n}(\ell) & \text{if } m = 1 \\ \mu_m(\ell) v_{m,n}(\ell) + c_{m-1}(\ell) v_{m-1,n}(\ell) & \text{if } m \geq 2, \end{cases} \tag{2}$$

where $v_{m,n}(\ell) = u_{\ell+(m-1)k,n}$, $\mu_m(\ell) = \lambda_{\ell+(m-1)k}$ for all $n, m \in \mathbb{N}$ with $m \geq 1$ and $c_m(\ell) = c_{\ell+(m-2)k}$ for all $m \in \mathbb{N}$ with $m \geq 2$.

System (1) can also be written in the following matrix form

$$\hat{u}_{n+1} = A_{\mathbf{c},\lambda}\hat{u}_n, \tag{3}$$

where $A_{\mathbf{c},\lambda}$ is the infinite lower matrix

$$A_{c,\lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 & 0 & \cdots \\ \vdots & 0 & \ddots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & 0 & \cdots \\ c_1 & 0 & 0 & 0 & \lambda_{k+1} & \vdots \\ 0 & c_2 & 0 & \ddots & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{4}$$

and $\hat{u}_n$ is the real column sequence $\hat{u}_n = (u_{1,n}, u_{2,n}, \ldots)$. The matrix formulation of the problem is of paramount importance because it will allow us to show the development of the delay in the structure of the coefficient matrix for different iterations of the system (1).

We can also restate in matrix notation each of the k 1-delay systems by using the following algorithm, which is illustrated in an example below:

Algorithm 2.1 (*Decoupling*)**.** Generation of k 1-delay discrete systems.

For each value of ℓ from $1, 2, 3, \ldots, k$ perform the following steps; each value of ℓ will give rise to a 1-delay discrete system.

1. Replace the variable u_1 with u_ℓ. This amounts to interchanging rows 1 and ℓ and columns 1 and ℓ in matrix (4).
2. Move the c_ℓ value that is located at the entry $k+\ell, 1$ to the entry $2, 1$ by interchanging rows $k+\ell$ and 2 and also columns $k+\ell$ and 2.

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This step alters the lower triangular structure of the matrix, since the value c_2 will be located in the entry $k+2, k+\ell$, which will be corrected in the next step.

3. (*Intermediate step*) Interchange rows $k+2$ and $k+3$ and columns $k+2$ and $k+3$ in order to recover the lower triangular form of the coefficient matrix.
4. Continue moving all subsequent nonzero subdiagonal values as in steps 2 and 3. That is, for $m = 1, 2, 3, \dots$ move the $c_{\ell+mk}$ value by interchanging rows $m+2$ and $\ell+mk+3$ and columns $m+2$ and $\ell+mk+3$ then perform the intermediate step by exchanging rows $m+3$ and $\ell+mk+3$ and columns $m+3$ and $\ell+mk+3$.

The following example with $k = \ell = 3$ illustrates the steps of the algorithm.

Example (*Decoupling algorithm*).

$$A_{c,\lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_3 & 0 & 0 & \cdots \\ c_1 & 0 & 0 & \lambda_4 & 0 & \cdots \\ 0 & c_2 & 0 & 0 & \lambda_5 & \vdots \\ 0 & 0 & c_3 & \ddots & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5}$$

After the first step (interchanging row 1 and row 3 and column 1 by column 3) matrix (5) becomes

$$A_{c,\lambda} = \begin{pmatrix} \lambda_3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & c_1 & \lambda_4 & 0 & \cdots \\ 0 & c_2 & 0 & 0 & \lambda_5 & \vdots \\ c_3 & 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{6}$$

After the second step (interchanging rows $k+\ell = 6$ and 2) matrix (6) becomes

$$A_{c,\lambda} = \begin{pmatrix} \lambda_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ c_3 & \lambda_6 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & c_1 & \lambda_4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \lambda_5 & c_2 & 0 \\ 0 & 0 & 0 & \ddots & 0 & \lambda_2 & 0 \\ \vdots & 0 & \vdots & c_4 & \ddots & 0 & \ddots \\ \vdots & 0 & \vdots & 0 & c_5 & \ddots & \ddots \\ \vdots & c_6 & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}. \tag{7}$$

Notice that the value c_2 is the upper half position of the matrix breaking the lower structure of the matrix. This is where the intermediate step corrects the structure of the matrix, thus obtaining

$$A_{c,\lambda} = \begin{pmatrix} \lambda_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ c_3 & \lambda_6 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & c_1 & \lambda_4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & c_2 & \lambda_5 & 0 \\ \vdots & 0 & \vdots & c_4 & \ddots & 0 & \ddots \\ \vdots & 0 & \vdots & 0 & c_5 & \ddots & \ddots \\ \vdots & c_6 & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}. \tag{8}$$

Finally, step 4 moves all the values $c_6, c_9,\ c_{12}$ to its respective position giving rise to the coefficient matrix sought, namely

$$A_{c,\lambda} = \begin{pmatrix} \lambda_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ c_3 & \lambda_6 & 0 & 0 & 0 & 0 & \cdots \\ 0 & c_6 & \lambda_9 & 0 & 0 & 0 & \cdots \\ 0 & 0 & c_9 & \lambda_{12} & 0 & 0 & \cdots \\ 0 & 0 & 0 & c_{12} & \lambda_{15} & 0 & 0 \\ 0 & 0 & 0 & \ddots & c_{15} & \lambda_{18} & 0 \\ \vdots & 0 & \vdots & 0 & \ddots & c_{18} & \ddots \\ \vdots & 0 & \vdots & 0 & 0 & \ddots & \ddots \\ \vdots & 0 & \vdots & 0 & \ddots & \ddots & \ddots \end{pmatrix}.$$

Algorithm 2.1 resembles the Gaussian elimination algorithm [14] in the sense that both deal with the entries below the diagonal; in this algorithm by moving the nonzero values to its higher position below the diagonal and in the Gaussian elimination by generating zeros under the diagonal. The main difference between both algorithms is in the intermediate step in Algorithm 2.1. It is important to remark that there are several techniques to move the nonzero entries below the diagonal to their right position but most of them destroy the geometry of the lower triangular matrix.

Remark. Although the algorithm was formulated for a matrix with the special structure as in (4), it is clear that an analogous algorithm can be applied to decouple any system whose coefficient matrix is infinite lower triangular and has at most one nonzero entry in each column and in each row of its main diagonal.

We can replace Algorithm 2.1 by a more precise procedure by performing the following change of variables in (3):

$$\hat{u}_n = F_\ell E \hat{v}_n(\ell), \quad \ell = 1, 2, \ldots, k, \tag{9}$$

where

$$F_\ell = \prod_{m=0}^{\infty} E_{mk+1,mk+\ell} \quad \text{and} \quad E = \prod_{m=2}^{\infty} \left(E_{m,m+k-1} E_{m+k,mk+1}\right)$$

here $E_{s,t}$ denotes the identity infinity matrix with the columns s and t permuted. The reason for having a double row–column interchange in each step (see the double factors in the product of the matrix E) is to keep the lower triangular structure on the coefficient matrix. On the other hand, the matrix F_ℓ basically is used to change the ordered set of variables

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \ldots)$$

into the ordered set

$$(\hat{u}_\ell, \hat{u}_{\ell+k}, \hat{u}_{\ell+2k}, \hat{u}_{\ell+3k}, \hat{u}_{\ell+4k}, \ldots) \quad \text{for } \ell = 1, 2, \ldots, k.$$

Thus each 1-delay system, there are k of them, can be written as:

$$\hat{v}_{n+1}(\ell) = B_{\mathbf{c},\lambda,\ell} \hat{v}_n(\ell), \quad \ell = 1, 2, 3, \ldots, k, \tag{10}$$

where $B_{\mathbf{c},\lambda,\ell}$ is the infinite lower matrix

$$B_{\mathbf{c},\lambda,\ell} = \begin{pmatrix} \lambda_\ell & 0 & 0 & 0 & 0 & \cdots \\ c_\ell & \lambda_{\ell+k} & 0 & 0 & 0 & \cdots \\ 0 & c_{\ell+k} & \lambda_{\ell+2k} & 0 & 0 & \cdots \\ 0 & 0 & c_{\ell+2k} & \ddots & 0 & \cdots \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{11}$$

and $\hat{v}_n(\ell)$ is the real column sequence

$$\hat{v}_n(\ell) = (u_{\ell,n}, u_{\ell+k,n}, \ldots). \tag{12}$$

Notice that in matrix (11), once that we fixed the values of ℓ, the main diagonal elements form a subsequence of $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ where the original generic subindex j has been replaced by $\ell + (j-1)k$. The previous results can be summarized in the following result.

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Theorem 2.1. *The discrete k-delay dynamical system*

$$\hat{u}_{n+1} = A_{c,\lambda}\hat{u}_n, \tag{13}$$

with $A_{c,\lambda}$ given as in (4), can be written as k discrete 1-delay dynamical systems of the form:

$$\hat{v}_{n+1}(\ell) = B_{c,\lambda,\ell}\hat{v}_n(\ell), \tag{14}$$

where ℓ is a fixed natural number satisfying $1 \le \ell \le k$, $B_{c,\lambda,\ell}$ is given by (11) and the sequence $\hat{v}_n(\ell)$ is given by (12).

Once we get written the family of systems (14) as in (10), then the value of each sequence $\hat{v}_n$ is known since $\hat{v}_n = B^n_{\mathbf{c},\lambda,\ell}\hat{v}_0$ where $\hat{v}_0$ is an initial sequence. Therefore we need to compute the powers of the infinite matrix $B_{\mathbf{c},\lambda,\ell}$. From [11] we get that:

Corollary 2.2. *The entries of $B^n_{\mathbf{c},\lambda,\ell}$ are given by*

$$(B^n_{\mathbf{c},\lambda,\ell})_{i,j} = \begin{cases} 0 & \text{if } i < j \\ \lambda^n_{\ell+(i-1)k} & \text{if } i = j \\ \left(\prod_{m=j}^{i-1} c_{\ell+(m-1)k}\right) \cdot \mathbf{P}_{n-i+j}\left[\lambda_{\ell+(j-1)k}, \ldots, \lambda_{\ell+(i-1)k}\right] & \text{if } i > j \end{cases}$$

where we have used the following notation (obtained from [12]): Given any three natural numbers ℓ, m and n, let $\mathbf{P}_\ell[\lambda_m, \lambda_{m+2}, \lambda_{m+4}, \ldots, \lambda_{m+2(n-1)}]$ denote the homogeneous polynomial of degree ℓ consisting of the sum of all possible monomials with unitary coefficients in the n variables $\lambda_m, \lambda_{m+2}, \ldots, \lambda_{m+2(n-1)}$. By convention $\mathbf{P}_0[\lambda_m, \lambda_{m+2}, \ldots, \lambda_{m+2(n-1)}] = 1$ and $\mathbf{P}_{-\ell}[\lambda_m, \lambda_{m+2}, \ldots, \lambda_{m+2(n-1)}] = 0$, for all $\ell \in \mathbb{N}$.

So in essence we have decomposed the system into k decoupled 1-delay discrete dynamical systems and theoretically the solutions to each system can be computed and then integrate them into a general solution. When the value of k is small the previous procedure is very practical but for larger values of k it is impractical. Thus, we have to find a different strategy to compute the solutions for (14) as a complete set.

2.2. Entries of the operator $(A^n_{\mathbf{c},\lambda})$

The previous result gives us an explicit suggestion for the functional form for the entries of the operator $(A^n_{\mathbf{c},\lambda})$, but we need to take in consideration the delay. Thus we get the following result, which generalizes Theorem 3.5 of [13]. That result was stated and proved only for the special case $k = 1$.

Theorem 2.3. *The entries $(A^n_{\mathbf{c},\lambda})_{i,j}$ for $i, j \in \mathbb{N}$ are given by*

$$\begin{cases} 0 & \text{if } i < j \\ \lambda^n_i & \text{if } i = j \\ 0 & \text{if } i > j \text{ and } i \not\equiv j \bmod (k) \\ \left(\prod_{m=0}^{s-1} c_{j+mk}\right) \cdot \mathbf{P}_{n-s}\left[\lambda_j, \lambda_{j+k}, \ldots, \lambda_i\right] & \text{if } i > j \text{ and } i \equiv j \bmod (k) \end{cases}$$

where $s = (i-j)/k$.

Proof. The cases when $i \le j$ or when $i > j$ and $i \not\equiv j \bmod (k)$ are straightforward. For $i > j$ and $i \equiv j \bmod (k)$ the proof is achieved by induction over n. In this case $i = j + ks$ for some natural number $s > 1$. When $n = 1$ we have that $(A_{c,\lambda})_{i,j} = c_j$ and since $\mathbf{P}_{1-s}$ is different than zero only if $s = 1$ then we get:

$$\left(\prod_{m=0}^{s-1} c_{j+km}\right) \cdot \mathbf{P}_{1-s}\left[\lambda_j, \ldots, \lambda_{j+k}\right] = \left(\prod_{m=0}^{0} c_{j+km}\right) \cdot \mathbf{P}_0\left[\lambda_j, \lambda_{j+k}\right] = c_j.$$

Suppose now that $(A^n_{c,\lambda})_{i,j} = \left(\prod_{m=0}^{s-1} c_{j+mk}\right) \cdot \mathbf{P}_{n-s}\left[\lambda_j, \lambda_{j+k}, \ldots, \lambda_i\right]$ is valid for a fixed $n \in \mathbb{N}$ for all $i, j \in \mathbb{N}$, then

$$\begin{aligned} (A^{n+1}_{c,\lambda})_{i,j} &= \lambda_i (A^n_{c,\lambda})_{i,j} + c_{i-k}(A^n_{\mathbf{c},\lambda})_{i-k,j} \\ &= \lambda_i \left[\left(\prod_{l=0}^{s-1} c_{j+l}\right) \mathbf{P}_{n-s}[\lambda_j, \lambda_{j+k}, \ldots, \lambda_i]\right] + c_{i-k}\left[\left(\prod_{l=0}^{\frac{i-k-j}{k}-1} c_{j+l}\right) \mathbf{P}_{n-\frac{1}{k}(i-k-j)}[\lambda_j, \lambda_{j+k}, \ldots, \lambda_{i-k}]\right] \\ &= \left(\prod_{l=0}^{s-1} c_{j+l}\right)\left[\lambda_i \mathbf{P}_{n-s}[\lambda_j, \lambda_{j+k}, \ldots, \lambda_i] + \mathbf{P}_{n+1-s}[\lambda_j, \lambda_{j+k}, \ldots, \lambda_{i-k}]\right] \\ &= \left(\prod_{l=0}^{s-1} c_{j+l}\right) \mathbf{P}_{n+1-s}[\lambda_j, \lambda_{j+k}, \ldots, \lambda_i]. \qquad \blacksquare \end{aligned}$$

Notice that Theorem 2.3 establishes the value of the entries of the powers of the infinite matrix $A_{\mathbf{c},\lambda}$. Therefore the dynamics of the whole system is completely determined. Moreover the delay affects the entries of the matrix by setting $k-1$ zero subdiagonals in between each nonzero subdiagonal.

3. Particular cases

It is important to understand the theoretical implications of Theorem 2.3. So in order to obtain information of such result we will analyze two particular cases. The first one is the case when the two defining sequences λ and $\mathbf{c}$ are constant sequences. In the second one, we will consider general sequences $\mathbf{c}$ but the λ sequence will be required to have different elements. For the first case we will show the iterates of the infinite matrix $A_{\mathbf{c},\lambda}$ to show the effect of the delay and then we will provide with sufficient conditions on uniform convergence to guarantee that the zero sequence becomes an attractor. For the second case we will compute explicitly the entries of the powers of the infinite matrix $A_{\mathbf{c},\lambda}$ with a different approach as in the proof of Theorem 2.3 to show the direct connection of such entries with interpolation theory.

3.1. Constant case

Let us show a particular case with constant sequences given by $\{\lambda_j\}_{j=1}^{\infty} = \{\lambda\}_{j=1}^{\infty}$ and $\{c_j\}_{j=1}^{\infty} = \{c\}_{j=1}^{\infty}$ for every j. Then the matrix $A^3_{\mathbf{c},\lambda}$ (for simplicity we choose $n=3$ and $k=3$) takes the form:

$$\begin{pmatrix}
\lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & \cdots \\
3c\lambda^2 & 0 & 0 & \lambda^3 & 0 & 0 & 0 & 0 & \cdots \\
0 & 3c\lambda^2 & 0 & 0 & \lambda^3 & 0 & 0 & 0 & \cdots \\
0 & 0 & 3c\lambda^2 & 0 & 0 & \lambda^3 & 0 & 0 & \cdots \\
3c^2\lambda & 0 & 0 & 3c\lambda^2 & 0 & 0 & \lambda^3 & 0 & \cdots \\
0 & 3c^2\lambda & 0 & 0 & 3c\lambda^2 & 0 & 0 & \lambda^3 & \ddots \\
0 & 0 & 3c^2\lambda & 0 & 0 & 3c\lambda^2 & 0 & 0 & \ddots \\
c^3 & 0 & 0 & 3c^2\lambda & 0 & 0 & 3c\lambda^2 & 0 & \ddots \\
0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & \ddots
\end{pmatrix}. \tag{15}$$

Observe that in this particular case only four subdiagonals (including the main diagonal) are nonzero and each one is equal to the terms in the binomial expansion of $(\lambda+c)^3$. Starting in the main diagonal with λ^3. Following with the subdiagonals with the common entry given by $3\lambda^2 c$ and $3\lambda c^2$ and finishing with c^3. Therefore the $\hat{u}_3$ sequence consists in the multiplication of such infinite matrix with a given initial condition $\hat{u}_0$. For general $n \in \mathbb{N}$, the matrix $A^n_{\mathbf{c},\lambda}$ will have $n+1$ nonzero subdiagonals separated by $k-1$ zero subdiagonals. Each nonzero diagonal has a common entry given by the term corresponding to the development of $(\lambda+c)^n$.

In the following we will provide some sufficient conditions on convergence to illustrate the use of the results from Theorem 2.3. The convergence depends on the norm used, and for the sake of applicability we use the sup norm to guarantee uniform convergence. Let $\mathbb{S}$ be the Banach space consisting of all convergent sequences together with the sup norm $\|\cdot\|_\infty$, and $\mathbb{S}_0$ be its closed subspace formed by those sequences converging to 0. Taking an arbitrary sequence $\mathbf{y} \in \mathbb{S}$, we have that

$$\|A^n_{\mathbf{c},\lambda}\mathbf{y}\|_\infty \le (\,|c|+|\lambda|\,)^n \cdot \|\mathbf{y}\|_\infty.$$

Thus a sufficient condition for the limiting sequence (the limit is taken entry-wise) $\lim_{n\to\infty} A^n_{\mathbf{c},\lambda} y$ to be in $\mathbb{S}_0$ is that $|c|+|\lambda| < 1$. Therefore the zero sequence becomes an attractor for all sequences in $\mathbb{S}$.

For a general case if we assume that $\left(\prod_{k=0}^{\infty} c_{j+k}\right) < \infty$ for all j with a uniform bound given by C and if $\lambda = \{\lambda_j\}_{j=1}^{\infty} \in \ell^1$, the subspace of all real sequences consisting of all sequences λ satisfying $\sum_m |\lambda_m| < \infty$, then taking an arbitrary sequence $\mathbf{y} \in \mathbb{S}$, we have that

$$\begin{aligned}
\|A^n_{\mathbf{c},\lambda}\mathbf{y}\|_\infty &\le C\|\mathbf{y}\|_\infty \sum_{\beta=1}^{\alpha_n} |\mathbf{P}_{n+1-\frac{\alpha_n-\beta}{k}}[\lambda_\beta,\ldots,\lambda_{\alpha_n}]| \\
&\le C_1 \sum_{\beta=1}^{\alpha_n} \|\lambda\|_1^{n+1-\frac{\alpha_n-\beta}{k}} = C_1 \|\lambda\|_1^{n+1-\frac{\alpha_n-1}{k}} \left(\frac{1-\|\lambda\|_1^{\frac{\alpha_n}{k}}}{1-\|\lambda\|_1^{\frac{1}{k}}}\right),
\end{aligned}$$

where $C_1 = C\|\mathbf{y}\|_\infty$, α_n is the row where $\|A^n_{\mathbf{c},\lambda}\mathbf{z}\|_\infty$ is achieved for the unitary sequence $\mathbf{z} = \{1,1,\ldots\}$ and $\|\cdot\|_1$ denotes the ℓ^1 norm. Therefore assuming that $\|\lambda\|_1 < 1$ we have that the sequence $\{A^n_{\mathbf{c},\lambda}\mathbf{y}\}$ approaches the zero sequence as n goes to infinity for any $\mathbf{y} \in \mathbb{S}$.

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3.2. Different element case

As a next step we will consider another particular case: we will consider a special case where all the eigenvalues of $A_{\mathbf{c},\lambda}$ are different, that is, the sequence λ has different elements.

Theorem 3.1. *Assume that all eigenvalues of $A_{c,\lambda}$ are different. Let $(\Phi_{c,\lambda})_{ij}$ denote the i,j entry of $\Phi_{c,\lambda}$ with*

$$(\Phi_{c,\lambda})_{ij} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ 0 & \text{if } i > j \text{ and } i \not\equiv j \bmod (k) \\ \dfrac{1}{(\lambda_j - \lambda_i)} \displaystyle\prod_{m=0}^{s-1} \frac{c_{j+mk}}{\lambda_{j+mk} - \lambda_j} & \text{if } i > j \text{ and } i \equiv j \bmod (k), \end{cases}$$

where $s = \frac{i-j}{k}$. Let Ω_w be an infinite diagonal matrix with $(\Omega_w)_{ii} = \omega(\lambda_i)$, where $\omega : \mathbb{R} \to \mathbb{R}$ is an arbitrary but fixed function. Then the matrix $\Phi_{c,\lambda}$ is an invertible infinite lower triangular matrix such that $(A^n_{c,\lambda})_{ij} = (\Phi_{c,\lambda} \circ \Omega^n_w \circ \Phi^{-1}_{c,\lambda})_{ij}$ for any natural number $n \geq 0$, where the entries $(\Phi_{c,\lambda} \circ \Omega_w \circ \Phi^{-1}_{c,\lambda})_{ij}$ are given by

$$\begin{cases} 0 & \text{if } i < j \\ \omega(\lambda_i) & \text{if } i = j \\ 0 & \text{if } i > j \text{ and } i \not\equiv j \bmod(k) \\ \left(\dfrac{1}{s} \displaystyle\prod_{m=0}^{s-1} c_{j+mk} \right) \cdot \dfrac{d^s}{d\mu^s} \mathcal{L}_\omega[\lambda_j, \lambda_{j+k}, \dots, \lambda_i](\mu) & \text{if } i > j \text{ and } i \equiv j \bmod (k) \end{cases}$$

where $\mathcal{L}_\omega[\lambda_j, \lambda_{j+k}, \dots, \lambda_i](\lambda)$ is the Lagrange polynomial in the variable μ of degree s at the nodes $\{\lambda_j, \lambda_{j+k}, \dots, \lambda_i\}$ for the function $\omega(\mu)$ [15].

Proof. It is straightforward to verify that the matrix $\Phi^{-1}_{c,\lambda}$ exists. Moreover its entries are given by

$$(\Phi^{-1}_{c,\lambda})_{i,j} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ 0 & \text{if } i > j \text{ and } i \not\equiv j \bmod (k) \\ \displaystyle\prod_{m=0}^{\frac{i-j}{k}-1} \frac{c_{j+mk}}{\lambda_i - \lambda_{j+mk}} & \text{if } i > j \text{ and } i \equiv j \bmod (k). \end{cases}$$

Now since all three matrices, Φ, Ω and Φ^{-1} are lower triangular, their product is also lower triangular. Thus there are several cases to show, but the only one that is not straightforward is if $i > j$ and $i \equiv j \bmod (k)$ then $i = j + ks$ for some $s \in N$. Thus the $\Phi_{c,\lambda} \circ \Omega_w \circ \Phi^{-1}_{c,\lambda_{ij}}$ entries are given by

$$\begin{aligned}
\sum_{m=j}^{i} \left(\Phi_{c,\lambda} \circ \Omega_w\right)_{i,m} \left(\Phi^{-1}_{c,\lambda}\right)_{m,j} &= \sum_{m=0}^{s} \left(\Phi_{c,\lambda} \circ \Omega_w\right)_{i,j+mk} \left(\Phi^{-1}_{c,\lambda}\right)_{j+mk,j} \\
&= \omega(\lambda_i) \prod_{\ell=0}^{s-1} \frac{c_{j+k\ell}}{\lambda_i - \lambda_{j+k\ell}} + \frac{\omega(\lambda_j)}{\lambda_j - \lambda_i} \prod_{\ell=0}^{s-1} \frac{c_{j+k\ell}}{\lambda_j - \lambda_{j+k\ell}} \\
&\quad + \sum_{m=1}^{s-1} \frac{\omega(\lambda_{j+km})}{\lambda_{j+km} - \lambda_i} \left(\prod_{\ell=0}^{s-m-1} \frac{c_{j+km+k\ell}}{\lambda_{j+km} - \lambda_{j+km+k\ell}} \right) \left(\prod_{\ell=0}^{m-1} \frac{c_{j+k\ell}}{\lambda_{j+km} - \lambda_{j+k\ell}} \right) \\
&= \left(\prod_{\ell=0}^{s-1} c_{j+k\ell} \right) \left(\omega(\lambda_i) \prod_{\ell=0}^{s-1} \frac{1}{\lambda_i - \lambda_{j+k\ell}} + \frac{\omega(\lambda_j)}{\lambda_j - \lambda_i} \prod_{\ell=1}^{s-1} \frac{1}{\lambda_j - \lambda_{j+k\ell}} \right) \\
&\quad + \sum_{m=1}^{s-1} \frac{\omega(\lambda_{j+km}) c_{j+km}}{\lambda_{j+km} - \lambda_i} \left(\prod_{l=m+1}^{s-1} \frac{c_{j+k\ell}}{\lambda_{j+km} - \lambda_{j+kl}} \right) \left(\prod_{l=0}^{m-1} \frac{c_{j+kl}}{\lambda_{j+km} - \lambda_{j+kl}} \right) \\
&= \left(\prod_{\ell=0}^{s-1} c_{j+k\ell} \right) \left[\sum_{m=0}^{s} \omega(\lambda_{j+km}) \left(\prod_{\ell=0, \ell \neq m}^{s} \frac{1}{\lambda_{j+km} - \lambda_{j+k\ell}} \right) \right] \\
&= \left(\prod_{\ell=0}^{s-1} c_{j+k\ell} \right) \frac{d^s}{d\mu^s} \mathcal{L}_\omega[\lambda_j, \lambda_{j+k}, \dots, \lambda_i](\mu).
\end{aligned}$$

Now let us prove that $\Phi_{c,\lambda} \circ \Omega_w \circ \Phi_{c,\lambda}^{-1} = (A_{c,\lambda})$. We will prove such equality proving that equality is achieved entry-wise. Let us notice that we only have to prove the equality for the entries $i > j$ with $i \equiv j \bmod (k)$.

Since the polynomial $p(x) = x$ interpolates at the nodes given by the eigenvalues of $A_{c,\lambda}$ and by uniqueness of the Lagrange polynomial, we conclude that $\mathcal{L}_w[\lambda_j, \lambda_{j+k}, \dots, \lambda_i](\mu)$ is a polynomial of degree one. Therefore its s-derivative is zero except when $i - j = k$. For these entries we have

$$(\Phi_{c,\lambda} \circ \Omega \circ \Phi_{c,\lambda}^{-1})_{j+k,j} = \prod_{\ell=0}^{0} c_{j+k\ell} \frac{d}{d\mu} \mathcal{L}_\omega[\lambda_j, \lambda_{j+k}](\mu)$$
$$= c_j \frac{d}{d\mu} \left(\frac{\lambda_j(\mu - \lambda_{j+k})}{\lambda_j - \lambda_{j+k}} + \frac{\lambda_{j+k}(\mu - \lambda_j)}{\lambda_{j+k} - \lambda_j} \right) = c_j. \quad \blacksquare$$

In general the eigenvalues of $A_{c,\lambda}$ appear in the entries of $A_{c,\lambda}^n$ as nodes of an interpolation polynomial for the function $w(x) = x^n$ and the product of the subdiagonal elements can be interpreted as correction coefficients or weights. These two facts show a direct connection with interpolation theory. Notice that we also obtain the following double identity:

Corollary 3.2. *If $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ is a sequence with different elements and $w(\lambda) = \lambda$, then $\forall i, j, n \in \mathbb{N}$ with $i > j$ and $i \equiv j \bmod (k)$*

$$s\mathbf{P}_{n-s}\left[\lambda_j, \lambda_{j+k}, \dots, \lambda_i\right] = \mathcal{L}_w^{(s)}[\lambda_j, \lambda_{j+k}, \dots, \lambda_i] = sf_n[\lambda_j, \lambda_{j+k}, \dots, \lambda_i],$$

where $f_n[\lambda_j, \lambda_{j+k}, \dots, \lambda_i]$ denotes the divided differences (see [16]) of the function $f_n(x) = x^n$ at the nodes $\lambda_j, \lambda_{j+k}, \dots, \lambda_i$, see [12] for details on this last equality.

4. Conclusions

One goal of this study was to analyze the result of introducing a multiple delay in the advection–reaction operators. We discovered that introducing a k multiple delay in a discrete dynamical system generated by such operators is equivalent to introducing a delay in k dynamical systems. Moreover, we were able to compute explicitly the iterates of such operator with two different approaches showing their dynamical behavior and their connection with interpolation theory.

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