Nonlinear transport network design *

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Abstract

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In this paper two problems of optimal design of the nonlinear transportation networks will be presented. The goal functions in each of them are so complicated that the special approach for the determination of gradient and special attention for relaxation of the criterion function had to be reworked. Our considerations concern the water supply networks mainly.

Keywords: Water distribution system, optimal control, optimal design.

1. Introduction

In this paper we present two mathematical models of the transport network design problem. In both of them the idea of the optimization problem is to obtain the parameters of the elements for which the operation cost of the system is minimal subject to previously known restrictions of the installation costs.

Our considerations concern the water supply networks mainly. In this case we assume that the demands of water consumers and the topology of the network structure are previously known. In practice an operation cost equals the cost of consumed electric energy in the pumping stations.

Under the first assumption presented above this cost is constant, because for most of the types of the pump units, the level of consumed energy is simply a linear function of the total, previously known, output flow. That is why we proposed another shape of the goal function which is effectively more useful for our purposes than the previous one. Namely, we took into consideration the total amount of the energy wastes in the transportation network.

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2. Simulation model of the water supply system

2.1. Mathematical relations

Mathematical model for the simulations of flows in the water supply systems, which consists of n arcs and $s+1$ nodes, contains the following set of equations:

$$Ay=\sigma, \tag{1}$$

$$Bx=0, \tag{2}$$

$$x_i=r_i y_i^2 \operatorname{sgn} y_i + d_i, \quad i=1,\dots,n. \tag{3}$$

We presented this model in details in our previous papers [6,8]. Equation (1) expresses the material continuity at nodes (first Kirchhoff law), where $\sigma \in \mathbb{R}^s$ is a vector of flows from network nodes, A is the incidence matrix, and $y \in \mathbb{R}^n$ is the vector of flows in the network arcs. Equation (2) expresses path equations (second Kirchhoff law). B is a loop matrix and $x \in \mathbb{R}^n$ is a vector of head differences between two ends of the pipeline. The relation connecting y_i and x_i (Bernoulli law) is expressed by (3), where d_i is the difference of altitudes between two ends of the ith pipeline and r_i is the resistance of the ith pipeline. The value of r_i depends on the length, diameter and smoothness coefficient of the pipeline and can be calculated from the Hazen–Williams law.

2.2. Optimization

In [8] we have shown that the solution of (1)–(3) is identical with the solution of the following static optimization problem:

$$p(y)=\sum_{i=1}^{n} p_i(y_i) \to \min, \tag{4}$$

where y fulfils (1) and

$$p_i(y_i)=x_i y_i = r_i y_i^3 \operatorname{sgn} y_i, \quad 1,\dots,n. \tag{5}$$

After improving the notation,

$$R(r,\ y)=\operatorname{diag}\{r_i y_i \operatorname{sgn} y_i, \quad i=1,\dots,n\}, \tag{6}$$

one can transform (1)–(3) to

$$Ay=\sigma, \tag{7}$$

$$BR(r,\ y)y=0, \tag{8}$$

and (4) to

$$y^{\mathrm{T}}R(r,\ y)y \to \min. \tag{9}$$

An equivalence of (1)–(3) and (9) subject to (1) is very interesting from the physical point of view. Values of (9) denote the total wastes of energy in the set of pipelines connected with their resistances. This remark allows to obtain the nature of the second Kirchhoff law. The following property is true in each network for which the flow of the transported medium fulfils (1), (2).

In [5], on the base of the sensitivity analysis, we presented the relation between r and y in the form of a Jacobian matrix:

$$\frac{\partial y}{\partial r} = -\tfrac{1}{2} R(y, y) B^{\mathrm{T}} \left(B R(r, y) B^{\mathrm{T}} \right)^{-1} B. \tag{10}$$

So, for calculating the Jacobian matrix of the operator $y = y(r)$ we have to simulate the network (solve (1)–(3) or (9) subject to (1)) for given r and put the obtained vector of flows y to (10).

3. Mathematical model of the transport network design problem

3.1. Technological restrictions

We start elaborations of this problem from the optimal design of the single network arc.

Let us assume that there are m types of pipelines in their store with given unit linear resistance ρ_j and unit linear cost κ_j. The problem leads to design the minimal-resistance single pipeline with given length l_0 and a total cost no greater than k_0. The mathematical model of this topic is as follows:

$$\sum_{j=1}^{m} \rho_j z_j \to \min \tag{11}$$

subject to

$$\sum_{j=1}^{m} z_j = l_0, \tag{12}$$

$$\sum_{j=1}^{m} \kappa_j z_j \leqslant k_0, \tag{13}$$

$$z_j \geqslant 0, \quad j = 1, \dots, m, \tag{14}$$

where $\{z_j\}$ denotes lengths of particular types of pipelines. In practice $\rho' > \rho'' \Rightarrow \kappa' < \kappa''$, so we can enumerate variables in (11)–(14) in such a manner that the following relations

$$\rho_1 > \rho_2 > \cdots > \rho_m, \tag{15}$$

$$\kappa_1 < \kappa_2 < \cdots < \kappa_m, \tag{16}$$

will be fulfilled.

On the base of the linear programming theory we now have the following properties.

Property 1. At least two variables z_j are greater than 0 in the optimal solution of (11)–(14).

Property 2. If

$$\forall_j \in \{1, \dots, m\}, \quad \left[D_{tj} = \begin{vmatrix} 1 & 1 & 1 \\ \kappa_t & \kappa_{t+1} & \kappa_j \\ \rho_t & \rho_{t+1} & \rho_j \end{vmatrix} \geqslant 0 \right] \tag{17}$$

and

$$\kappa_t l_0 \leqslant k_0 \leqslant \kappa_{t+1} l_0, \tag{18}$$

then

$$Q_B = \begin{vmatrix} 1 & 1 \\ \kappa_t & \kappa_{t+1} \end{vmatrix} \tag{19}$$

is the the optimal basis matrix for the problem (11)–(14).

From (18) we have

$$z_B = (z_t,\ z_{t+1})^{\mathrm{T}} = \frac{1}{\kappa_{t+1} - \kappa_t}(\kappa_{t+1} l_0 - k_0,\ k_0 - \kappa_t l_0)^{\mathrm{T}} \geqslant 0 \tag{20}$$

and

$$f(k_0) = \frac{\rho_t \kappa_{t+1} - \rho_{t+1}\kappa_t}{\kappa_{t+1} - \kappa_t} + \frac{\rho_{t+1} - \rho_t}{\kappa_{t+1} - \kappa_t} k_0. \tag{21}$$

Under the previous assumption $f(k)$ is a decreasing and convex function for all $k_0 \geqslant \kappa_1 l_0$.

Let $r_0 = f(k_0)$. For $r_0 \leqslant \rho_m l_0$ an inverse function to f also exists. We will denote this function in the next parts of this paper by g. Function g is also decreasing and convex.

3.2. A general version of the optimization problem

Let us assume that we know the forecasted consumers need σ, and the required topology of the system of pipelines (matrix A). In the general case, on the base of parameters (15), (16) for available types of material, our aim is to build a water-supply network, optimal in the sense of the criterion function (9). The total cost of this investment must not exceed k_0. The mathematical model of this problem has the following form:

$$F(r) = y^{\mathrm{T}} R(r,\ y) y \to \min, \tag{22}$$

$$Ay = \sigma, \tag{23}$$

$$\sum_{i=1}^{n} g_i(r_i) \leqslant k_0. \tag{24}$$

Equations (22)–(24) form a nonlinear static optimization problem. It is rather difficult, or impossible, to discuss even very basic properties like existence and uniqueness of the optimal solution or its convexity. We return to these problems in Section 3.3.

In the real cases the complexity of (22)–(24) is connected with the available techniques for the calculation of the goal function (22) value for a given vector of resistances r. An obvious approach leads to a two-level procedure: for a given r solve (1)–(3) (or (9) subject to (1)) to obtain $y = y(r)$ and put this result to (22). So, the final optimization program may be time consuming if the simulation procedure will not be good.

3.3. A few remarks on the convexity

The aim of this section is to proof that the convexity of (22)–(24) even in very single cases strictly depends on some extra relations between parameters (15), (16).

We will try to show it in a very simple example, i.e., the system that consists of one source and one consumer with needs $\sigma \in \mathbb{R}^1$, connected by two parallel pipes with unique length. Let us approximate functions $g_1(r_1)$ and $g_2(r_2)$ by the following formula:

$$g(r) = g_0 r^{\alpha} \tag{25}$$

for any positive r_1, r_2.

In such a simple case all the results can be easily obtained theoretically. First, for any r_1, $r_2 > 0$, we have from (1)–(3)

$$y_1 = \frac{r_2^{0.5}\sigma}{r_1^{0.5} + r_2^{0.5}}, \qquad y_2 = \frac{r_1^{0.5}\sigma}{r_1^{0.5} + r_2^{0.5}}, \tag{26}$$

and after putting (26) to (22) and (25) to (24) we have the following problem:

$$F(r_1, r_2) = \frac{r_1 r_2}{\left(r_1^{0.5} + r_2^{0.5}\right)^2}\sigma^3 \to \min, \tag{27}$$

$$r_1^{\alpha} + r_2^{\alpha} \leqslant k_0. \tag{28}$$

This problem is convex when $\alpha > -0.5$, and

$$r_1^* = r_2^* = \left(\frac{k_0}{2g_0}\right)^{1/\alpha}. \tag{29}$$

When $\alpha = -0.5$, each pair (r_1, r_2) that fulfils (28) as equality is an optimal solution of (27), (28). Finally, when $\alpha \in (-0.5, 0)$, an optimal solution goes to infinity on the curve (28).

So the presented problem is convex iff a unit price of the pipe decreases sufficiently quickly with the growing values of its resistance (big absolute value of α). This is a property of an economy rather than an optimization problem itself.

4. Algorithms

4.1. Linear case

Problem (22)–(24) has a special form when the graph of a network is a tree (there is no loop in it). A is a square matrix which allows to rewrite (23) to $y_0 = A^{-1}\sigma$. Under some simple rules from the linear programming theory, (22)–(24) can be transformed to the following linear program:

$$\sum_{i=1}^{n} y_{0i}^3 r_i \to \min, \tag{30}$$

$$\sum_{i=1}^{n} k_i \leqslant k_0, \tag{31}$$

$$(r_i - \rho_j l_i)(\kappa_{j+1} - \kappa_j) \leqslant (k_i - \kappa_j l_i)(\rho_{j+1} - \rho_j), \quad i = 1, \ldots, n, \ j = 1, \ldots, m-1, \tag{32}$$

$$\rho_m l_i \leqslant r_i \leqslant \rho_1 l_i, \quad i = 1, \ldots, n, \tag{33}$$

$$k_i \geqslant 0, \quad i = 1, \ldots, n. \tag{34}$$

Elements of vector r and $k=(k_1, k_2, \dots, k_n)$ are decision variables. We denote by $l=(l_1, l_2, \dots, l_n)$ a vector of particular lengths of pipes in the designed network.

This problem can be solved by the procedure presented below.

Algorithm 1.
Step 1 (Setting initial values). Put $r_i = \rho_m l_i$ and $t_i = m$ for $i = 1, \dots, n$. Put $k_0 = k_0 - \Sigma_{i=1}^{n} \kappa_m l_i$. If $k_0 < 0$, STOP. The problem has no feasible solution. Otherwise go to Step 2.
Step 2. If $k_0 = 0$ or $\max_i t_i = 1$, STOP. An optimal solution r_i, $i = 1, \dots, n$, was found. Otherwise go to Step 3.
Step 3. Obtain $\theta_u = \max_{t_i > 1} \theta_i$, where

$$\theta_i = y_{0i}^3 \frac{\rho_{t_i} - \rho_{t_i - 1}}{\kappa_{t_i - 1} - \kappa_{t_i}}. \tag{35}$$

Put $\tilde{k} = \min(k_0, \theta_u l_u)$, $k_0 = k_0 - \tilde{k}$, $t_u = t_u - 1$ and

$$r_u = r_u - \frac{\rho_{t_i} - \rho_{t_i - 1}}{\kappa_{t_i - 1} - \kappa_{t_i}} l_u \tilde{k}. \tag{36}$$

Return to Step 2.

The maximal number of iterations in Algorithm 1 equals $n \times m$. So, it has a polynomial complexity.

4.2. General case

If any loop exists in the graph of the network structure, it complicates an optimization problem (22)–(24) radically. Thanks to (10) the gradient of the goal function $F(r)$ is as follows:

$$\frac{\partial F}{\partial r} = \left[\frac{\partial y}{\partial r}\right]^{\mathrm{T}} \frac{\partial p}{\partial y} + \frac{\partial p}{\partial r} = 3\left(\left[\frac{\partial y}{\partial r}\right]^{\mathrm{T}} R(r, y) + R(1, y)\right) y. \tag{37}$$

One can show that

$$\forall r > 0 \ \frac{\partial F}{\partial r} > 0. \tag{38}$$

On the base of (35) the numerical procedure for calculating $\partial F / \partial r$ can be obtained. So, if the gradient of (22) is available, the whole optimization problem (22)–(24) can be solved on the base of the Rosen gradient method. In such a case this method is very convenient because the set of the feasible solutions is given by the system of linear inequalities.

As usually, the time of calculation hardly depends on the initial point r_0. Good values of the elements of the vector r_0 can be obtained with the help of Algorithm 2, presented below. This algorithm is a natural modification of Algorithm 1.

Algorithm 2.
Step 1 (Setting initial values). Put $r_i = \rho_m l_i$ and $t_i = m$ for $i = 1, \dots, n$. Put $k_0 = k_0 - \Sigma_{i=1} \kappa_m l_i$. If $k_0 < 0$, STOP. The problem has no feasible solution. Otherwise go to Step 2.
Step 2. If $k_0 = 0$ or $\max_i t_i = 1$, put $r_0 = r$ and STOP. Otherwise go to Step 3.

Step 3. Determinate $y_0 = \partial h/\partial r$ according to (37) for current values of r. Next put $\theta_u = \max_{y_i > 1} \theta_i$, where θ_i is given by (35) and $\tilde{k} = \min(k_0, \theta_u l_u)$, $k_0 = k_0 - \tilde{k}$, $t_u = t_u - 1$. Determinate r_u according to (36). Go to Step 2.

The course of action presented above results from (38).

5. Final remarks

In the present paper we elaborated two mathematical models of the water supply network design problem. Under the base of some of their mathematical properties we obtained optimization procedures and checked their complexity. In both cases it is sufficiently low (polynomial) for solving a real problem with high dimension (over hundreds arcs).

These procedures were programmed in C on the IBM PC/AT computer.

In the Introduction we discussed the shape of the goal function. Let us return to this problem again in detail. An optimal solution of the optimization problem (22)–(24) has some extra very useful properties that allow to fulfil water consumer demands easily. For any pump one can find in the catalogue the shape of the so-called characteristic curve. This is the relation between an output flow y_p from the pump and the necessary head of the water $H(y_p)$. For the given network and the simulated vector of flows y one can determinate the necessary head of the water at the pumping station v. Such a flow can be obtained in practice iff $H(y_p) > v$. It is easy to show that for an optimal solution of (22)–(24) v is minimal. This property allows also to increase the total wastes of the leaked water, because in this situation the heads of it are relatively small in each node of the network.

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