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The evaluation of geometric Asian power options under time changed mixed fractional Brownian motion

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ABSTRACT

In this paper, the geometric Asian option pricing problem is investigated under the assumption that the underlying stock price is assumed following a mixed fractional subdiffusive Black–Scholes model, and the geometric average Asian option pricing formula is derived under this assumption. We then apply the results to value Asian power options on the stocks that pay constant dividends when the payoff is a power function. Finally, lower bound of Asian options and some special cases are provided.

1. Introduction

A standard option is a financial contract which gives the owner of the contract the right, but not the obligation, to buy or sell a specified asset at a prespecified time (maturity) for a prespecified price (strike price). The specified asset (underlying asset) can be for example stocks, indexes, currencies, bonds or commodities. The option can be either a call option, which gives the owner the right to buy the underlying asset, or it can be a put option, which gives the owner the right to sell the underlying asset. There are several types of options that are traded in a market. American option allows the owner to exercise his option at any time up to and including the strike date. European options can be exercised only on the strike date. European options are also called vanilla options. Their payoffs at maturity depend on the spot value of the stock at the time of exercise. There are other options whose values depend on the stock prices over a predetermined time interval. For an Asian option, the payoff is determined by the average value over some predetermined time interval. The average price of the underlying asset can either determine the underlying settlement price (average price Asian options) or the option strike price (average strike Asian options). Furthermore, the average prices can be calculated using either the arithmetic mean or the geometric mean. The type of Asian option that will be examined throughout this research is geometric Asian option.

Over the past three decades, academic researchers and market practitioners have developed and adopted different models and techniques for option valuation. The most popular model on option pricing was introduced by Black and Scholes (*BS*) [1] in 1973. In the *BS* model it has been assumed that the asset price dynamics are governed by a geometric Brownian motion. However, a large number of empirical studies have shown that the distributions of the logarithmic returns of financial asset usually exhibit properties of self-similarity, heavy tails, long-range dependence in both auto-correlations and cross-correlations, and volatility clustering [2–4]. Actually, the most popular stochastic process that exhibits long-range

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dependence is of course the fractional Brownian motion. Moreover, the fractional Brownian motion produces a burstiness in the sample path behavior, which is the important behavior of financial time series. Since fractional Brownian motion is neither a Markov process nor a semi-martingale, we cannot use the usual stochastic calculus to analyze it. Further, fractional Brownian motion admits arbitrage in a complete and frictionless market. To get around this problem and to take into account the long memory property, it is reasonable to use the mixed fractional Brownian motion (*mfBm*) to capture the fluctuations of the financial asset [5–7].

The *mfBm* is a linear combination of the Brownian motion and fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$, defined on the filtered probability $(\Omega, \mathscr{F}, \mathbb{P})$ for any $t \in \mathbb{R}^+$ by:

$$M_t^H(a, b) = aB(t) + bB^H(t), \tag{1.1}$$

where $B(t)$ is a Brownian motion, and $B^H(t)$ is an independent fractional Brownian motion with Hurst index H. Cheridito [7] proved that, for $H \in (\frac{3}{4}, 1)$, the mixed model is equivalent to the Brownian motion and hence it is also arbitrage free. For $H \in (\frac{1}{2}, 1)$, Mishura and Valkeila [8] demonstrated that the mixed model is arbitrage free. Rao [9] discussed geometric Asian power option under *mfBm*. To see more about the mixed model, one can refer to Refs. [6,7,10,11].

Analysis of various real-life data shows that many processes observed in economics display characteristic periods in which they stay motionless [12]. This feature is most common for emerging markets in which the number of participants, and thus the number of transactions, is rather low. Notably, similar behavior is observed in physical systems exhibiting subdiffusion. The constant periods of financial processes correspond to the trapping events in which the subdiffusive test particle gets immobilized [13]. Subdiffusion is a well known and established phenomenon in statistical physics. Its usual mathematical description is in terms of the celebrated Fractional Fokker–Planck equation (*FFPE*). This equation was first derived from the continuous-time random walk scheme with heavy-tailed waiting times [14,15,10], and since then became fundamental in modeling and analysis of complex systems exhibiting slow dynamics. Following this line, and to model the observed long range dependence and fluctuations in the financial price time series, we introduce a time-changed mixed fractional *BS* model to value Asian power option when the underlying stock is

$$\begin{aligned} S_t &= X(T_\alpha(t)) \\ &= S_0 e^{(r-q)T_\alpha(t) + M_\alpha^H(t)(\sigma,\sigma) - \frac{1}{2}\sigma^2 \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{1}{2}\sigma^2 \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}}, \quad S_0 = X(0) > 0, \end{aligned} \tag{1.2}$$

where $M_\alpha^H(t)(\sigma, \sigma) = \sigma B(T_\alpha(t)) + \sigma B^H(T_\alpha(t))$, $\alpha \in (\frac{1}{2}, 1)$, $2\alpha - \alpha H > 1$ and $T_\alpha(t)$ is the inverse α-stable subordinator.

We then apply the result to price geometric Asian power options that pay constant dividends when the payoff is a power function. We also provide some special cases and lower bound for the Asian option price. The rest of the paper is organized as follows. In Section 2, some useful concepts and theorems of time changed mixed fractional process are introduced. In Section 3, a brief introduction of Asian options is given. Analytical valuation formula for geometric Asian options is derived in Section 4 and then applied to geometric Asian power options in Section 5. The lower bound on the price of the Asian option is proposed in Section 6.

2. Auxiliary facts

In this section, we recall some definitions and results about mixed fractional time changed process. More information about mixed fractional process can be found in [16,10].

The time-changed process $T_\alpha(t)$ is the inverse α-stable subordinator defined as below

$$T_\alpha(t) = \inf\{\tau > 0, U_\alpha(t) \geq t\}$$

here $U_\alpha(\tau)_{\tau \geq 0}$ is a strictly increasing α-stable Lévy process [17] with Laplace transform: $\mathbb{E}(e^{-uU_\alpha(\tau)}) = e^{-\tau u^\alpha}$, $\alpha \in (0, 1)$.

$U_\alpha(t)$ is $\frac{1}{\alpha}$ self-similar and $T_\alpha(t)$ is α self-similar, that is, for every $h > 0$, $U_\alpha(ht) \triangleq h^{\frac{1}{\alpha}} U_\alpha(t)$ $T_\alpha(ht) \triangleq h^\alpha T_\alpha(t)$, here $\triangleq$ indicates that the random variables on both sides have the same distribution. Specially, when $\alpha \uparrow 1$, $T_\alpha(t)$ reduces to the physical time t. You can find more details about subordinator and its inverse processes in [18,19].

Consider the subdiffusion process

$$M_\alpha^H(t)(a, b) = aW_\alpha(t) + bW_\alpha^H(t) = aB(T_\alpha(t)) + bB^H(T_\alpha(t)),$$

where $B(\tau)$ is a Brownian motion, $B^H(\tau)$ is a fractional Brownian motion with Hurst index H and $T_\alpha(t)$ is inverse α-subordinator which are supposed to be independent. When $a = 0, b = 1$, then it is the process considered in [20] and if $b = 0, a = 1$, then it is the process considered in [21]. In this research, we assume that $H \in (\frac{3}{4}, 1)$ and $(a, b) = (1, 1)$.

Remark 2.1. When $\alpha \uparrow 1$, the processes $W_\alpha(t)$ and $W_\alpha^H(t)$ degenerate to $B(t)$ and $B^H(t)$, respectively. Then, $M_\alpha^H(t)(a, b)$ reduces to the *mfBm* in Eq. (1.1).

Remark 2.2. From [20,21], we know that $\mathbb{E}(T_\alpha(t)) = \frac{t^\alpha}{\Gamma(\alpha+1)}$. Then, by applying α-self-similar and non-decreasing sample path of $T_\alpha(t)$, we have

$$\mathbb{E}[(B(T_\alpha(t)))^2] = \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{2.1}$$

$$\mathbb{E}[(B^H(T_\alpha(t)))^2] = \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}. \tag{2.2}$$

3. Asian options

The payoff of an Asian option is based on the difference between an asset's average price over a given time period, and a fixed price called the strike price. Asian options are popular because they tend to have lower volatility than options whose payoffs are based purely on a single price point. It is also harder for big traders to manipulate an average price over an extended period than a single price, so Asian option offers further protection against risk. The Asian call and put options have a payoff that is calculated with an average value of the underlying asset over a specific period. The payoff for an Asian call and put option with strike price K and expiration time T is $(\bar{S}(T) - K)_+$ and $(K - \bar{S}(T))_+$ respectively, where $\bar{S}(T)$ is the average price of the underlying asset over the prespecified interval. Since Asian options are less expensive than their European counterparts, they are attractive to many different investors. Apart from the regular Asian option there also exists Asian strike option. An Asian strike call option guarantees the holder that the average price of an underlying asset is not higher than the final price. The option will not be exercised if the average price of the underlying asset is greater than the final price. The holder of an Asian strike put option makes sure that the average price received for the underlying asset is not less than what the final price will provide. The payoff for an Asian strike call and put option is $(\bar{S}(T)-S(T))_+$ and $(S(T)-\bar{S}(T))_+$ respectively, where $S(T)$ is the value of underlying stock at maturity date T.

Asian options are divided into two different types, when calculating the average, the geometric Asian option

$$G(T) = \exp\left\{\frac{1}{T}\int_0^T \ln S(t)dt\right\},$$

and the arithmetic Asian option.

$$A(T) = \frac{1}{T}\int_0^T S(t)dt.$$

We assume that the prespecified interval $[0, T]$ is fixed, then will price the geometric Asian option in the continuous average case under time changed mixed fractional Brownian motion environment.

4. Pricing model of geometric Asian option

In order to derive an Asian option pricing formula in a time changed mixed fractional market, we make the following assumptions:

(i) The price of underlying stock at time t is given by Eq. (1.2).
(ii) There are no transaction costs in buying or selling the stocks or option.
(iii) The risk free interest rate r and dividend rate q are known and constant through time.
(iv) The option can be exercised only at the maturity time.

From Eq. (1.2), we know that $\ln S_t \simeq N(u, v)$, where

$$u = \ln S(0) + (r-q)T_\alpha(t) - \frac{1}{2}\sigma^2\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{1}{2}\sigma^2\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H} \tag{4.1}$$

$$v = \sigma^2\frac{t^\alpha}{\Gamma(\alpha+1)} + \sigma^2\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}. \tag{4.2}$$

Let $C(S(0), T)$ be the price of a European call option at time 0 with strike price K and that matures at time T. Then, from [16], we can get

$$C(S(0), T) = S(0)e^{-qT}\phi(d_1) - Ke^{-rT}\phi(d_2),$$

where

$$d_1 = \frac{\ln\frac{S_0}{K} + (r - q + \frac{\hat{\sigma}^2}{2})T}{\hat{\sigma}\sqrt{T}}, \quad d_2 = d_1 - \hat{\sigma}\sqrt{T},$$

$$\hat{\sigma}^2 = \sigma^2 \frac{T^{\alpha-1}}{\Gamma(\alpha)} + \sigma^2 \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H},$$

and $\phi(.)$ denotes cumulative normal density function.

Under the above assumptions (i)–(iv), we obtain the value of the geometric Asian call option by the following theorem

Theorem 4.1. *Suppose the stock price S_t satisfied Eq. (1.2). Then, under the risk-neutral probability measure, the value of geometric Asian call option $C(S(0), T)$ with strike price K and maturity time T is given by*

$$C(S(0), T) = S(0) \exp\left\{ -rT + (r-q)\frac{T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{2\Gamma(\alpha+3)} \right.$$
$$\left. - \frac{\sigma^2 T^{2\alpha H}}{4(2\alpha H+1)(\alpha H+1)(\Gamma(\alpha+1))^{2H}} \right\} \phi(d_1) - Ke^{-qT}\phi(d_2), \tag{4.3}$$

where

$$d_2 = \frac{\mu_G - \ln K}{\sigma_G}, \quad d_1 = d_2 + \sigma_G,$$
$$\mu_G = \ln S(0) + (r - q - \frac{\sigma^2}{2})\frac{T^\alpha}{\Gamma(\alpha+2)} - \frac{\sigma^2 T^{2\alpha H}}{2(2\alpha H+1)(\Gamma(\alpha+1))^{2H}},$$
$$\sigma_G^2 = \frac{\sigma^2 T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{\Gamma(\alpha+3)} + \frac{\sigma^2 T^{2\alpha H}}{(2\alpha H+2)(\Gamma(\alpha+1))^{2H}},$$

the interest rate r and the dividend rate q are constant over time and $\phi(.)$ denotes cumulative normal density function.

Proof. Suppose

$$L(T) = \frac{1}{T}\int_0^T \ln S(t)dt.$$

Then

$$G(T) = e^{L(T)}. \tag{4.4}$$

We know that $\ln S_t \simeq N(u, v)$, then it is clear that the random variable $L(T)$ has Gaussian distribution under the risk-neutral probability measure. We will now compute its mean and variance under the risk-neutral probability measure. Let $\mathbb{E}$ denote the expectation and, μ_G and σ_G^2 denote the mean and the variance of the random variable $\mathbb{E}$ under the risk-neutral probability measure. Note that

$$\mu_G = \mathbb{E}[L(T)] = \frac{1}{T}\int_0^T \mathbb{E}[\ln S(t)]dt$$
$$= \ln S(0) + \frac{1}{T}\int_0^T (r-q)\frac{t^\alpha}{\Gamma(\alpha+1)}dt - \frac{\sigma^2}{2T}\int_0^T \left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha H}}{(\Gamma(\alpha+1))^{2H}}\right]dt$$
$$= \ln S(0) + (r-q)\frac{T^\alpha}{\Gamma(\alpha+2)} - \frac{\sigma^2 T^\alpha}{2\Gamma(\alpha+2)} - \frac{\sigma^2 T^{2\alpha H}}{(4\alpha H+2)(\Gamma(\alpha+1))^{2H}},$$

and

$$\sigma_G^2 = Var[L(T)] = \mathbb{E}[(L(T) - \mu_G)^2]$$
$$= \frac{\sigma^2}{T^2}\int_0^T\int_0^T \left(\mathbb{E}[W_\alpha(t)W_\alpha(\tau)] + \mathbb{E}[W_\alpha^H(t)W_\alpha^H(\tau)]\right) dtd\tau,$$

by independence of the processes $B(t)$, $B^H(t)$ and $T_\alpha(t)$, we obtain

$$= \frac{\sigma^2}{T^2}\int_0^T\int_0^T \left(|\frac{t^\alpha}{\Gamma(\alpha+1)}| + |\frac{\tau^\alpha}{\Gamma(\alpha+1)}| - |\frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)}|\right) dtd\tau$$
$$+ \frac{\sigma^2}{T^2}\int_0^T\int_0^T \left(|\frac{t^\alpha}{\Gamma(\alpha+1)}|^{2H} + |\frac{\tau^\alpha}{\Gamma(\alpha+1)}|^{2H} - |\frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)}|^{2H}\right) dtd\tau$$
$$= \frac{\sigma^2 T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{\Gamma(\alpha+3)} + \frac{\sigma^2 T^{2\alpha H}}{(2\alpha H+2)(\Gamma(\alpha+1))^{2H}}.$$

From (4.4), we know that the random variable $G(T)$ is log-normally distributed, then $\ln G(T) \simeq N(\mu_G, \sigma_G^2)$. Let $I = \{x : e^x > K\}$ and $\phi(.)$ be the probability density function of a standard normal distribution, then the price of geometric Asian call option is given by the following computations

$$\begin{aligned}
C(S(0),T) &= e^{-rT}\mathbb{E}[(G(T)-K)^+]\\
&= e^{-rT}\int_I (e^x-K)\frac{1}{\sqrt{2\pi}\sigma_G}\exp\left\{-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right\}dx\\
&= e^{-rT}\int_I (e^{\mu_G+z\sigma_G}-K)\frac{1}{\sqrt{2\pi}\sigma_G}\exp\left\{-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right\}\varphi(z)dz\\
&= e^{-rT+\mu_G+\frac{1}{2}\sigma_G^2}\int_{-d_2}^{\infty} e^{-\frac{1}{2}(z-\sigma_G)^2}dz - Ke^{-rT}\int_{-d_2}^{-\infty}\varphi(z)dz\\
&= e^{-rT+\mu_G+\frac{1}{2}\sigma_G^2}\int_{-d_2-\sigma_G}^{\infty}\varphi(z)dz - Ke^{-rT}\int_{-\infty}^{d_2}\varphi(z)dz\\
&= e^{-rT+\mu_G+\frac{1}{2}\sigma_G^2}\int_{-\infty}^{d_2+\sigma_G}\varphi(z)dz - Ke^{-rT}\int_{-\infty}^{d_2}\varphi(z)dz\\
&= e^{-rT+\mu_G+\frac{1}{2}\sigma_G^2}\phi(d_1) - Ke^{-rT}\phi(d_2),\\
&= S(0)\exp\left\{-rT+(r-q)\frac{T^\alpha}{\Gamma(\alpha+2)}+\sigma^2\frac{(-T)^\alpha}{2\Gamma(\alpha+3)}\right.\\
&\quad\left.-\sigma^2\frac{T^{2\alpha H}}{4(2\alpha H+1)(\alpha H+1)(\Gamma(\alpha+1))^{2H}}\right\}\phi(d_1)-Ke^{-qT}\phi(d_2),
\end{aligned}$$

here

$$\begin{aligned}
I &= \{x : e^x > K\} = \{z : e^{\mu_G+z\sigma_G} > K\}\\
&= \{z : \mu_G + z\sigma_G > \ln K\} = \{z : z > -d_2\},
\end{aligned}$$

thus we obtain the pricing formula. □

Moreover, using the put–call parity, the valuation model for a geometric Asian put option under time changed mixed fractional *BS* model can be written

$$\begin{aligned}
P(S(0),T) &= Ke^{-qT}\phi(-d_2) - S(0)\exp\left\{-rT+(r-q)\frac{T^\alpha}{\Gamma(\alpha+2)}+\frac{\sigma^2(-T)^\alpha}{2\Gamma(\alpha+3)}\right.\\
&\quad\left.-\frac{\sigma^2T^{2\alpha H}}{4(2\alpha H+1)(\alpha H+1)(\Gamma(\alpha+1))^{2H}}\right\}\phi(-d_1),
\end{aligned}\tag{4.5}$$

where d_1 and d_2 are defined previously.

Letting $\alpha \uparrow 1$, then the stock price follows the *mfBm* shown below

$$\begin{aligned}
S_t &= S_0\exp\left\{(r-q)T+\sigma B(t)+\sigma B^H(t)\right.\\
&\quad\left.-\frac{1}{2}\sigma^2 t-\frac{1}{2}\sigma^2t^{2H}\right\},\quad 0<t<T,
\end{aligned}\tag{4.6}$$

and the result is presented below.

Corollary 4.1. *The value of geometric Asian call option with maturity T and strike K, whose stock price follows Eq. (4.6), is given by*

$$\begin{aligned}
&C(S(0),T) =\\
&S(0)\exp\left\{-\frac{1}{2}(r+q)T-\frac{\sigma^2T}{12}-\frac{\sigma^2T^{2H}}{4(2H+1)(H+1)}\right\}\phi(d_1)-Ke^{-qT}\phi(d_2),
\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}
d_2 &= \frac{\mu_G-\ln K}{\sigma_G},\quad d_1 = d_2+\sigma_G,\\
\mu_G &= \ln S(0)+\frac{1}{2}(r-q-\frac{\sigma^2}{2})T-\frac{\sigma^2T^{2H}}{2(2H+1)},
\end{aligned}$$

$$\sigma_G^2 = \frac{\sigma^2 T}{3} + \frac{\sigma^2 T^{2H}}{(2H+2)},$$

which is consistent with result in [9].

5. Pricing model of Asian power option

In this section, we consider the pricing model of Asian power call option with strike price K and maturity time T under time changed mixed fractional *BS* model where the payoff function is $(G^n(T) - K)^+$ for some constant integer $n \geq 1$.

Theorem 5.1. *Suppose the stock price S_t satisfied Eq. (1.2). Then, under the risk-neutral probability measure the value of geometric Asian power call option $C(S(0), T)$ with strike price K, maturity time T and payoff function $(G^n(T) - K)^+$ is given by*

$$\begin{aligned} C(S(0), T) &= S(0)\exp\Bigg\{-rT + (r-q)\frac{nT^\alpha}{\Gamma(\alpha+2)} - \frac{(n-n^2)\sigma^2 T^\alpha}{2\Gamma(\alpha+2)} + \frac{n^2\sigma^2(-T)^\alpha}{2\Gamma(\alpha+3)} \\ &\quad - \frac{n\sigma^2 T^{2\alpha H}}{(4\alpha H+2)(\Gamma(\alpha+1))^{2H}} - \frac{n^2\sigma^2 T^{2\alpha H}}{(4\alpha H+4)(\Gamma(\alpha+1))^{2H}}\Bigg\}\phi(f_1) \\ &\quad - Ke^{-qT}\phi(f_2), \end{aligned} \tag{5.1}$$

where

$$f_2 = \frac{\mu_G - \frac{1}{n}\ln K}{\sigma_G}, \quad f_1 = f_2 + n\sigma_G,$$

$$\mu_G = \ln S(0) + (r - q - \frac{\sigma^2}{2})\frac{T^\alpha}{\Gamma(\alpha+2)} - \frac{\sigma^2 T^{2\alpha H}}{2(2\alpha H+1)(\Gamma(\alpha+1))^{2H}},$$

$$\sigma_G^2 = \frac{\sigma^2 T^\alpha}{\Gamma(\alpha+2)} + \frac{\sigma^2(-T)^\alpha}{\Gamma(\alpha+3)} + \frac{\sigma^2 T^{2\alpha H}}{(2\alpha H+2)(\Gamma(\alpha+1))^{2H}},$$

the interest rate r and the dividend rate q are constant over time and $\varphi(.)$ denotes cumulative normal density function.

Proof. The payoff function for Asian power option is $(G^n(T) - K)^+ = (e^{nL(T)} - K)^+$, then applying similar computation in Theorem 4.1, we obtain

$$\begin{aligned} C(S(0), T) &= e^{-rT}\mathbb{E}[(G^n(T) - K)^+] \\ &= e^{-rT}\int_I (e^{nx} - K)\frac{1}{\sqrt{2\pi}\sigma_G}\exp\left\{-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right\}dx \\ &= e^{-rT}\int_I (e^{n(\mu_G + z\sigma_G)} - K)\frac{1}{\sqrt{2\pi}\sigma_G}\exp\left\{-\frac{(x-\mu_G)^2}{2\sigma_G^2}\right\}\varphi(z)dz \\ &= e^{-rT+n\mu_G+\frac{1}{2}n^2\sigma_G^2}\int_{-f_2}^{\infty} e^{-\frac{1}{2}(z-n\sigma_G)^2}dz - Ke^{-rT}\int_{-f_2}^{-\infty}\varphi(z)dz \\ &= e^{-rT+n\mu_G+\frac{1}{2}n^2\sigma_G^2}\int_{-f_2-n\sigma_G}^{\infty}\varphi(z)dz - Ke^{-rT}\int_{-\infty}^{f_2}\varphi(z)dz \\ &= e^{-rT+n\mu_G+\frac{1}{2}n^2\sigma_G^2}\int_{-\infty}^{f_2+n\sigma_G}\varphi(z)dz - Ke^{-rT}\int_{-\infty}^{f_2}\varphi(z)dz \\ &= e^{-rT+n\mu_G+\frac{1}{2}n^2\sigma_G^2}\phi(f_1) - Ke^{-rT}\phi(f_2), \\ &= S(0)\exp\Bigg\{-rT + (r-q)\frac{nT^\alpha}{\Gamma(\alpha+2)} - \frac{(n-n^2)\sigma^2 T^\alpha}{2\Gamma(\alpha+2)} + \frac{n^2\sigma^2(-T)^\alpha}{2\Gamma(\alpha+3)} \\ &\quad - \frac{n\sigma^2 T^{2\alpha H}}{(4\alpha H+2)(\Gamma(\alpha+1))^{2H}} - \frac{n^2\sigma^2 T^{2\alpha H}}{(4\alpha H+4)(\Gamma(\alpha+1))^{2H}}\Bigg\}\phi(f_1) \\ &\quad - Ke^{-qT}\phi(f_2), \end{aligned}$$

here

$$\begin{aligned} I &= \{x : e^{nx} > K\} = \{z : e^{n(\mu_G + z\sigma_G)} > K\} \\ &= \{z : \mu_G + z\sigma_G > \frac{1}{n}\ln K\} = \{z : z > -f_2\}, \end{aligned}$$

thus the proof is completed. □

The time changed mixed fractional *BS* model includes the jump behavior of price process because the subordinator process is a pure jump Levy process so it can capture the random variations in volatility. Also, it can be used for data with long range dependence and visible constant time periods characteristic for processes delayed by inverse subordinators.

6. Lower bound of the Asian option price

The aim of this section is to obtain the lower bound on the price of the Asian option. The next theorem shows that the normal distribution is stable when the random variables are jointly normal.

Theorem 6.1 ([22]). *The conditional distribution of* $\ln S_{t_i}$ *given* $\ln G(T)$ *is a normal distribution*

$$(\ln S_{t_i} | \ln G(T) = z) \simeq N(\mu_i + (z - \mu_G)\frac{\lambda_i}{\sigma_G^2}, \sigma_i^2 - \frac{\lambda_i^2}{\sigma_G^2}), \quad i = 1, \dots, n,$$

where

$$\begin{aligned} \mu_i &= \ln S(0) + (r-q)T_\alpha(t_i) - \frac{1}{2}\sigma^2 \frac{t_i^\alpha}{\Gamma(\alpha+1)} - \frac{1}{2}\sigma^2 \left(\frac{t_i^\alpha}{\Gamma(\alpha+1)}\right)^{2H} \\ \sigma_i^2 &= \sigma^2 \frac{t_i^\alpha}{\Gamma(\alpha+1)} + \sigma^2 \left(\frac{t_i^\alpha}{\Gamma(\alpha+1)}\right)^{2H}, \end{aligned}$$

$\lambda_i = Cov(\ln S_{t_i}, \ln G(T))$, $0 \le t_1 < t_2 < \cdots < t_n \le T$, $T_\alpha(t)$ *is inverse* α*-stable subordinator and,* μ_G *and* σ_G^2 *are defined in* Theorem 4.1.

Moreover, $(S_{t_i} | \ln G(T))$ *has a lognormal distribution and*

$$\begin{aligned} &\mathbb{E}\left[S_{t_i} | \ln G(T) = z\right] \\ &= \exp\left\{\mu_i + (z - \mu_G)\frac{\lambda_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\lambda_i^2}{\sigma_G^2})\right\} \quad i = 1, \dots, n. \end{aligned} \tag{6.1}$$

Now, we condition on the geometric average $G(T)$ in the pricing expression of the Asian option

$$\begin{aligned} C(S(0), T) &= e^{-rT}\mathbb{E}[(A(T) - K)^+] = e^{-rT}\mathbb{E}[\mathbb{E}[(A(T) - K)^+ | G(T)]] \\ &= e^{-rT}\int_0^\infty \mathbb{E}[(A(T) - K)^+ | G(T) = z]g(z)dz, \end{aligned}$$

where g is the lognormal density function of G. Let

$$\begin{aligned} C_1 &= \int_0^K \mathbb{E}[(A(T) - K)^+ | G(T) = z]g(z)dz, \\ C_2 &= \int_K^\infty \mathbb{E}[(A(T) - K)^+ | G(T) = z]g(z)dz, \end{aligned}$$

then $C(S(0), T) = e^{-rT}(C_1 + C_2)$. Since the geometric average is less than arithmetic average $A(T) \ge G(T)$,

$$C_2 = \int_K^\infty \mathbb{E}[A(T) - K | G(T) = z]g(z)dz, \tag{6.2}$$

from Theorem 6.1, we can calculate C_2. Applying Jensen's inequality we obtain a lower bound on C_1

$$\begin{aligned} C_1 &= \int_0^K \mathbb{E}[(A(T) - K)^+ | G(T) = z]g(z)dz \\ &\ge \int_0^K (E[A(T) - K | G(T) = z])^+ g(z)dz \\ &= \int_{\tilde{K}}^K \mathbb{E}[A(T) - K | G(T) = z]g(z)dz = \tilde{C}_1 \end{aligned} \tag{6.3}$$

where $\tilde{K} = \{z | \mathbb{E}[A(T) | G(T) = z] = K\}$.

Eq. (6.1) enables us to obtain $\tilde{K}$, then we calculate the following expectation

$$\begin{aligned}\mathbb{E}[A(T)|G(T)=z] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} S_{t_i}|G(T)=z\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[S_{t_i}|G(T)=z\right] \\ &= \frac{1}{n}\sum_{i=1}^{n}\exp\left(\mu_i + (\log z - \mu_G)\frac{\lambda_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\lambda_i^2}{\sigma_G^2})\right).\end{aligned}$$

Theorem 6.2. *A lower bound on the price of the Asian option with strike price K and maturity time T is given by*

$$\begin{aligned}\tilde{C}(S(0),T) &= e^{-rT}(\tilde{C}_1 + C_2) \\ &= e^{-rT}\left\{\frac{1}{n}\sum_{i=1}^{n}\exp(\mu_i + \frac{1}{2}\sigma_i^2)\phi\left(\frac{\mu_G - \ln\tilde{K} + \gamma_i}{\sigma_G}\right)\right. \\ &\quad \left. - K\phi\left(\frac{\mu_G - \ln\tilde{K}}{\sigma_G}\right)\right\},\end{aligned}$$

where all parameters are defined previously.

Proof. Collecting Eqs. (6.2) and (6.3) gives

$$\begin{aligned}\tilde{C}_1 + C_2 &= \int_{\tilde{K}}^{\infty}\mathbb{E}[A(T)-K|G(T)=z]g(z)dz \\ &= \int_{\tilde{K}}^{\infty}\mathbb{E}[A(T)|G(T)=z]g(z)dz - K\int_{\tilde{K}}^{\infty}g(z)dz \\ &= \int_{\tilde{K}}^{\infty}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}S_{t_i}|G(T)=z\right]g(z)dz - K\int_{\tilde{K}}^{\infty}g(z)dz \\ &= \int_{\tilde{K}}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[S_{t_i}|G(T)=z\right]g(z)dz - K\int_{\tilde{K}}^{\infty}g(z)dz \\ &= \frac{1}{n}\sum_{i=1}^{n}\int_{\tilde{K}}^{\infty}\mathbb{E}\left[S_{t_i}|\ln G(T)=\ln z\right]g(z)dz - K\int_{\tilde{K}}^{\infty}g(z)dz.\end{aligned}$$

From the proof of Theorem 4.1, we obtain

$$K\int_{\tilde{K}}^{\infty}g(z)dz = K\phi\left(\frac{\mu_G - \ln\tilde{K}}{\sigma_G}\right),$$

and from Eq. (6.1)

$$\begin{aligned}&\int_{\tilde{K}}^{\infty}\mathbb{E}\left[S_{t_i}|\ln G(T)=\ln z\right]g(z)dz \\ &= \int_{\tilde{K}}^{\infty}\exp\left(\mu_i + (\ln z - \mu_G)\frac{\lambda_i}{\sigma_G^2} + \frac{1}{2}(\sigma_i^2 - \frac{\lambda_i^2}{\sigma_G^2})\right)g(z)dz \\ &= \exp\left(\mu_i + \frac{1}{2}\sigma_i^2\right)\int_{\tilde{K}}^{\infty}\exp\left((\ln z - \mu_G)\frac{\lambda_i}{\sigma_G^2} - \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2}\right)g(z)dz.\end{aligned}$$

Using the density of the lognormal distribution, we have

$$\int_{\tilde{K}}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_G z}\exp\left((\ln z - \mu_G)\frac{\lambda_i}{\sigma_G^2} - \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2} - \frac{1}{2}(\frac{\mu_G - \ln z}{\sigma_G})^2\right)dz.$$

Making the change of variables $y = \frac{\mu_G - \ln z + \lambda_i}{\sigma_G}$ and $\frac{dy}{dz} = -\frac{1}{\sigma_G z}$, then we have

$$\begin{aligned}&\int_{\frac{\mu_G - \ln z + \lambda_i}{\sigma_G}}^{-\infty} -\frac{1}{\sqrt{2\pi}}\exp\left((\frac{\lambda_i}{\sigma_G} - y)\frac{\lambda_i}{\sigma_G} - \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2} - \frac{1}{2}(y - \frac{\lambda_i}{\sigma_G})^2\right)dy \\ &= \int_{-\infty}^{\frac{\mu_G - \ln z + \lambda_i}{\sigma_G}}\frac{1}{\sqrt{2\pi}}\exp\left(-y\frac{\lambda_i}{\sigma_G} + \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2} - \frac{1}{2}y^2 - \frac{1}{2}\frac{\lambda_i^2}{\sigma_G^2} + y\frac{\lambda_i}{\sigma_G}\right)dy \\ &= \int_{-\infty}^{\frac{\mu_G - \ln z + \lambda_i}{\sigma_G}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right)dy = \phi\left(\frac{\mu_G - \ln\tilde{K} + \gamma_i}{\sigma_G}\right),\end{aligned}$$

by collecting $\tilde{C}_1$ and C_2 the proof is completed. □

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