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Iterative methods for least-square problems based on proper splittings ☆

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Abstract

For the linear-squares problems $\min_{\boldsymbol{x}} \|\boldsymbol{b}-A\boldsymbol{x}\|_2$, where A is large and sparse, straightforward application of Cholesky or QR factorization will lead to catastrophic fill in factor R. We consider handling such problems by a iterative methods based on proper splittings. We establish the convergence, to the least-square solution $\boldsymbol{y}=A^{\dagger}\boldsymbol{x}$, for the sequential two-stage iterative method and for the parallel stationary iterative method.

Keywords: Iterative method; Proper splitting; Multisplitting; Moore–Penrose inverse; Least-square problem

1. Introduction

Consider the linear system

$$A\boldsymbol{x}=\boldsymbol{b}, \tag{1}$$

where A is an $m \times n$ real, large, and sparse matrix, $\boldsymbol{x}$ is the unknown real n-vector, and $\boldsymbol{b}$ is a given real m-vector.

If (1) is inconsistent, then the least-square solution, that is, the vector $\boldsymbol{y}$ of $\mathbb{R}^n$ such that

$$\|\boldsymbol{b}-A\boldsymbol{y}\|_2 = \min_{\boldsymbol{x}} \|\boldsymbol{b}-A\boldsymbol{x}\|_2$$

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is given by $\boldsymbol{y}=A^\dagger\boldsymbol{b}$ where $A^\dagger$ is the Moore–Penrose generalized inverse of matrix A. The vector $A^\dagger\boldsymbol{b}$ is also the solution of the normal equation $A^{\mathrm{T}}A\boldsymbol{x}=A^{\mathrm{T}}\boldsymbol{b}$, but the matrix $A^{\mathrm{T}}A$ has the disadvantage that it is frequently ill-conditioned and influenced greatly by roundoff error (see [5]). To avoid this disadvantage, Berman and Plemmons [2] introduce the following iterative method similar to the iterative method for nonsingular linear systems:

$$\boldsymbol{x}^{(k+1)}=M^\dagger N\boldsymbol{x}^{(k)}+M^\dagger\boldsymbol{b}, \quad k=0,1,2,\ldots, \tag{2}$$

where $A=M-N$ is a proper splitting of A; that is M and N are $m\times n$ matrices such that

$$\mathscr{R}(M)=\mathscr{R}(A) \quad \text{and} \quad \ker(M)=\ker(A).$$

Here, $\mathscr{R}(A)$ is the range or column space of A and $\ker(A)$ is the nullspace of A. Note that for the nonsingular case, a proper splitting is a splitting (see [8]).

Berman and Plemmons [2] introduce the following convergence result.

Theorem 1 (Berman and Plemmons [2, Corollary 1]). *Let A be an $m\times n$ real matrix and let $A=M-N$ be a proper splitting. Then iterative process* (2) *converges to $A^\dagger\boldsymbol{b}$ for each initial vector $\boldsymbol{x}^{(0)}$ if and only if $\rho(M^\dagger N)<1$.*

We say that matrix $A=[a_{ij}]$ is nonnegative ($A\geqslant 0$) if $a_{ij}\geqslant 0$ for $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$. Also we say that $A\leqslant B$ if $B-A\geqslant 0$. Similar for vectors.

Definition 1. Let A be an $m\times n$ real matrix and let $A=M-N$ be a proper splitting. We say that $A=M-N$ is regular if $M^\dagger\geqslant 0$ and $N\geqslant 0$. Furthermore, we say that $A=M-N$ is weak nonnegative of the first type if $M^\dagger\geqslant 0$ and $M^\dagger N\geqslant 0$.

Climent et al. [3] extend the convergence results introduced by Berman and Plemmons [2], proving the following theorem that we quote for further references.

Theorem 2 (Climent et al. [3, Theorem 3]). *Let A be an $m\times n$ real matrix with complete rank and let $A=M-N$ be a weak nonnegative proper splitting of the first type. Then the following conditions are equivalent*:

(i) $A^\dagger\geqslant 0$,
(ii) $A^\dagger\geqslant M^\dagger$,
(iii) $A^\dagger M\geqslant 0$,
(iv) $\rho(M^\dagger N)=(\rho(A^\dagger M)-1)/\rho(A^\dagger M)$,
(v) $\rho(M^\dagger N)=\rho(NM^\dagger)<1$,
(vi) $(I-M^\dagger N)^{-1}\geqslant 0$,
(vii) $A^\dagger N\geqslant 0$,
(viii) $A^\dagger N\geqslant M^\dagger N$,
(ix) $\rho(M^\dagger N)=\rho(A^\dagger N)/(1+\rho(A^\dagger N))<1$.

If in the above theorem we remove the hypothesis "A has complete rank", then we only can establish the equivalence between parts (i), (ii) and (v)–(ix), obtaining in this way the extension of [2, Theorem 3]. That hypothesis is necessary as we can see in [3, Example 1].

It is very usual to use, to obtain the least-square solution of system (1), one of the several existing codes for the sparse Cholesky factorization of $A^{\mathrm{T}}A$ or the QR factorization of matrix A, than the iterative method (2). However, when matrix A is sparse except for a few dense rows, it is an open problem that a straightforward application of Cholesky factorization or the QR factorization will lead to catastrophic fill in the factor R. Recently, authors as Adlers and Björck [1] introduce a new technique by using a matrix stretching by row splittings.

In this paper, with the purpose to introduce alternative solution to this problem, we introduce two iterative methods to obtain the least-square solution of system (1), based on the concept of proper splitting. The first one is a two-stage method and the second one is a parallel iterative method.

2. Convergence results

Let $A = M - N$ be a proper splitting of A and let $M = F - G$ be a proper splitting of M. If we consider q inner iterations with splitting $M = F - G$ we obtain the two-stage iterative method (see [6,4] for the nonsingular case)

$$\boldsymbol{x}^{(k+1)} = (F^{\dagger}G)^{q}\boldsymbol{x}^{(k)} + \sum_{j=0}^{q-1}(F^{\dagger}G)^{j}F^{\dagger}(N\boldsymbol{x}^{(k)} + \boldsymbol{b}). \tag{3}$$

Now, using Theorems 1 and 2 we can establish the following convergence result.

Theorem 3. *Let $A = M - N$ be a regular proper splitting, and let $M = F - G$ be a weak nonnegative proper splitting of the first type. Then the stationary two-stage iterative method* (3) *converges to the least-square solution of system* (1) *for any initial vector* $\boldsymbol{x}^{(0)}$.

Proof. Since $MM^{\dagger}N = N$, $F^{\dagger}M = F^{\dagger}F - F^{\dagger}G$, and $F^{\dagger}F = M^{\dagger}M$ we have, from Eq. (3), that

$$\begin{aligned}
T_p &= (F^{\dagger}G)^{p} + \sum_{j=0}^{p-1}(F^{\dagger}G)^{j}F^{\dagger}N \\
&= (F^{\dagger}G)^{p} + \sum_{j=0}^{p-1}(F^{\dagger}G)^{j}F^{\dagger}MM^{\dagger}N \\
&= (F^{\dagger}G)^{p} + \sum_{j=0}^{p-1}(F^{\dagger}G)^{j}(F^{\dagger}F - F^{\dagger}G)M^{\dagger}N \\
&= (F^{\dagger}G)^{p} + (F^{\dagger}F - (F^{\dagger}G)^{p})M^{\dagger}N \\
&= (F^{\dagger}G)^{p} + F^{\dagger}FM^{\dagger}N - (F^{\dagger}G)^{p}M^{\dagger}N \\
&= (F^{\dagger}G)^{p} + M^{\dagger}MM^{\dagger}N - (F^{\dagger}G)^{p}M^{\dagger}N \\
&= (F^{\dagger}G)^{p} + M^{\dagger}N - (F^{\dagger}G)^{p}M^{\dagger}N
\end{aligned}$$

$$
\begin{aligned}
&= (F^\dagger G)^p + (I - (F^\dagger G)^p)M^\dagger N \\
&= I - I + (F^\dagger G)^p + (I - (F^\dagger G)^p)M^\dagger N \\
&= I - (I - (F^\dagger G)^p)(I - M^\dagger N) \\
&= I - \sum_{j=0}^{p-1} (F^\dagger G)^j (I - F^\dagger G)(I - M^\dagger N).
\end{aligned}
$$

Now, taking into account that $A = M - N$ and $M = F - G$ are convergent, if $\boldsymbol{y} \geqslant 0$, then $\boldsymbol{x} = (I - M^\dagger N)^{-1}(I - F^\dagger G)^{-1}\boldsymbol{y} \geqslant 0$ and, consequently,

$$0 \leqslant T_p \boldsymbol{x} = \boldsymbol{x} - \sum_{j=0}^{p-1} (F^\dagger G)^j \boldsymbol{y} < \boldsymbol{x},$$

which implies that $\rho(T_p) < 1$. □

Next, in a similar way that for weak nonnegative splittings of the first type of nonsingular matrices (see [7]), we can obtain a least-square solution of system (1) using a parallel iterative process based on a multisplitting of A.

Definition 2. Let A be an $m \times n$ real matrix. We say that $\{(M_l, N_l, E_l)\}_{l=1}^{p}$ is a proper multisplitting of A if $A = M_l - N_l$, for $l = 1, \ldots, p$, is a proper splitting, E_l, for $l = 1, \ldots, p$, is a nonnegative and diagonal $m \times m$ matrix, and $\sum_{l=1}^{p} E_l = I_m$ where I_m is the $m \times m$ identity matrix.

As a generalization of Definition 1, we say that a proper multisplitting is regular or weak nonnegative of the first type, if each one of the proper splittings is regular or weak nonnegative of the first type, respectively.

If $\{(M_l, N_l, E_l)\}_{l=1}^{p}$ is a proper multisplitting of A, we can consider the iterative scheme

$$\boldsymbol{x}^{(k+1)} = H\boldsymbol{x}^{(k)} + G\boldsymbol{b}, \quad k = 0, 1, 2, \ldots, \tag{4}$$

where

$$H = \sum_{l=1}^{p} E_l M_l^\dagger N_l \quad \text{and} \quad G = \sum_{l=1}^{p} E_l M_l^\dagger.$$

Before establishing the convergence result given in Theorem 4 for the above iterative scheme, we need the following result.

Lemma 1. *Let A be an $m \times n$ real matrix and let $\{(M_l, N_l, E_l)\}_{l=1}^{p}$ be a proper multisplitting of the first type of A. Then*

(i) $H \geqslant 0$ *and therefore* $H^j \geqslant 0$, *for* $j = 0, 1, \ldots,$
(ii) $\sum_{l=1}^{p} E_l M_l^\dagger A = (I - H)A^\dagger A$,
(iii) $(I + H + H^2 + \cdots + H^m)(I - H) = I - H^{m+1}$.

Proof. Part (i) is a consequence from the definition of matrix H and the definition of weak nonnegative proper splitting of the first type. Part (iii) is a consequence of part (i).

To prove part (iii) we proceed as follows. From $M_l M_l^\dagger N_l = N_l = N_l M_l^\dagger M_l$ and $M_l^\dagger M_l = A^\dagger A$ for $l = 1, 2, \ldots, p$ we have that

$$A = M_l - N_l = M_l(I - M_l^\dagger N_l), \quad l = 1, 2, \ldots, p$$

and then

$$\begin{aligned}
\sum_{l=1}^{p} E_l M_l^\dagger A &= \sum_{l=1}^{p} E_l M_l^\dagger M_l (I - M_l^\dagger N_l) \\
&= \sum_{l=1}^{p} E_l (M_l^\dagger M_l - M_l^\dagger M_l M_l^\dagger N_l) \\
&= \sum_{l=1}^{p} E_l (M_l^\dagger M_l - M_l^\dagger N_l M_l^\dagger M_l) \\
&= \sum_{l=1}^{p} E_l ((I - M_l^\dagger N_l) M_l^\dagger M_l) \\
&= \left(I - \sum_{l=1}^{p} E_l M_l^\dagger N_l \right) A^\dagger A \\
&= (I - H) A^\dagger A,
\end{aligned}$$

that is, part (ii) holds. □

Theorem 4. *Let A be an $m \times n$ real matrix and let $\{(M_l, N_l, E_l)\}_{l=1}^{p}$ be a proper multisplitting of the first type of A. Then iterative method* (4) *converges to the least-square solution of system* (1) *for any initial vector $\boldsymbol{x}^{(0)}$.*

Proof. Using Lemma 1 and taking into account that $M_l^\dagger M_l M_l^\dagger = M_l^\dagger$, and $M_l M_l^\dagger = AA^\dagger$ for $l = 1, 2, \ldots, p$, and $A^\dagger A A^\dagger = A^\dagger$, we obtain

$$\begin{aligned}
0 &\leqslant (I + H + H^2 + \cdots + H^m) \sum_{l=1}^{p} E_l M_l^\dagger \\
&= (I + H + H^2 + \cdots + H^m) \sum_{l=1}^{p} E_l M_l^\dagger M_l M_l^\dagger \\
&= (I + H + H^2 + \cdots + H^m) \sum_{l=1}^{p} E_l M_l^\dagger A A^\dagger \\
&= (I + H + H^2 + \cdots + H^m)(I - H) A^\dagger A A^\dagger
\end{aligned}$$

$$= (I + H + H^2 + \cdots + H^m)(I - H)A^\dagger$$
$$= (I - H^{m+1})A^\dagger \leqslant A^\dagger.$$

Therefore, the elements of H^m must remain bounded, and therefore H is convergent, that is, iterative method (4) converges to the least-square solution of system (1) for any initial vector $\boldsymbol{x}^{(0)}$. □

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