



# Fractional Cauchy transforms

Thomas H. MacGregor

*Department of Mathematics and Statistics, The University at Albany, Albany, NY 12222, USA*

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

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## Abstract

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## 1. Introduction

This paper gives a survey of research about fractional Cauchy transforms.

We begin with some definitions. Let  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  and let  $\mathcal{A} = \{z \in \mathbb{C} : |z| = 1\}$ . Let  $\mathcal{M}$  denote the set of complex-valued Borel measures on  $\mathcal{A}$ . For each  $\alpha > 0$  a family of functions denoted  $\mathcal{F}_\alpha$  is defined in the following way. A function  $f \in \mathcal{F}_\alpha$  provided that there exists  $\mu \in \mathcal{M}$  such that

$$f(z) = \int_{\mathcal{A}} \frac{1}{(1 - \bar{\zeta}z)^\alpha} d\mu(\zeta) \quad (1)$$

for  $|z| < 1$ . The power function in (1), as well as each logarithm in this paper, is the principal branch. We call a function defined by (1) a fractional Cauchy transform, and when  $\alpha = 1$  such a function is called a Cauchy transform or a Cauchy–Stieltjes integral.

Each function in  $\mathcal{F}_\alpha$  is analytic in  $\Delta$ .  $\mathcal{F}_\alpha$  is a vector space with respect to ordinary addition of functions and multiplication by complex numbers. For each  $f \in \mathcal{F}_\alpha$  we set

$$\|f\|_{\mathcal{F}_\alpha} = \inf_{\mu} \|\mu\|, \quad (2)$$

where  $\mu$  varies over the set of measures in  $\mathcal{M}$  for which (1) holds and  $\|\mu\|$  denotes the total variation of  $\mu$ . It can be shown that there is  $\mu \in \mathcal{M}$  such that (1) holds and  $\|\mu\| = \|f\|_{\mathcal{F}_\alpha}$ . Also (2) defines a norm on  $\mathcal{F}_\alpha$  and with respect to this norm  $\mathcal{F}_\alpha$  is a Banach space. Convergence in this norm implies convergence that is uniform on compact subsets of  $\Delta$ .

The case  $\alpha = 1$  is of special importance because of the Cauchy formula. For example, if  $f$  is analytic in  $\bar{\Delta}$  then

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{f(\zeta)}{\zeta - z} d\zeta \quad (3)$$

for  $|z| < 1$ . In other words, (1) holds where  $d\mu(\zeta) = (f(\zeta)/2\pi i \zeta) d\zeta$ . More generally, (3) holds whenever  $f$  belongs to the Hardy space  $H^1$ . In this case, a measure representing  $f$  is given as above where

$$f(\zeta) \equiv \lim_{r \rightarrow 1^-} f(r\zeta). \quad (4)$$

Limit (4) exists for almost all  $\theta$  ( $\zeta = e^{i\theta}$ ,  $-\pi \leq \theta \leq \pi$ ) and defines a function in  $L^1([-\pi, \pi])$  [12, p. 41].

The set of measures which represent a function in  $\mathcal{F}_\alpha$  by (1) has the description  $\mu = \underline{\mu}_0 + \nu$ , where  $\mu_0 \in \mathcal{M}$ ,  $\mu_0$  represents  $f$  and  $\nu$  varies over the set of measures given by  $d\nu(\zeta) = g(e^{i\theta}) d\theta$  where  $g \in H^1$  and  $g(0) = 0$ . This is consequence of a theorem of  $F$  and Riesz [12, p. 41].

The research described in this paper goes back to work on Cauchy transforms beginning with the paper [29] by Havin in 1958. Other contributors to research on Cauchy transforms include Aleksandrov, Hrušev, Goluzina and Vinogradov. Most of the research we present about  $\mathcal{F}_\alpha$  for general  $\alpha$  was done in the last ten years. This began with the paper [44] by the author and the main contributors to this development are Hallenbeck, Hirschweiler and Samotij. Formula (1) occurs earlier in various places in the literature (for example, see [46]), usually where the measure is absolutely continuous with respect to Lebesgue measure. The subfamily of  $\mathcal{F}_\alpha$  given by (1) where  $\mu$  varies over the probability measures in  $\mathcal{M}$  was introduced by Brickman, Hallenbeck, Wilken and the author in [7] in connection with questions about extreme points and closed convex hulls of families of functions.

There is a definition of  $\mathcal{F}_\alpha$  for  $\alpha \leq 0$ . It is given in [34] for  $\alpha = 0$  and in [37] for  $\alpha < 0$ . A function  $f \in \mathcal{F}_0$  provided that there exists  $\mu \in \mathcal{M}$  such that

$$f(z) = f(0) + \int_{\Delta} \log \left[ \frac{1}{1 - \bar{\zeta}z} \right] d\mu(\zeta) \quad (5)$$

for  $|z| < 1$ .

This paper is an expansion of material presented by the author at the conference on Continued Fractions and Geometric Function Theory held at the Norwegian University of Science and Technology in Trondheim, Norway, on 24–28 June 1997. The conference was held in honor of Haakon Waadeland and in celebration of his seventieth birthday. The author congratulates Professor Waadeland and also thanks Lisa Lorentzen, Olav Njåstad and Frode Rønning for organizing such a successful conference.

## 2. Relations with other Banach spaces

There are several connections between  $\mathcal{F}_\alpha$  and other spaces of analytic functions. We shall describe some of them with Hardy spaces, Besov spaces and Dirichlet spaces.

For  $p > 0$  let  $H^p$  denote the Hardy space; that is,  $f \in H^p$  provided that  $f$  is analytic in  $\Delta$  and

$$\|f\|_{H^p} \equiv \sup_{0 < r < 1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty. \quad (6)$$

Also,  $H^\infty$  consists of all functions that are analytic and bounded in  $\Delta$  and

$$\|f\|_{H^\infty} \equiv \sup_{|z| < 1} |f(z)|. \quad (7)$$

Three references about  $H^p$  spaces are [12, 15, 42].

Our first theorem gives set theoretic relations between  $\mathcal{F}_\alpha$  and  $H^p$ .

**Theorem 1.** *If  $0 < \alpha \leq 1$  then  $\mathcal{F}_\alpha \subset H^p$  for  $0 < p < 1/\alpha$ .  $\mathcal{F}_0 \subset H^p$  for all  $p > 0$ . If  $0 < p \leq 1$  then  $H^p \subset \mathcal{F}_{1/p}$ .*

Theorem 1 is in [44]. It is useful to have a relation  $\mathcal{F}_\alpha \subset H^p$  since the results established for Hardy spaces become applicable to  $\mathcal{F}_\alpha$ .

There are a number of so-called Besov spaces. The spaces which concern us are defined as follows for each  $\alpha > 0$ . A function  $f \in \mathcal{B}_\alpha$  provided that  $f$  is analytic in  $\Delta$  and

$$\|f\|_{\mathcal{B}_\alpha} \equiv |f(0)| + \int_0^1 \int_{-\pi}^{\pi} |f'(re^{i\theta})|(1-r)^{\alpha-1} d\theta dr < \infty. \quad (8)$$

$\mathcal{B}_\alpha$  is a Banach space with respect to the norm defined by (8).

**Theorem 2.**  *$\mathcal{B}_\alpha \subset \mathcal{F}_\alpha$  for all  $\alpha > 0$ .  $\mathcal{F}_\alpha \subset \mathcal{B}_\beta$  for all  $\beta > \alpha$ .*

Theorem 2 is in [21]. The argument showing that  $\mathcal{B}_\alpha \subset \mathcal{F}_\alpha$  depends on a suitable transformation between  $\mathcal{F}_\alpha$  and  $\mathcal{F}_1$  and the fact that  $H^1 \subset \mathcal{F}_1$ . That (8) implies  $f \in \mathcal{F}_\alpha$  is useful because it gives an analytic condition for membership in  $\mathcal{F}_\alpha$ .

For each  $\alpha > 0$  the Dirichlet space denoted  $\mathcal{D}_\alpha$  is defined as the set of analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (|z| < 1) \quad (9)$$

such that

$$\|f\|_{\mathcal{D}_\alpha} \equiv |a_0| + \left\{ \sum_{n=1}^{\infty} n^\alpha |a_n|^2 \right\}^{1/2} < \infty. \quad (10)$$

$\mathcal{D}_\alpha$  is a Banach space with respect to the norm defined by (10). There are relations between fractional Cauchy transforms and  $\mathcal{D}_\alpha$  as well as between Besov spaces and  $\mathcal{D}_\alpha$ . We mention one attractive fact

below which holds for inner functions. Recall that a function  $f$  is called an inner function provided that  $f \in H^\infty$  and  $|f(e^{i\theta})| = 1$  for almost all  $\theta$ .

**Theorem 3.** *Let  $0 < \alpha < 1$  and suppose that  $f$  is an inner function. The following statements are equivalent to each other: (a)  $f \in \mathcal{F}_\alpha$ , (b)  $f \in \mathcal{B}_\alpha$  and (c)  $f \in \mathcal{D}_{1-\alpha}$ .*

The fact that (b) and (c) are equivalent for inner functions is due to Ahern [1]. The proof of the remaining parts of Theorem 3 are in [21].

The argument for Theorems 1–3 yield comparisons of norms. Next, we give examples which apply these results.

A (infinite) Blaschke product is a function  $B$  having the form

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}, \quad (11)$$

where  $\{z_n\}$  is a sequence of nonzero complex numbers in  $\Delta$  satisfying

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty \quad (12)$$

and  $m$  is a nonnegative integer. The infinite product converges as a consequence of (12) and  $B$  is an inner function. Because  $B \in H^\infty$  it follows that  $B \in \mathcal{F}_1$ . If the zeros of  $B$  are more restricted than (12) then  $B$  belongs to a smaller family  $\mathcal{F}_\alpha$ . More specifically, let  $0 < \alpha < 1$  and suppose that  $\{z_n\}$  is a sequence in  $\Delta$  such that

$$\sum_{n=1}^{\infty} (1 - |z_n|)^\alpha < \infty. \quad (13)$$

If  $B$  is defined by (11) for some  $m$ , then  $B \in \mathcal{F}_\alpha$ . This follows from Theorem 2 and a result of Protas [48] that  $B \in \mathcal{B}_\alpha$ . Since  $0 < \alpha < 1$ ,  $B \in \mathcal{F}_\alpha$  is stronger than  $B \in \mathcal{F}_1$ . In general,  $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$  if  $0 \leq \alpha < \beta$ .

An important inner function is defined by

$$S(z) = \exp \left[ -\frac{1+z}{1-z} \right]. \quad (14)$$

In [21] it is shown that  $S \in \mathcal{F}_\alpha$  if and only if  $\alpha > \frac{1}{2}$ . Another fact about inner functions and membership in  $\mathcal{F}_\alpha$  is the following theorem.

**Theorem 4.** *If  $f$  is an inner function and  $f \in \mathcal{F}_0$ , then  $f$  is a finite Blaschke product.*

Theorem 4 is proved in [21, 22]. The argument in [21] uses the fact that any inner function which belongs to  $\mathcal{D}_1$  must be a finite Blaschke product. Proofs of this fact are in [14, 47]. More recently

K. Samotij proved this using a more geometric argument based on the fact that  $f \in \mathcal{D}_1$  corresponds to  $f(\Delta)$  having finite area (counting multiple coverings).

### 3. Boundary values

Suppose that  $f \in \mathcal{F}_\alpha$  for some  $\alpha$  where  $0 < \alpha \leq 1$ . Then Theorem 1 implies  $f \in H^p$  for  $0 < p < 1/\alpha$ . Consequently  $f(e^{i\theta})$  exists for almost all  $\theta$  and defines a function which belongs to  $L^p([-\pi, \pi])$  for  $0 < p < 1/\alpha$ . Theorem 5 below gives an improvement of this result.

We recall that a measurable function  $F : [-\pi, \pi] \rightarrow \mathbb{C}$  is called weak  $L^p$  provided that there is a constant  $A > 0$  such that

$$m(\{\theta: |f(\theta)| > t\}) \leq \frac{A}{t^p} \quad (15)$$

for  $t > 0$ , where  $m(\Gamma)$  denotes the Lebesgue measure of the set  $\Gamma$ . If  $F \in L^p$  then  $F$  is weak  $L^p$ , and conversely if  $F$  is weak  $L^p$  then  $f \in L^q$  for every  $q < p$ .

**Theorem 5.** Suppose that  $0 < \alpha \leq 1$ ,  $f \in \mathcal{F}_\alpha$  and let  $F(\theta) = f(e^{i\theta})$ . Then  $F$  is weak  $L^{1/\alpha}$ .

When  $\alpha = 1$  Theorem 5 is essentially the same as a theorem due to Kolmogoroff [41, p. 66]. The proof of Theorem 5 for general  $\alpha$  is due to the author and is unpublished. The corresponding result in the case  $\alpha = 0$  is the following assertion. Suppose that  $f \in \mathcal{F}_0$  and  $f(0) = 0$ . There are positive constants  $A$  and  $B$  such that

$$m(\{\theta: |f(e^{i\theta})| > t\}) \leq A \exp(-Bt) \quad (16)$$

for  $t > 0$ .

The next result strengthens the fact that if  $f \in \mathcal{F}_\alpha$  for some  $\alpha$ ,  $0 \leq \alpha \leq 1$ , then  $f(e^{i\theta})$  exists for almost all  $\theta$ . It asserts that exceptional sets of measure zero can be replaced by exceptional sets having zero  $\alpha$ -capacity.

We recall that if  $0 < \alpha < 1$  then a Borel set  $E \subset [-\pi, \pi]$  is said to have positive  $\alpha$ -capacity provided that there exists a probability measure  $\mu$  supported on  $E$  such that

$$\sup_{\theta} \int_{-\pi}^{\pi} \frac{1}{|\sin(1/2)(\theta - t)|^\alpha} d\mu(t) < \infty. \quad (17)$$

If  $E$  does not have positive  $\alpha$ -capacity we say that  $E$  has zero  $\alpha$ -capacity and write  $C_\alpha(E) = 0$ . When  $\alpha = 0$  we have the idea of positive logarithmic capacity which is defined as above where the kernel  $1/|\sin(1/2)\theta|^\alpha$  is replaced by  $\log(1/|\sin(1/2)\theta|)$ . Every set of zero  $\alpha$ -capacity has Lebesgue measure zero but not conversely. Also if  $C_\alpha(E) = 0$  and  $\beta > \alpha$  then  $C_\beta(E) = 0$ .

**Theorem 6.** If  $0 \leq \alpha < 1$  and  $f \in \mathcal{F}_\alpha$  then  $f(e^{i\theta}) = \lim_{r \rightarrow 1-} f(re^{i\theta})$  exists except possibly for a set having zero  $\alpha$ -capacity.

Theorem 6 is in [19]. The argument depends upon the results listed below as Theorems 7 and 8. Theorem 7 is a local result. In general, the behavior of a fractional Cauchy transform in  $\Delta$  and near  $e^{i\theta}$  depends on the behavior of a representing measure in the neighborhood of  $\theta$ . Since (1) can be rewritten as a Stieltjes integral with respect to some function of bounded variation, a number of results ultimately depend on suitable facts about nondecreasing functions. In particular, this is how Theorems 7 and 8 yield Theorem 6. Theorem 8 is due to Twomey [53].

**Theorem 7.** *Suppose that  $\alpha > 0$ ,  $g$  is a complex-valued function of bounded variation on  $[-\pi, \pi]$  and let*

$$f(z) = \int_{-\pi}^{\pi} \frac{1}{(1 - e^{it}z)^{\alpha}} dg(t) \quad (18)$$

for  $|z| < 1$ . If

$$\int_{-\pi}^{\pi} \frac{|g(\theta + t) - g(\theta)|}{|t|^{\alpha+1}} dt < \infty$$

then  $\lim_{r \rightarrow 1-} f(re^{i\theta})$  exists.

**Theorem 8.** *Suppose that  $g: [-\pi, \pi] \rightarrow \mathbb{R}$  is nondecreasing and  $0 \leq \alpha < 1$ . Then*

$$\int_{-\pi}^{\pi} \frac{g(\theta + t) - g(\theta - t)}{t^{\alpha+1}} dt < \infty$$

except possibly for a set having zero  $\alpha$ -capacity.

If  $f \in \mathcal{F}_{\alpha}$  and  $\alpha > 1$  then  $f(e^{i\theta})$  may fail to exist for all  $\theta$ . This is discussed below after Theorem 12.

Another question concerns the growth of a function in  $\mathcal{F}_{\alpha}$  and what exceptional sets can be associated with a given growth. This question is of interest for all  $\alpha \geq 0$ .

If  $f \in \mathcal{F}_{\alpha}$  then (1) implies that  $|f(z)| \leq \|\mu\|/(1 - |z|)^{\alpha}$ . This maximal growth can be achieved on at most a countable set [18].

**Theorem 9.** *If  $f \in \mathcal{F}_{\alpha}$  and  $\alpha > 0$  then  $\lim_{r \rightarrow 1-} (1 - r)^{\alpha} f(re^{i\theta}) = 0$  for all  $\theta$  in  $[-\pi, \pi]$  except possibly for a finite or countable set.*

Another example of the interplay between growth and exceptional sets is the following result.

**Theorem 10.** *If  $f \in \mathcal{F}_{\alpha}$  and  $\alpha > 1$  then  $\lim_{r \rightarrow 1-} (1 - r)^{\alpha-1} f(re^{i\theta}) = 0$  for almost all  $\theta$  in  $[-\pi, \pi]$ .*

Theorem 10 was first proved in [18]. Another proof is given in [20]. The second argument uses the following local result and the fact that a nondecreasing function is differentiable almost everywhere.

**Theorem 11.** Suppose that  $\alpha > 1$ ,  $g: [-\pi, \pi] \rightarrow \mathbb{R}$  is nondecreasing and

$$f(z) = \int_{-\pi}^{\pi} \frac{1}{(1 - e^{-it}z)^\alpha} dg(t)$$

for  $|z| < 1$ . If  $g$  is differentiable at  $\theta$  then  $\lim_{r \rightarrow 1-} (1-r)^{\alpha-1} f(re^{i\theta}) = 0$ .

Theorem 10 is sharp in the following sense.

**Theorem 12.** Suppose that  $\varphi$  is a positive function on  $(0, 1)$  such that  $\lim_{r \rightarrow 1-} \varphi(r) = 0$  and let  $\alpha > 1$ . Then there exists  $f \in \mathcal{F}_\alpha$  such that

$$\overline{\lim}_{r \rightarrow 1-} \left\{ \frac{(1-r)^{\alpha-1} \min_{|z|=r} |f(z)|}{\varphi(r)} \right\} = \infty. \quad (19)$$

Theorem 12 is especially strong due to the minimum in (19). A proof is in [18], and the argument gives a construction of a suitable lacunary series depending upon the function  $\varphi$ . By choosing  $\varphi$  such that  $\lim_{r \rightarrow 1-} ((1-r)^{\alpha-1}/\varphi(r)) = 0$  we obtain  $f \in \mathcal{F}_\alpha$  such that  $\overline{\lim}_{r \rightarrow 1-} \min_{|z|=r} |f(z)| = \infty$ . Thus, if  $\alpha > 1$  there exists  $f \in \mathcal{F}_\alpha$  such that  $\overline{\lim}_{r \rightarrow 1-} |f(re^{i\theta})| = \infty$  for all  $\theta$ . In particular,  $f(e^{i\theta})$  fails to exist for all  $\theta$ . This gives the fact that for each  $\alpha > 1$  there exists  $f \in \mathcal{F}_\alpha$  such that  $f \notin H^p$  for all  $p > 0$ .

#### 4. Zeros

Suppose that  $f \in \mathcal{F}_\alpha$  for some  $\alpha$ ,  $0 \leq \alpha \leq 1$  and  $f \neq 0$ . Then Theorem 1 implies  $f \in H^p$  for suitable  $p$ . Hence if  $\{z_n\}$  denotes the zeros of  $f$ , counting multiplicities, then the Blaschke condition (12) holds.

Since the Blaschke product having the zeros  $\{z_n\}$  belongs to  $\mathcal{F}_1$ , the Blaschke condition characterizes the set of zeros of a nonzero function in  $\mathcal{F}_1$ .

Not much is known about the zeros of a function in  $\mathcal{F}_\alpha$  when  $0 \leq \alpha < 1$ . Recall that if  $0 < \alpha < 1$  and (13) holds then  $B$  in (11) satisfies  $B \in \mathcal{F}_\alpha$ . Another piece of information is that nothing better than (13) depending only on  $\{|z_n|\}$  is possible in the sense of the next theorem.

**Theorem 13.** Suppose that  $0 < \alpha < 1$ ,  $0 < r_n < 1$  and  $\sum_{n=1}^{\infty} (1-r_n)^\alpha = \infty$ . There exists a real sequence  $\{\theta_n\}$  such that if  $z_n = r_n e^{i\theta_n}$  and  $f \in \mathcal{F}_\alpha$  satisfies  $f(z_n) = 0$  for  $n = 1, 2, \dots$  then  $f = 0$ .

A proof of Theorem 13 depends upon a modification of a result in [46, see p. 359] as pointed out by Samotij.

Much more is known about the zeros of functions in  $\mathcal{F}_\alpha$  when  $\alpha > 1$ .

**Theorem 14.** Suppose that  $\alpha > 1$ ,  $f \in \mathcal{F}_\alpha$  and  $f \neq 0$ . Let  $\{z_n\}$  denote the nonzero zeros of  $f$  ordered so that  $\{|z_n|\}$  is nondecreasing. Then

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{n^{\alpha-1}} \prod_{k=1}^n \frac{1}{|z_k|} \right\} = 0. \quad (20)$$

Theorem 14 is proved in [18]. Condition (20) is less restrictive than (12). Theorem 14 is sharp in a sense described in [18] and the argument for this uses the same kind of function constructed to prove Theorem 12.

When the zeros of a function in  $\mathcal{F}_\alpha$  belong to a Stolz angle in  $\Delta$  (or a finite union of Stolz angles) with vertex on  $\partial\Delta$  then the Blaschke condition holds. This follows directly from a general result due to Hayman and Korenblum [31]. A construction due to Carleson [9] provides a converse of this fact, which is part of the next theorem.

**Theorem 15.** *Suppose that  $\alpha > 1$ ,  $f \in \mathcal{F}_\alpha$ ,  $f \neq 0$  and  $f(z_n) = 0$  for  $n = 1, 2, 3, \dots$  where  $\{z_n\}$  is in some Stolz angle in  $\Delta$  with vertex on  $\partial\Delta$ . Then (12) holds. Conversely, if  $\{z_n\}$  is a sequence in such a Stolz angle and if (12) holds then there exists a function  $f \in \mathcal{F}_\alpha$  for all  $\alpha \geq 0$  which has zeros precisely given by  $\{z_n\}$ .*

## 5. Multipliers

A function  $f$  is called a multiplier of  $\mathcal{F}_\alpha$  provided that  $fg \in \mathcal{F}_\alpha$  for every  $g \in \mathcal{F}_\alpha$ . We let  $\mathcal{M}_\alpha$  denote the set of multipliers of  $\mathcal{F}_\alpha$ . If  $f \in \mathcal{M}_\alpha$  then the mapping  $g \mapsto fg$  is a continuous, linear operator on  $\mathcal{F}_\alpha$  and  $\mathcal{M}_\alpha$  is a Banach space with respect to the norm given by this operator.

The study of  $\mathcal{M}_\alpha$  is a very rich line of research. A number of properties of functions in  $\mathcal{M}_\alpha$  are known and some of them are stated in the next result obtained in [36].

**Theorem 16.** *If  $f \in \mathcal{M}_\alpha$  for some  $\alpha > 0$  then  $f \in H^\infty$ . Also  $f$  has a finite radial variation and a nontangential limit in every direction. For every  $\alpha > 0$   $\mathcal{M}_\alpha \subset \mathcal{F}_\alpha$ , and  $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$  if  $\alpha < \beta$ .*

We recall that the radial variation of  $f$  in the direction  $\theta$  is given by the integral  $\int_0^1 |f'(re^{i\theta})| dr$ . These integrals are bounded in  $\theta$  for  $-\pi \leq \theta \leq \pi$  if  $f \in \mathcal{M}_\alpha$  for some  $\alpha$ .

There are a number of sufficient conditions for membership in  $\mathcal{M}_\alpha$ . The first one we state concerns the Taylor coefficients.

**Theorem 17.** *Suppose that  $f(z) = \sum_{n=0}^\infty a_n z^n$  for  $|z| < 1$ . Each of the following conditions implies  $f \in \mathcal{M}_\alpha$ :*

$$\sum_{n=1}^\infty n^{1-\alpha} |a_n| < \infty, \tag{21}$$

when  $0 < \alpha < 1$ ;

$$\sum_{n=0}^\infty [\log(n+2)] |a_n| < \infty, \tag{22}$$



when  $\alpha = 1$ ;

$$\sum_{n=0}^{\infty} |a_n| < \infty, \quad (23)$$

when  $\alpha > 1$ .

Theorem 17 was proved in the case  $0 < \alpha < 1$  by Dansereau [11] and independently by Hallenbeck, Samotij and the author [21]. The case  $\alpha = 1$  is due to Vinogradov [54] and the case  $\alpha > 1$  is in [21]. The arguments depend on finding suitable estimates on  $\|z^n/(1 - \bar{\zeta}z)^\alpha\|_{\mathcal{F}_\alpha}$  where  $|\zeta| = 1$  and  $n \geq 0$ . This is the same as estimating the multiplier norm of  $z^n$  as  $n \rightarrow \infty$ . For example, when  $\alpha > 1$ , Theorem 17 depends on the fact that there is a positive constant  $A$  (depending on  $\alpha$ ) such that

$$\left\| \frac{z^n}{(1 - \bar{\zeta}z)^\alpha} \right\|_{\mathcal{F}_\alpha} \leq A \quad (24)$$

for  $|\zeta| = 1$  and  $n = 0, 1, 2, \dots$ .

We outline the argument that (24) yields the result when  $\alpha > 1$ . First note that for each  $n$

$$\left\| \left( \sum_{k=0}^n a_k z^k \right) \frac{1}{(1 - \bar{\zeta}z)^\alpha} \right\|_{\mathcal{F}_\alpha} \leq \sum_{k=0}^n |a_k| \left\| \frac{z^k}{(1 - \bar{\zeta}z)^\alpha} \right\|_{\mathcal{F}_\alpha} \leq \sum_{k=0}^n |a_k| A \leq A \sum_{k=0}^{\infty} |a_k| \equiv B < \infty.$$

By letting  $n \rightarrow \infty$  we find that

$$\left\| f(z) \frac{1}{(1 - \bar{\zeta}z)^\alpha} \right\|_{\mathcal{F}_\alpha} \leq B \quad \text{for } |\zeta| = 1. \quad (25)$$

The conclusion that  $f \in \mathcal{M}_\alpha$  is a consequence of the following result [36], which serves as a basic lemma for several arguments about multipliers.

**Theorem 18.** Suppose that  $f$  is analytic in  $\Delta$  and  $\alpha > 0$ . Then  $f \in \mathcal{M}_\alpha$  if and only if  $f(z)/(1 - \bar{\zeta}z)^\alpha \in \mathcal{F}_\alpha$  for  $|\zeta| = 1$  and there is a positive constant  $B$  such that (25) holds.

If  $f \in H^\infty$  and  $f(e^{i\theta})$  is sufficiently smooth then  $f \in \mathcal{M}_\alpha$ . A particular result of this kind is stated next. It was proved in [21] for  $0 < \alpha < 1$  and in [55] for  $\alpha = 1$ .

**Theorem 19.** Suppose that  $f \in H^\infty$  and  $0 < \alpha \leq 1$ . If

$$\sup_{t \in \mathbb{R}} \int_{-\pi}^{\pi} \frac{|f(e^{i(t+s)}) - f(e^{it})|}{|s|^{2-\alpha}} ds < \infty, \quad (26)$$

then  $f \in \mathcal{M}_\alpha$ .

A recent result concerning  $\mathcal{M}_\alpha$  is the following theorem in [43]. It actually implies Theorem 19 when  $0 < \alpha < 1$  and has other consequences.

**Theorem 20.** *Let  $0 < \alpha < 1$  and let  $dA$  denote two-dimensional Lebesgue measure. If  $f \in H^\infty$  and*

$$\sup_{|\zeta|=1} \int \int_A \frac{|f'(z)|(1-|z|)^{\alpha-1}}{|z-\zeta|^\alpha} dA(z) < \infty \quad (27)$$

*then  $f \in \mathcal{M}_\alpha$ .*

A sufficient condition for membership in  $\mathcal{M}_\alpha$  which does not depend on  $\alpha$  is the following theorem from [21].

**Theorem 21.** *If  $f \in \mathcal{F}_0$  and the Taylor coefficients of  $f$  satisfy  $\sum_{n=0}^\infty |a_n| < \infty$  then  $f \in \mathcal{M}_\alpha$  for all  $\alpha > 0$ .*

In the case  $\alpha > 1$  Theorem 21 is given by Theorem 17 without the assumption  $f \in \mathcal{F}_0$ . Here is an outline of the argument for Theorem 21. Let  $f$  satisfy the stated conditions, assume that  $g \in \mathcal{F}_\alpha$  and let  $h = fg$ . Because  $g \in \mathcal{F}_\alpha$  it follows that  $g' \in \mathcal{F}_{\alpha+1}$ . From the case  $\alpha > 1$  of Theorem 17 we conclude that  $fg' \in \mathcal{F}_{\alpha+1}$ . Also  $f \in \mathcal{F}_0$  implies  $f' \in \mathcal{F}_1$ . From  $f' \in \mathcal{F}_1$  and  $g \in \mathcal{F}_\alpha$  a product theorem in [44] yields  $f'g \in \mathcal{F}_{\alpha+1}$ . We have  $fg' \in \mathcal{F}_{\alpha+1}$  and  $f'g \in \mathcal{F}_{\alpha+1}$ . Thus  $h' = fg' + f'g \in \mathcal{F}_{\alpha+1}$ . From  $h' \in \mathcal{F}_{\alpha+1}$  it follows that  $h \in \mathcal{F}_\alpha$ . Therefore  $f \in \mathcal{M}_\alpha$ .

**Theorem 22.** *If  $f' \in H^1$  then  $f \in \mathcal{M}_\alpha$  for all  $\alpha > 0$ .*

Theorem 21 implies Theorem 22. This can be seen in the following way. The condition  $f' \in H^1$  implies  $f' \in \mathcal{F}_1$  which yields  $f \in \mathcal{F}_0$ . Also the condition  $f' \in H^1$  implies  $\sum_{n=0}^\infty |a_n| < \infty$  for the Taylor coefficients of  $f$  due to an inequality of Hardy [12, p. 48].

There are other proofs of Theorem 22 in the case  $\alpha = 1$ . That result was first obtained by Vinogradov in [54]. Another argument is given in [36], where Theorem 22 was first proved.

Next, some facts are presented about membership in  $\mathcal{M}_\alpha$  for inner functions. The following theorem is due to Hruščev and Vinogradov [40].

**Theorem 23.** *An inner function belongs to  $\mathcal{M}_1$  if and only if it is a Blaschke product (finite or infinite) and its zeros  $\{z_n\}$  satisfy*

$$\sup_{|\zeta|=1} \sum_n \frac{1-|z_n|}{|1-\bar{\zeta}z_n|} < \infty. \quad (28)$$

Theorem 23 is a significant result and its proof is difficult and long. Condition (28) is associated with the work of Frostman. In a precise way it asserts that the zeros cannot accumulate too much toward any particular radial direction.

Below we state a partial generalization of Theorem 23 for  $\mathcal{M}_\alpha$  where  $0 < \alpha < 1$ . First note that if  $f$  is an inner function and  $f \in \mathcal{M}_\alpha$  for some  $\alpha$  where  $0 < \alpha < 1$  then  $f \in \mathcal{M}_1$ . Hence Theorem 23 implies  $f$  is a Blaschke product. Whether the zeros must satisfy condition (29) given below is not yet resolved. Theorem 24 is in [21].

**Theorem 24.** *Suppose that  $f$  is an infinite Blaschke product having the set of zeros  $\{z_n\}$ . If*

$$\sup_{|\zeta|=1} \sum_{n=1}^{\infty} \left\{ \frac{1 - |z_n|}{|1 - \bar{\zeta} z_n|} \right\}^{\alpha} < \infty \quad (29)$$

*for some  $\alpha$ , where  $0 < \alpha < 1$ , then  $f \in \mathcal{M}_\alpha$ .*

Our last remark about inner functions concerns  $S(z) = \exp[-(1+z)/(1-z)]$ . The next theorem is proved in [21] and one-half of the assertion follows from Theorem 23 and  $\mathcal{M}_\alpha \subset \mathcal{M}_1$  for  $\alpha < 1$ .

**Theorem 25.**  *$S \in \mathcal{M}_\alpha$  if and only if  $\alpha > 1$ .*

## 6. Compositions

Suppose that  $\varphi: \Delta \rightarrow \Delta$  is analytic. We consider the composition  $f \circ \varphi$  where  $f \in \mathcal{F}_\alpha$ . If this composition belongs to  $\mathcal{F}_\alpha$  for every  $f \in \mathcal{F}_\alpha$  then the mapping  $f \mapsto f \circ \varphi$  defines a continuous linear operator on  $\mathcal{F}_\alpha$ . Two basic facts about compositions and  $\mathcal{F}_\alpha$  are stated next.

**Theorem 26.** *If  $\alpha > 0$  and  $\varphi$  is a conformal automorphism of  $\Delta$ , then  $f \circ \varphi \in \mathcal{F}_\alpha$  for every  $f \in \mathcal{F}_\alpha$ .*

**Theorem 27.** *If  $\alpha \geq 1$  and  $\varphi: \Delta \rightarrow \Delta$  is analytic, then  $f \circ \varphi \in \mathcal{F}_\alpha$  for every  $f \in \mathcal{F}_\alpha$ .*

Theorems 26 and 27 are proved in [34]. Theorem 27 was proved earlier in [44] for the case  $\alpha = 2$  and in [5] for the case  $\alpha = 1$ . Further information about composition operators and  $\mathcal{F}_\alpha$  are obtained by Bourdon and Cima in [5] and by Hirschweiler and Nordgren in [33, 39]. The statement of Theorem 27 is not valid in general when  $0 < \alpha < 1$ . The question of characterizing the functions  $\varphi$  for which that statement holds when  $0 < \alpha < 1$  is not resolved.

We shall give outlines of the proofs of Theorems 26 and 27. In order to prove Theorem 26, suppose that (1) holds where  $\mu \in \mathcal{M}$  and let  $\varphi(z) = x(z+w)/(1+\bar{w}z)$  where  $|x| = 1$  and  $|w| < 1$ . Then

$$f[\varphi(z)] = (1 + \bar{w}z)^\alpha \int_{\Gamma} \frac{1}{[1 - (\bar{\zeta}x - \bar{w})/(1 - \bar{\zeta}xw)]^\alpha} \frac{1}{(1 - xw\bar{\zeta})^\alpha} d\mu(\zeta). \quad (30)$$

If we begin by noting that  $|(\bar{\zeta}x - \bar{w})/(1 - \bar{\zeta}xw)| = 1$  for  $|\zeta| = 1$  we find that a suitable change of variables can be introduced so that (30) can be rewritten

$$f[\varphi(z)] = (1 + \bar{w}z)^\alpha \int_A \frac{1}{(1 - \bar{s}z)^\alpha} dv(s) \quad (31)$$

where  $v \in \mathcal{M}$ . The argument is completed by using the fact that the mapping  $z \mapsto (1 + \bar{w}z)^\alpha$  gives a multiplier of  $\mathcal{F}_\alpha$  for every  $w$  ( $|w| < 1$ ).

A consequence of Theorem 26 shown in [36] is the fact that  $\mathcal{M}_\alpha$  is closed under compositions with conformal automorphisms of  $\Delta$ .

Theorem 27 is proved in the following way. There are two main steps in the argument. The first one uses Theorem 26 to show that it suffices to further assume that  $\varphi(0) = 0$ . The second step depends on decomposing the measure representing a function  $f \in \mathcal{F}_\alpha$  into a linear combination of probability measures and then appealing to the following result of Brannan et al. [6].

**Theorem 28.** *Let  $\mathcal{G}_\alpha$  denote the set of functions that are subordinate to  $F_\alpha(z) = 1/(1 - z)^\alpha$  in  $\Delta$ . If  $\alpha \geq 1$  then a function  $f$  belongs to the closed convex hull of  $\mathcal{G}_\alpha$  if and only if there is a probability measure  $\mu \in \mathcal{M}$  such that (1) holds.*

## 7. Geometric function theory

There are a number of results in geometric function theory which concern fractional Cauchy transforms. For example, the Riesz–Herglotz formula [13, p. 22] is one of them and more generally we have Theorem 28.

Let  $U$  denote the set of functions that are analytic and univalent in  $\Delta$ . Let  $S$  denote the subset of  $U$  consisting of functions  $f$  normalized by  $f(0) = 0$  and  $f'(0) = 1$ . Also let  $S^*$  denote the subset of  $S$  consisting of functions  $f$  for which  $f(\Delta)$  is starlike with respect to the origin.

In [8] it was shown that each  $f \in S^*$  can be represented

$$f(z) = \int_A \frac{z}{(1 - \bar{\zeta}z)^2} dv(\zeta) \quad (32)$$

for  $|z| < 1$ , where  $v \in \mathcal{M}$  is a probability measure. Indeed, if  $v$  varies over all such measures then Eq. (32) gives the closed convex hull of  $S^*$ . We call any function given by (32), where  $v \in \mathcal{M}$ , a Koebe transform. When  $f(0) = 0$  formulas (1) and (32) are equivalent, where  $\mu \in \mathcal{M}$  and  $v \in \mathcal{M}$ .

Functions in several subsets of  $U$  can be represented as Koebe transforms. In particular, this is the case for the so-called close-to-convex functions and for the spirallike functions [5, 44]. Such a representation also holds when the function has a more restricted growth than the maximal growth  $|f(z)| = \mathcal{O}[1/(1 - |z|)^2]$ . This is stated in Theorem 29 below. A related fact is stated in Theorem 30 and the question of whether  $U \subset \mathcal{F}_2$  is answered by Theorem 31. These results are contained in [44].

**Theorem 29.** Suppose that  $f \in U$  and let  $M(r) = \max_{|z|=r} |f(z)|$  for  $0 < r < 1$ . If  $\int_0^1 (1-r)M(r) dr < \infty$  then  $f \in \mathcal{F}_2$ .

**Theorem 30.** If  $f \in U$  then  $f \in \mathcal{F}_\alpha$  for all  $\alpha > 2$ .

**Theorem 31.** There exist functions  $f \in U$  such that  $f \notin \mathcal{F}_2$ .

Theorem 29 is proved using the Prawitz inequality for univalent functions.

The argument for Theorem 31 depends on the construction of examples which conformally map  $\Delta$  onto the complement of a spiral which slowly turns toward  $\infty$ . The functions have a singularity at 1. Through an application of results due to S. Warschawski in relation to the Ahlfors's distortion theorem it was shown that such functions satisfy

$$|f(z)| \geq \frac{A}{|1-z|^2} \quad (33)$$

for  $|z| < 1$ , where  $A$  is a positive constant. The first proof that  $f$  does not belong to  $\mathcal{F}_2$  was long and technical and was considerably simplified in [35]. This simplification uses the following fact: if  $f \in \mathcal{F}_\alpha$  for some  $\alpha > 0$ ,  $|\zeta| = 1$ , and  $g(z) = (1 - \bar{\zeta}z)^\alpha f(z)$ , then the curve  $w = g(r\zeta)$ ,  $0 \leq r < 1$ , is rectifiable. Here is the argument that this fact implies that the functions above satisfy  $f \notin \mathcal{F}_2$ . Let  $g(z) = (1 - z)^2 f(z)$ . Then (33) gives  $|g(z)| \geq A$  for  $|z| < 1$ . Since the spiral  $\mathbb{C} \setminus f(\Delta)$  meets the positive and negative real axis infinitely often, the curve  $w = g(r)$ ,  $0 \leq r < 1$ , has the same property. This and  $|g(z)| \geq A$  ( $A > 0$ ) imply that the curve  $w = g(r)$ ,  $0 \leq r < 1$ , is not rectifiable. Therefore  $f \notin \mathcal{F}_2$ .

A number of facts about representation of functions in  $U$  are obtained by Bass [4]. One of the results there is the next statement.

**Theorem 32.** Suppose that  $f \in U \cap \mathcal{F}_2$  and (1) holds where  $\mu \in \mathcal{M}$ . Then  $\mu(\{\zeta\}) = 0$  for all but at most one number  $\zeta$  on  $\Delta$ . Also the continuous component of  $\mu$  is absolutely continuous.

A similar theorem is proved in [4] for  $f \in U \cap F_1$ .

## 8. Concluding remarks

A number of results about fractional Cauchy transforms are not included in this survey. For example, several contributions made by Hallenbeck and Samotij have not been mentioned. Another omission is the characterization of functions given by (1) when  $\alpha = 1$ , when viewed as functions defined in  $\mathbb{C} \setminus \Delta$  [2, see Section 5]. We have tried to remedy this somewhat by the inclusion of a comprehensive list of references. Besides the references directly quoted in the text this list includes most of the papers in this area which the author is aware of. Only a few of the earlier papers on Cauchy transforms which are published in Russian and have not been translated into English have been listed. Refs. [10, 45] are earlier survey articles about fractional Cauchy transforms.

## 9. For further reading

[3, 16, 17, 23–28, 30, 32, 38, 49–52]

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