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Time-harmonic acoustic propagation in the presence of a shear flow

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Abstract

This work deals with the numerical simulation, by means of a finite element method, of the time-harmonic propagation of acoustic waves in a moving fluid, using the Galbrun equation instead of the classical linearized Euler equations. This work extends a previous study in the case of a uniform flow to the case of a shear flow. The additional difficulty comes from the interaction between the propagation of acoustic waves and the convection of vortices by the fluid. We have developed a numerical method based on the regularization of the equation which takes these two phenomena into account. Since it leads to a partially full matrix, we use an iterative algorithm to solve the linear system.

Keywords: Aeroacoustics; Shear flow; Fredholm formulation; FEM

1. Introduction

Understanding of the propagation of sound in a moving fluid is of particular importance in several industries. A large part of the efforts made in that domain is devoted to the computation of the noise generated and radiated by engines. In this work, we are interested in the simulation of acoustic propagation in the presence of a shear flow, using the so-called Galbrun equation [5].

This peculiar model assumes small perturbations of an isentropic flow of a perfect fluid and is based on a Lagrangian–Eulerian description of the perturbations, in the sense that Lagrangian perturbations of the quantities are expressed in terms of Eulerian variables with respect to the mean flow. It consists of a linear partial differential equation of second order in time and space on the Lagrangian displacement perturbation, which is amenable to variational methods. However, the numerical solution of Galbrun equation by standard (i.e., nodal) finite element methods is subject to difficulties quite similar to those observed for Maxwell's equations in electromagnetism.

In [2], we proposed a regularized formulation of the time-harmonic Galbrun equation in presence of a uniform mean flow that allowed the use of nodal finite elements for the discretization of the problem. The application of this method to the case of a shear mean flow is investigated here. Following [2] we consider an artificial problem set in a

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bounded domain. To deal with more realistic situations, the regularization technique could be coupled with perfectly matched layers (PML) as done in [1].

The outline is the following. The problem and the framework used to solve it are presented in Section 2. Section 3 is devoted to the mathematical study of a weak formulation of this problem. Finally, Section 4 is concerned with numerical applications.

2. Presentation of the problem

2.1. Physical setting

We consider an infinite two-dimensional rigid duct of height ℓ, set in the x_1x_2 plane and filled with a compressible fluid. We are interested in the linear propagation of waves in the presence of a subsonic shear mean flow of velocity $\boldsymbol{v_0}(\boldsymbol{x}) = v(x_2)\boldsymbol{e_1}$, $\boldsymbol{e_1}$ being the unit vector in the x_1 direction, and assuming a time-harmonic dependence of the form $\exp(-\mathrm{i}\omega t)$, $\omega > 0$ being the pulsation. In terms of the perturbation of the Lagrangian displacement $\boldsymbol{u}$, this problem is modelled by the following equation and boundary condition: *find a displacement* $\boldsymbol{u}$ *satisfying*

$$\mathrm{D}^2\boldsymbol{u} - \nabla(\operatorname{div}\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \mathbb{R} \times [0, \ell], \tag{1}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{for } x_2 = 0 \text{ and } x_2 = \ell, \tag{2}$$

and an adequate radiation condition at infinity, $\boldsymbol{n}$ being the unit outward normal to the duct walls. Eq. (1) is the Galbrun equation, in which the letter D stands for the material derivative in the mean flow with time-harmonic dependence, that is $\mathrm{D}\boldsymbol{u} = -\mathrm{i}k\boldsymbol{u} + M\partial_{x_1}\boldsymbol{u}$, with $k = \omega/c_0$ the acoustic wave number and $M = v/c_0$ the Mach number, c_0 being the sound velocity. The right-hand side term $\boldsymbol{f}$ represents an acoustic source placed in the duct. Note that we restrict ourselves to a subsonic shear flow whose Mach number profile $M(x_2)$ is a nonvanishing $C^1([0, \ell])$ function. The previous study dealt with the uniform flow case which corresponds to a constant profile (i.e., $M' \equiv 0$). Considering a less regular or vanishing profile would bring up difficulties which are beyond the scope of this paper.

As solving the problem in an unbounded domain necessitates, as previously mentioned, the determination of a radiation condition (see [1] in the uniform flow case) and since this article focuses on the finite element method used to compute a solution to Galbrun's equation, we consider from now on an artificial problem set in a bounded portion of the duct of length L. In what follows, Ω denotes the domain $[0, L] \times [0, \ell]$.

Boundary conditions have now to be prescribed on the vertical boundaries $\Sigma_- = \{0\} \times [0, \ell]$ and $\Sigma_+ = \{L\} \times [0, \ell]$. By analogy with the no flow case, we impose the value of $\boldsymbol{u} \cdot \boldsymbol{n}$. Notice that, by linearity of Eq. (1), we can choose this boundary condition to be a homogeneous one, and we now have

$$\mathrm{D}^2\boldsymbol{u} - \nabla(\operatorname{div}\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{4}$$

instead of previous Eqs. (1) and (2). However, this last problem is not well posed. We will indeed see in Section 2.2.2 that, due to the presence of flow, a supplementary boundary condition is required on the vertical boundaries Σ_- and Σ_+.

2.2. Variational framework

When considering the discretization of Galbrun's equation (3) by a finite element method, one is confronted with the choice of a variational formulation of the problem, which will, in turn, lead to the use of suitable finite element spaces. As for the second-order form of Maxwell's equations appearing in computational electromagnetism, two main strategies can be considered, both of which are presented on a model problem of acoustic propagation in a fluid at rest. In the following, it is assumed that the reader is familiar with such spaces as $L^2(\Omega)$, $H(\mathrm{curl}; \Omega)$, $H(\mathrm{div}; \Omega)$ and $H^1(\Omega)$, and their respective subspaces $H_0(\mathrm{div}; \Omega)$ and $H_0^1(\Omega)$.

2.2.1. The no mean flow case

When the fluid is initially at rest, problem (3)–(4) becomes: *find* $\boldsymbol{u}$ *such that*

$$-k^2\boldsymbol{u} - \nabla(\operatorname{div}\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{5}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{6}$$

This type of problem arises in several acoustic fluid-structure interaction problems of interest (see [7] for instance.) For the sake of simplicity, we furthermore assume that the vector field $\boldsymbol{f}$ is such that $\operatorname{curl}\boldsymbol{f} = 0$ in Ω, which amounts to saying that the source only generates acoustic (i.e., irrotational) perturbations. Note that, as the wave number k is non zero, the displacement field $\boldsymbol{u}$ satisfies the following constraint:

$$\operatorname{curl}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{7}$$

which is simply a consequence of Eq. (5).

Since Eq. (5) does not exhibit ellipticity properties, proper care has to be taken when writing a weak formulation of problem (5)–(6) and two different approaches can be followed.

First, dropping constraint (7), which may be difficult to impose on the numerical approximation, leads to a straightforward variational formulation in the Hilbert space $U = H_0(\operatorname{div};\Omega)$: *find* $\boldsymbol{u}$ *in* U *such that*

$$\int_\Omega (\operatorname{div}\boldsymbol{u}\operatorname{div}\overline{\boldsymbol{v}} - k^2\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} = (\boldsymbol{f},\boldsymbol{v})_{L^2(\Omega)^2}, \quad \forall \boldsymbol{v}\in U. \tag{8}$$

However, attempts to solve this problem by Lagrange finite element methods (each field component being represented on nodal basis functions) have proved to be ill-suited, the computed solutions being affected by the occurrence of "non-physical" modes, related to the fact that the space U is not compactly imbedded in $L^2(\Omega)^2$. Nevertheless, the above curl-free constraint can be enforced by means of a Lagrange multiplier. This leads to the following mixed formulation of problem (5)–(6): *find* $(\boldsymbol{u}, p)$ *in* $U \times H_0^1(\Omega)$ *such that*

$$\begin{aligned}&\int_\Omega (\operatorname{div}\boldsymbol{u}\operatorname{div}\overline{\boldsymbol{v}} - k^2\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathbf{curl}\,p\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} = (\boldsymbol{f},\boldsymbol{v})_{L^2(\Omega)^2}, \quad \forall \boldsymbol{v}\in U,\\ &\int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\overline{q}\,\mathrm{d}\boldsymbol{x} = 0, \quad \forall q\in H_0^1(\Omega),\end{aligned} \tag{9}$$

the unknown p being the aforementioned multiplier. Convergence of approximations of problem (9) requires the use of so-called mixed finite elements (Raviart–Thomas elements for instance), which respect some necessary features such as the inf-sup condition and discrete compactness property [8]. Additionally, one sees that a solution to (8) is a solution to (9) with $p = 0$. As a consequence, if an adequate discretization is used, the approximated multiplier can be regarded as a "hidden" variable and thus dropped.

Another possibility consists in modifying the weak problem (8) in order to make it account directly for the constraint (7). In our case, this is done by adding a $(\operatorname{curl}\cdot, \operatorname{curl}\cdot)_{L^2(\Omega)}$ product to the formulation, which yields the more general *regularized* or *augmented*, with respect to (8), variational problem: *find* $\boldsymbol{u}\in V$ *such that*

$$\int_\Omega (\operatorname{div}\boldsymbol{u}\operatorname{div}\overline{\boldsymbol{v}} + s\operatorname{curl}\boldsymbol{u}\operatorname{curl}\overline{\boldsymbol{v}} - k^2\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} = (\boldsymbol{f},\boldsymbol{v})_{L^2(\Omega)^2}, \quad \forall \boldsymbol{v}\in V, \tag{10}$$

where V is the Hilbert space $V = H_0(\operatorname{div};\Omega)\cap H(\operatorname{curl};\Omega)$, equipped with the graph norm, and s is a given positive real number. The space V being compactly imbedded into $L^2(\Omega)^2$ and the new sesquilinear form having coercivity properties on it, one can classically make use of the Riesz–Fredholm theory to prove existence and uniqueness of a solution to problem (10).

What is more, when the domain Ω is convex or the boundary $\partial\Omega$ is smooth, observe that we have $V = W$, where the space

$$W = H_0(\operatorname{div};\Omega)\cap H^1(\Omega)^2$$

is equipped with the $H^1(\Omega)^2$ norm. From the point of view of the numerical approximation, the fact that V is a subspace of $H^1(\Omega)^2$ in this case allows a suitable and convenient discretization of the problem by Lagrange finite elements.

2.2.2. The shear flow case

The presence of a flow complicates considerably the above analysis, as the convective terms appearing in the equation raise difficulties on several levels.

First, taking the curl of Eq. (3) leads to an ordinary linear differential equation on $\operatorname{curl}\boldsymbol{u}$ instead of the explicit constraint (7). The main consequence of this change is that, even if the source $\boldsymbol{f}$ is irrotational, the displacement field $\boldsymbol{u}$ is not curl-free, a notable exception being if the mean flow is uniform [2]. Also related is the fact that mixed finite elements adapted to this configuration are to be found yet in the literature. We nevertheless show in the next section that the regularization technique can be nontrivially extended to successfully solve the problem.

Second, the functional framework is not completely clear. For the convective terms to have a sense in $L^2(\Omega)^2$, the variational problem needs to be set a priori in a space smaller than V. We deliberately choose to work in the subspace W of $H^1(\Omega)^2$, which will suitably fit our needs for the regularization process.

Then, any solution to (3)–(4) satisfies the following weak formulation of the problem: *find $\boldsymbol{u}$ in W such that, for any $\boldsymbol{v}$ in W,*

$$\int_\Omega (\operatorname{div}\boldsymbol{u}\operatorname{div}\overline{\boldsymbol{v}} - M^2\partial_{x_1}\boldsymbol{u}\cdot\partial_{x_1}\overline{\boldsymbol{v}} - 2\mathrm{i}kM\partial_{x_1}\boldsymbol{u}\cdot\overline{\boldsymbol{v}} - k^2\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} \\ + \langle M^2\partial_{x_1}\boldsymbol{u}(\boldsymbol{n}\cdot\boldsymbol{e_1}),\boldsymbol{v}\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} = (\boldsymbol{f},\boldsymbol{v})_{L^2(\Omega)^2}.$$

Note that a supplementary boundary condition is needed in order to properly deal with the surface term in the left-hand side. Among several possible choices, we select the assumption that $\operatorname{curl}\boldsymbol{u}$ is known on the boundary $\partial\Omega$. We consequently add the boundary conditions

$$\operatorname{curl}\boldsymbol{u} = \psi_\pm \quad \text{on } \Sigma_\pm, \tag{11}$$

where ψ_+ (resp. ψ_-) belongs to $L^2(\Sigma_+)$ (resp. $L^2(\Sigma_-)$), to the set of equations (3)–(4) and close the problem to solve. We will see in Section 3.1 that this last condition will prove useful in obtaining an explicit constraint for the scalar field $\operatorname{curl}\boldsymbol{u}$.

Last, it is also obvious that the above sesquilinear form has no coerciveness properties on $H^1(\Omega)^2$, so that an augmented (or regularized) form of the variational problem is clearly required. This is the purpose of the next section.

3. Study of a regularized problem

In the next subsection, we derive from Galbrun's equation an explicit constraint for $\operatorname{curl}\boldsymbol{u}$ analogous to identity (7) and preliminary results are given. We then write a weak regularized problem, whose well-posedness and equivalence with the original problem are proved in Sections 3.2 and 3.3, respectively.

3.1. Derivation of a constraint for curl $\boldsymbol{u}$

Assume that the source $\boldsymbol{f}$ belongs to $H(\text{curl};\Omega)$ and formally apply the curl operator to Galbrun's equation (3). The Mach number M being a function of the x_2 variable, we obtain

$$\mathrm{D}^2(\operatorname{curl}\boldsymbol{u}) = 2M'\mathrm{D}(\partial_{x_1}u_1) + \operatorname{curl}\boldsymbol{f} \quad \text{on } \Omega. \tag{12}$$

For any fixed value of x_2 in $[0,\ell]$, the above equation is simply an ordinary, linear, constant coefficient differential equation with respect to the x_1 variable. Denoting $\psi = \operatorname{curl}\boldsymbol{u}$ and considering the following problem:

$$-k^2\psi - 2\mathrm{i}kM\partial_{x_1}\psi + M^2\partial^2_{x_1}\psi = g \quad \text{in } \Omega, \\ \psi = \psi_0 \quad \text{on } \Sigma_\pm, \tag{13}$$

where g and ψ_0 are given data, the solution to (12) can be computed easily if we conveniently choose the following decomposition:

$$\psi = \mathscr{A}\boldsymbol{u} + \psi_{\boldsymbol{f}}, \tag{14}$$

where the field $\psi_{\boldsymbol{f}}$ denotes the solution to problem (13) with $g = \operatorname{curl} \boldsymbol{f}$ and $\psi_0 = \psi_\pm$ on $\Sigma_\pm$, and $\mathscr{A}\boldsymbol{u}$ is the solution to (13) with $g = 2M'\mathrm{D}(\partial_{x_1} u_1)$ and vanishing on the boundaries $\Sigma_\pm$. Note that identity (14) replaces the simple constraint (7) in the presence of a shear flow. The explicit determination of the field $\psi_{\boldsymbol{f}}$ is tackled in Appendix A and we now state two results on the field $\mathscr{A}\boldsymbol{u}$ that will be needed for the subsequent study of the regularized problem.

Lemma 1. *For all $\boldsymbol{u}$ in W and (x_1, x_2) in Ω, we have*

$$\begin{aligned}\mathscr{A}\boldsymbol{u}(x_1, x_2) = {} & \frac{2\mathrm{i}kM'(x_2)}{M^2(x_2)} \frac{L - x_1}{L} \int_0^{x_1} u_1(z, x_2) \mathrm{e}^{\mathrm{i}(k(x_1 - z)/M(x_2))} \,\mathrm{d}z \\ & - \frac{2\mathrm{i}kM'(x_2)}{M^2(x_2)} \frac{x_1}{L} \int_{x_1}^{L} u_1(z, x_2) \mathrm{e}^{\mathrm{i}(k(x_1 - z)/M(x_2))} \,\mathrm{d}z \\ & + \frac{2M'(x_2)}{M(x_2)} u_1(x_1, x_2). \end{aligned} \tag{15}$$

The proof of this lemma is given in Appendix A.

Lemma 2. *The operator $\mathscr{A}$ is continuous from W onto $H^1(\Omega)$.*

Proof. We need to prove the existence of a positive constant C such that

$$\|\mathscr{A}\boldsymbol{u}\|_{H^1(\Omega)} \leqslant C \|\boldsymbol{u}\|_{H^1(\Omega)^2}, \quad \forall \boldsymbol{u} \in W.$$

Owing to the regularity of the Mach number profile, one can easily see that the term $[2M'(x_2)/M(x_2)]u_1(x_1, x_2)$ in expression (15) is continuous from W onto $H^1(\Omega)$. For the remaining terms, the use of the Cauchy–Schwarz inequality allows to show the existence of a strictly positive constant C' such that

$$\|\zeta\|_{H^1(\Omega)} \leqslant C' \|u_1\|_{H^1(\Omega)} \leqslant C' \|\boldsymbol{u}\|_{H^1(\Omega)^2}, \quad \forall \boldsymbol{u} \in W. \qquad \square$$

3.2. Well posedness of the regularized problem

A regularized variational formulation of the problem is given by: *find $\boldsymbol{u} \in W$ such that*

$$a_s(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in W, \tag{16}$$

where $a_s(\cdot, \cdot)$ denotes the sesquilinear form defined on $W \times W$ by

$$\begin{aligned} a_s(\boldsymbol{u}, \boldsymbol{v}) = {} & \int_\Omega (\operatorname{div} \boldsymbol{u} \operatorname{div} \overline{\boldsymbol{v}} + s \operatorname{curl} \boldsymbol{u} \operatorname{curl} \overline{\boldsymbol{v}} - M^2 \partial_{x_1} \boldsymbol{u} \cdot \partial_{x_1} \overline{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} \\ & - \int_\Omega (k^2 \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + 2\mathrm{i}kM \partial_{x_1} \boldsymbol{u} \cdot \overline{\boldsymbol{v}} + s \mathscr{A}\boldsymbol{u} \operatorname{curl} \overline{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{17}$$

with s a given strictly positive constant, and $l(\cdot)$ is an antilinear form on W given by

$$l(\boldsymbol{v}) = \int_\Omega (\boldsymbol{f} \cdot \overline{\boldsymbol{v}} + s \psi_{\boldsymbol{f}} \operatorname{curl} \overline{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} - \int_{\Sigma_\pm} M^2 \psi_\pm v_2 (\boldsymbol{n} \cdot \boldsymbol{e_1}) \,\mathrm{d}\sigma. \tag{18}$$

We have:

Theorem 1. *Variational problem* (16) *can be written as a Fredholm equation if $s \geqslant s_0$, where $s_0 = \max_{x_2 \in [0, \ell]} M^2(x_2)$.*

Proof. We prove that the sesquilinear form $a_s(\cdot,\cdot)$ defines, by means of the Riesz representation theorem, an operator on W which is the sum of an automorphism and a compact operator. To this end, consider the operators B and C defined on W by

$$(B\boldsymbol{u},\boldsymbol{v})_W = \int_\Omega (\boldsymbol{u}\cdot\boldsymbol{v} + \operatorname{div}\boldsymbol{u}\operatorname{div}\overline{\boldsymbol{v}} + s\operatorname{curl}\boldsymbol{u}\operatorname{curl}\overline{\boldsymbol{v}} - M^2\partial_{x_1}\boldsymbol{u}\cdot\partial_{x_1}\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x},$$

and

$$(C\boldsymbol{u},\boldsymbol{v})_W = \int_\Omega (-(k^2+1)\boldsymbol{u}\cdot\overline{\boldsymbol{v}} - 2\mathrm{i}kM\partial_{x_1}\boldsymbol{u}\cdot\overline{\boldsymbol{v}} - s\mathscr{A}\boldsymbol{u}\operatorname{curl}\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x},$$

where $(\cdot,\cdot)_W$ denotes the natural scalar product in W. Due to the remarkable identity (see [3])

$$\int_\Omega (|\operatorname{div}\boldsymbol{v}|^2 + |\operatorname{curl}\boldsymbol{v}|^2)\,\mathrm{d}\boldsymbol{x} = \int_\Omega |\nabla\boldsymbol{v}|^2\mathrm{d}\boldsymbol{x},\quad \forall\boldsymbol{v}\in W,$$

there exists a strictly positive constant α such that

$$\begin{aligned}(B\boldsymbol{u},\boldsymbol{u})_W &= \int_\Omega (|\boldsymbol{u}|^2 + (1-s_0)|\operatorname{div}\boldsymbol{u}|^2 + (s-s_0)|\operatorname{curl}\boldsymbol{u}|^2 + s_0|\nabla\boldsymbol{u}|^2 - M^2|\partial_{x_1}\boldsymbol{u}|^2)\,\mathrm{d}\boldsymbol{x}\\ &\geqslant \alpha\|\boldsymbol{u}\|_W^2,\end{aligned}$$

if $s>s_0$, since $M^2(x_2)\leqslant s_0<1$ for all $x_2\in[0,\ell]$.

Additionally, the operator C is compact on W. Indeed, introducing the operator $\mathscr{K}$ from W to W, such that

$$(\mathscr{K}\boldsymbol{u},\boldsymbol{v})_W = (\mathscr{A}\boldsymbol{u},\operatorname{curl}\boldsymbol{v})_{L^2(\Omega)},\quad \forall\boldsymbol{u}\in W,\ \forall\boldsymbol{v}\in W,$$

we just need to prove that $\mathscr{K}$ is compact, the two other terms defining C being obviously compact. Taking $\boldsymbol{v}=\mathscr{K}\boldsymbol{u}$ and using the Cauchy–Schwarz inequality, we get

$$\|\mathscr{K}\boldsymbol{u}\|_W \leqslant \|\mathscr{A}\boldsymbol{u}\|_{L^2(\Omega)},\quad \forall\boldsymbol{u}\in W.$$

We then conclude by virtue of the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ and the continuity of the operator $\mathscr{A}$ from W onto $H^1(\Omega)$. □

Owing to the Fredholm alternative, showing uniqueness of a solution to problem (16) gives its existence for any right-hand side F, and conversely.

3.3. Equivalence between the original and regularized problems

We end this study by proving that regularized variational problem (16) implies the original strong problem (3)–(4)–(11). Since it is quite obvious that any solution $\boldsymbol{u}$ to (16) is also a solution to: *find* $\boldsymbol{u}$ *such that*

$$\begin{cases}\mathrm{D}^2\boldsymbol{u} - \nabla(\operatorname{div}\boldsymbol{u}) + s\,\mathbf{curl}(\operatorname{curl}\boldsymbol{u} - \mathscr{A}\boldsymbol{u} - \psi_{\boldsymbol{f}}) = \boldsymbol{f} & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega,\\ \operatorname{curl}\boldsymbol{u} = \mathscr{A}\boldsymbol{u} + \psi_{\boldsymbol{f}} & \text{on } \partial\Omega,\end{cases} \tag{19}$$

we simply have to prove that $\operatorname{curl}\boldsymbol{u} = \mathscr{A}\boldsymbol{u} + \psi_{\boldsymbol{f}}$ in Ω. Following [2], we take test functions in the form $\boldsymbol{v} = \mathbf{curl}\,\varphi$, with φ a function of $H^2(\Omega)\cap H_0^1(\Omega)$, and obtain that $\operatorname{curl}\boldsymbol{u} - \mathscr{A}\boldsymbol{u} - \psi_{\boldsymbol{f}}$ is orthogonal to the range of the operator $H_{k,M,s}$, defined by

$$H_{k,M,s} = M^2\partial_{x_1}^2 - 2\mathrm{i}kM\partial_{x_1} - k^2 I - s\Delta,$$

and with domain $D(H_{k,M,s}) = H^2(\Omega) \cap H_0^1(\Omega)$. Since $H_{k,M,s}$ is a selfadjoint operator with domain dense in $L^2(\Omega)$, $\operatorname{curl} \boldsymbol{u} - \mathscr{A}\boldsymbol{u} - \psi_{\boldsymbol{f}}$ belongs to the kernel of this operator. Using compactness arguments, we then prove that, for fixed wave number k and profile M, there exists s^* such that for all $s > s^*$ the kernel of $H_{k,M,s}$ is $\{0\}$ (for a uniform flow, the value of s^* can be determined analytically [2].) Note that this result is valid only if Ω is convex or if $\partial\Omega$ is smooth. Indeed, due to singularities, the operator $H_{k,M,s}$ is not selfadjoint and its kernel is never reduced to $\{0\}$ (see [6] for instance.) In particular, our approach cannot be extended to the case in which a thin plate is placed in the flow.

4. Numerical applications

4.1. The technical difficulty

A direct consequence of Theorem 1 is that a Lagrange finite element approximation of problem (16) will converge. However, the main challenge lies in the implementation of the quantity $\psi = \mathscr{A}\boldsymbol{u} + \psi_{\boldsymbol{f}}$.

On the one hand, the field $\psi_{\boldsymbol{f}}$ is computed a priori and without difficulty from the data, using the explicit expression (A.3). On the other hand, the field $\mathscr{A}\boldsymbol{u}$ has to be split in two local and nonlocal contributions,

$$\mathscr{A}\boldsymbol{u} = \mathscr{A}_{\text{Loc}}\boldsymbol{u} + \mathscr{A}_{N\text{Loc}}\boldsymbol{u},$$

with $\mathscr{A}_{\text{Loc}}\boldsymbol{u}(x_1, x_2) = (2M'(x_2)/M(x_2))u_1(x_1, x_2)$, the difficulty being the computation of the nonlocal part $\mathscr{A}_{N\text{Loc}}\boldsymbol{u}$ with a finite element code. Indeed, we need to evaluate integrals over streamlines of the flow which are not necessarily lines of the finite element mesh. Even when working with a structured mesh, the implementation of this term remains difficult and costly, as each integral couples degrees of freedom which do not belong to the same (or even adjacent) finite element(s).

4.2. Implementation

Let $\mathscr{Q}_h$ be a quadrangulation of domain Ω such that $\overline{\Omega} = \bigcup_{Q\in\mathscr{Q}_h} Q$, h being the discretization step. We denote by $\mathscr{V}_h^p$ the finite-dimensional space of continuous functions which are polynomials of degree p over $\mathscr{Q}_h$, i.e.,

$$\mathscr{V}_h^p = \{v_h \in C^0(\overline{\Omega}) | v_{h|_Q} \in P_p,\ \forall Q \in \mathscr{Q}_h\},$$

its dimension being N_h^p. This space is obviously $H^1(\Omega)$ conforming, and we introduce $V_h^p = (\mathscr{V}_h^p)^2$. We denote the basis functions of $\mathscr{V}_h^p$ by $(l^j)_{j=1,\dots,N_h^p}$ and by $(\boldsymbol{w}_\alpha^j)_{\alpha=1,2}^{j=1,\dots,N_h^p}$ the ones of V_h^p, defined by $\boldsymbol{w}_\alpha^j = l^j \boldsymbol{e}_{\boldsymbol{\alpha}}$. An approximate solution $\boldsymbol{u}_h$ to problem (16) in V_h^p is then written as

$$\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{\alpha, j} u_{h\alpha}(\boldsymbol{x}^j)\boldsymbol{w}_\alpha^j(\boldsymbol{x}),$$

and the associated matricial problem is $\mathbb{A}\boldsymbol{U} = \mathbb{L}$, where $(\mathbb{A}_{\alpha,\gamma})^{i,j} = a_s(\boldsymbol{w}_\alpha^j, \boldsymbol{w}_\gamma^i)$, $(\mathbb{L}_\gamma)^i = l(\boldsymbol{w}_\gamma^i)$, a_s and l being defined in (17) and (18), respectively, and

$$\boldsymbol{U} = \begin{bmatrix} [(\boldsymbol{u}_{h1})^i]_{i\in I_h^p} \\ [(\boldsymbol{u}_{h2})^i]_{i\in I_h^p} \end{bmatrix} \quad \text{with } I_h^p = \{1, 2, \dots, N_h^p\}.$$

As previously stated, from a computational point of view, the difficulty lies in the numerical evaluation of coefficients

$$(\mathbb{C}_{\alpha,\gamma})^{i,j} = s \int_\Omega \mathscr{A}_{N\text{Loc}}\boldsymbol{w}_\alpha^j(\boldsymbol{x}) \operatorname{curl} \boldsymbol{w}_\gamma^i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

By interpolating $\mathscr{A}_{N\text{Loc}}\boldsymbol{w}_\alpha^j$, we have $(\mathbb{C}_{\alpha,\gamma})^{i,j} \simeq s\sum_m \mathscr{A}_{N\text{Loc}}\boldsymbol{w}_\alpha^j(\boldsymbol{x}^m) \int_\Omega l^m(\boldsymbol{x}) \operatorname{curl} \boldsymbol{w}_\gamma^i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$.

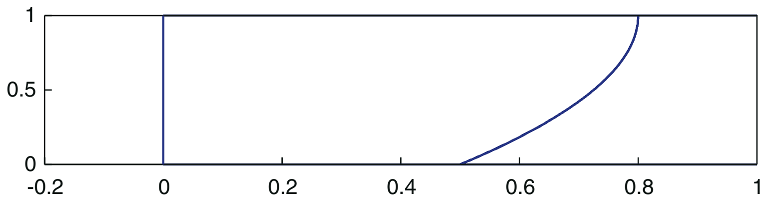

Fig. 1. Profile of the Mach number $M(x_2) = -0.3x_2^2 + 0.6x_2 + 0.5$, $x_2 \in [0, 1]$.

Moreover, setting

$$
\begin{aligned}
(\mathbb{I}_\alpha)^{m,j} &= \mathscr{A}_{N\mathrm{Loc}} \boldsymbol{w}_\alpha^j(\boldsymbol{x}^m),\\
&= \frac{2\mathrm{i}kM'(x_2^m)}{M^2(x_2^m)L}\left[(L - x_1^m)\int_0^{x_1^m} \delta_{1\alpha} l^j(z, x_2^m)\mathrm{e}^{(\mathrm{i}k(x_1^m - z)/M(x_2^m))}\,\mathrm{d}z,\right.\\
&\left.\quad - x_1^m \int_{x_1^m}^{L} \delta_{1\alpha} l^j(z, x_2^m)\mathrm{e}^{(\mathrm{i}k(x_1^m - z)/M(x_2^m))}\,\mathrm{d}z\right],
\end{aligned}
$$

we can compute exactly the matrix $\mathbb{I}$: it simply consists of evaluating integrals of the form $\int z^q \mathrm{e}^{-(\mathrm{i}kz/M(x_2))}\,\mathrm{d}z$, with $q = 0, \ldots, p$. This matrix is partially full since all degrees of freedom having the same x_2-coordinate are linked, and, as a consequence, the matrix $\mathbb{C} = \mathbb{P} \times \mathbb{I}$, with $(\mathbb{P}_\alpha)^{i,j} = \int_\Omega l^j(\boldsymbol{x})\mathrm{curl}\,\boldsymbol{w}_\alpha^i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, is also partially full.

4.3. Solution of the linear system

The matrix $\mathbb{C}$ being partially full, an iterative method is used for the solution of the linear system $\mathbb{A}\boldsymbol{U} = \mathbb{L}$ in order to avoid the inversion of the matrix $\mathbb{A} = \mathbb{B} + \mathbb{C}$. We use the following iterative scheme

$$\mathbb{B}\boldsymbol{U}^{n+1} = \mathbb{C}\boldsymbol{U}^n + \mathbb{L}.$$

For a uniform flow (i.e., when $M' \equiv 0$), the nonlocal term vanishes and the solution is obtained after a single iteration. In the shear flow case, we conjecture that the scheme converges when $\max_{x_2\in[0,\ell]}(|M'|/M^2)$ is small, if k does not belong to the set of the frequencies where $\mathbb{B}$ cannot be inverted. Indeed, we have $\|\mathbb{B}^{-1}\| \leqslant C_{k,M,s} < +\infty$, with $C_{k,M,s}$ a constant, and we can prove that $\|\mathbb{C}\| \leqslant 4ksL\max_{x_2\in[0,\ell]}(|M'|/M^2)$, hence $\|\mathbb{B}^{-1}\mathbb{C}\| \leqslant \|\mathbb{B}^{-1}\|\|\mathbb{C}\| \leqslant 4ksLC_{k,M,s}\max_{x_2\in[0,\ell]}$ $(|M'|/M^2)$. This allows to understand why, in practice, when M varies slowly enough, $\|\mathbb{B}^{-1}\mathbb{C}\| \leqslant 1$ and the iterative method works.

4.4. Numerical results

We validate the method with simulations of the propagation of guided modes, which are solutions of the form $\boldsymbol{u}(x_1, x_2) = \boldsymbol{w}(x_2)\mathrm{e}^{\mathrm{i}\beta x_1}$, β being a complex number, to the homogeneous version of Eq. (1). Values of the axial wave number β and of the vector function $\boldsymbol{w}$ are obtained semianalytically by computing solutions of the Pridmore–Brown equation by a Chebishev method, as done in [4], for the parabolic profile of a subsonic Mach number shown in Fig. 1, with $k = 6$ and $\ell = 1$.

The obtained values of β are plotted in Fig. 2. Axial wave numbers such that $\mathrm{Re}(\beta) \in [k/M_{\max}, k/M_{\min}] = [7.5, 12]$ and $\mathrm{Im}(\beta) = 0$ are associated with the so-called *hydrodynamic* modes. The remaining values are associated with the *acoustic* modes. Among these modes, we can again distinguish the *propagative* ones, which have a strictly real axial wave number, and the *evanescent* ones.

In the simulations, we consider two different combinations of modes in a two unit long piece of duct. The first combination, labelled A, combines two upstream modes (I_U and II_U in Fig. 2), while the second, labelled B, combines two downstream modes (0_D and II_D in Fig. 2). These combinations are imposed via a non-homogeneous boundary condition on the vertical boundaries for the normal displacement $\boldsymbol{u} \cdot \boldsymbol{n}$ and $\mathrm{curl}\,\boldsymbol{u}$. The iterative method is initialized with a null displacement field. All the simulations were done with the finite element library MÉLINA [9].

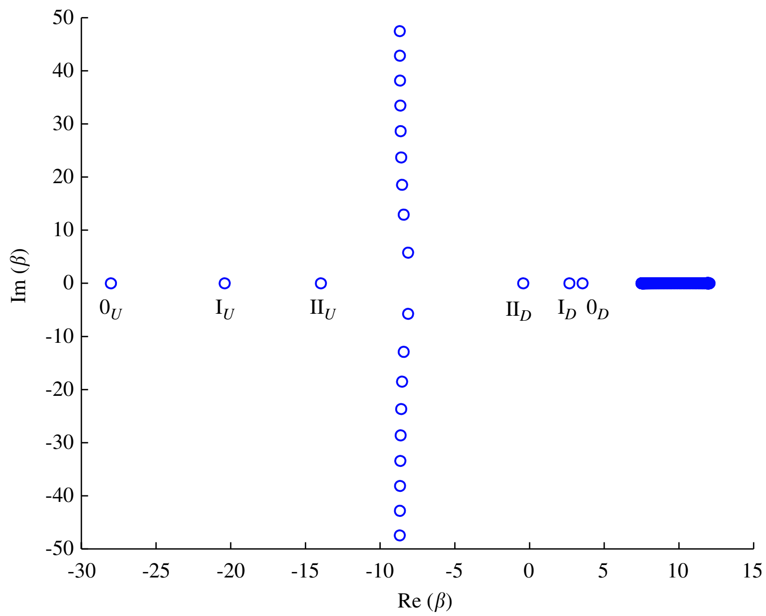

Fig. 2. Values of β in complex plane.

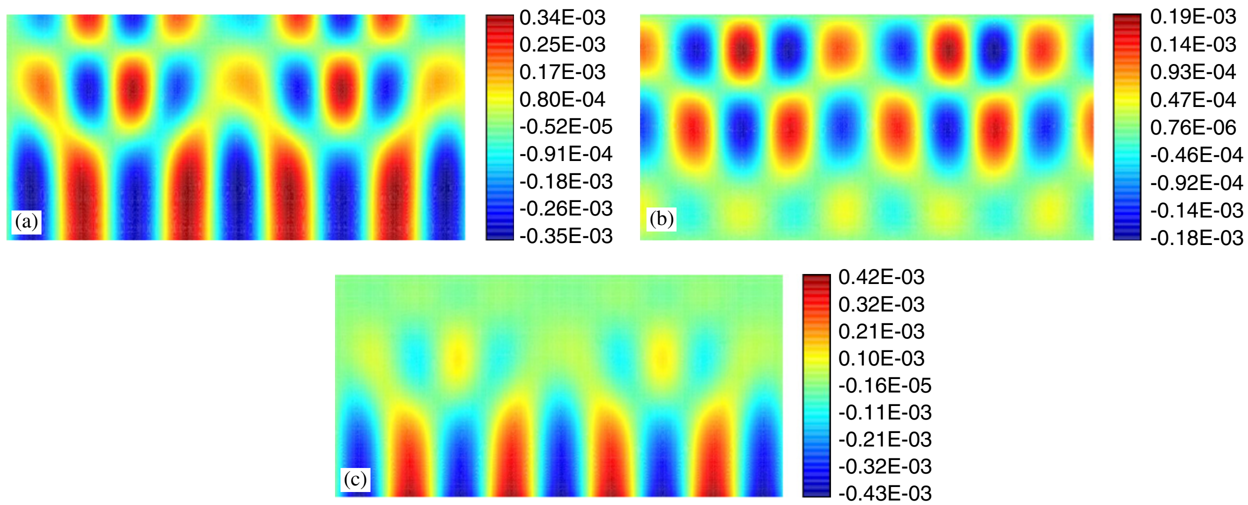

Fig. 3. Isovalues of the real part of the components of the computed displacement field and vorticity (combination A). (a) u_1, (b) u_2, (c) ψ.

The computed displacement field and associated curl field are shown in Figs. 3 and 4. The convective effect of the flow is clearly seen: the wavelengths of the upstream waves are shorter than those of the downstream ones. We also observe that the curl field ψ is localized where the shear of the flow is important (the function $|M'(x_2)|$ being maximum in $x_2 = 0$.) This is in accordance with expression (15).

Fig. 5 plots the relative error in $L^2(\Omega)^2$ (resp. $L^2(\Omega)$) norm between the computed and reference solutions (the reference solution being obtained by the solution of Pridmore–Brown's equation) for the displacement field (resp. ψ) versus the number of iterations. We note that seven iterations have been necessary to reach the stop condition, that the final error on the displacement is under one percent and that the error on $\boldsymbol{u}$ is decreasing faster than the one on ψ.

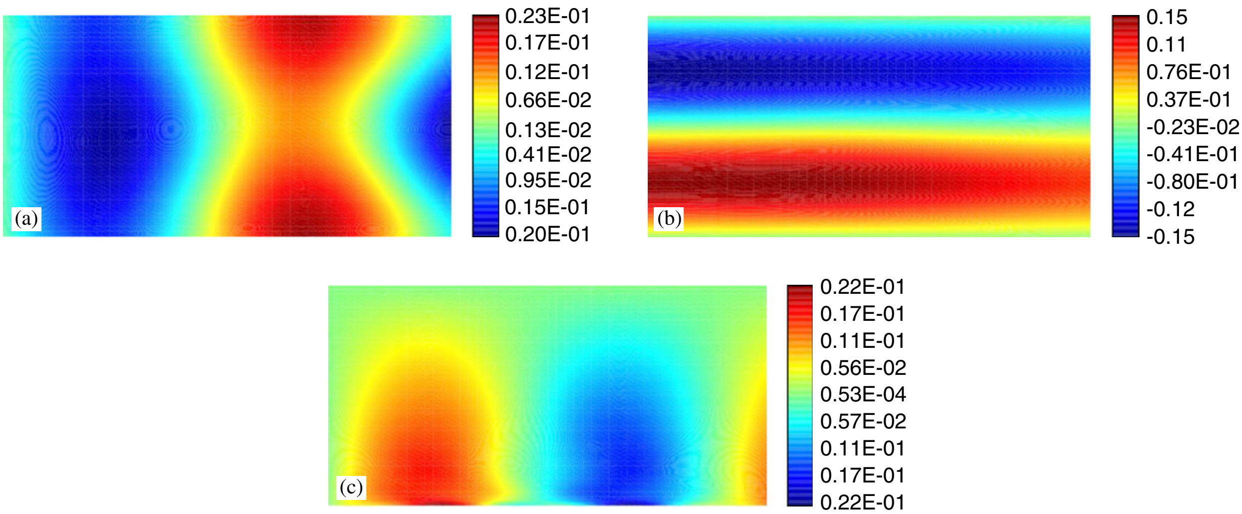

Fig. 4. Isovalues of the real part of the components of the computed displacement field and vorticity (combination B). (a) u_1, (b) u_2, (c) ψ.

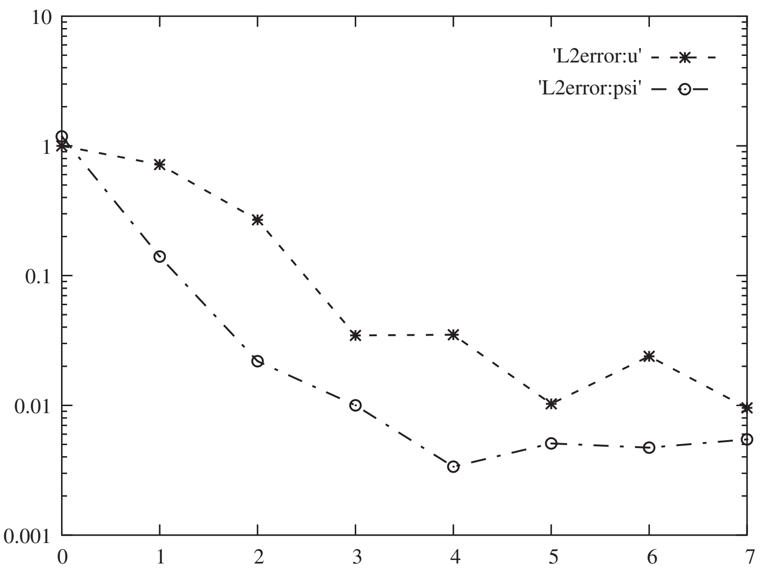

Fig. 5. Behavior of the logarithm of the relative L^2 error in $\boldsymbol{u}$ and ψ.

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Appendix A. Solution of differential equation (12)

A.1. The Green function

In order to solve problem (13), we introduce its associated Green function G_{x_2} ($x_2 \in [0, \ell]$ being here a fixed parameter), which satisfies, for all z in $[0, L]$,

$$(M(x_2)^2 \partial_{x_1}^2 - 2\mathrm{i}kM(x_2)\partial_{x_1} - k^2)G_{x_2}(x_1, z) = \delta(x_1 - z), \quad \forall x_1 \in [0, L], \tag{A.1}$$

where $\delta(x_1 - z)$ is the Dirac delta function at point z, along with the homogeneous Dirichlet boundary conditions

$$G_{x_2}(0, z) = G_{x_2}(L, z) = 0, \quad \forall z \in [0, L]. \tag{A.2}$$

The solution to problem (A.1)–(A.2) is the following:

$$G_{x_2}(x_1, z) = \begin{cases} -\dfrac{x_1(L-z)}{M^2(x_2)L} \mathrm{e}^{\mathrm{i}(k(x_1-z)/M(x_2))} & \text{if } x_1 \leqslant z, \\ -\dfrac{z(L-x_1)}{M^2(x_2)L} \mathrm{e}^{\mathrm{i}(k(x_1-z)/M(x_2))} & \text{if } x_1 > z. \end{cases}$$

A.2. The field $\psi_{\boldsymbol{f}}$

Theorem 2. *The solution to problem* (13) *with $g = \mathrm{curl}\boldsymbol{f}$ and $\phi_0 = \psi_\pm$ on $\Sigma_\pm$, denoted $\psi_{\boldsymbol{f}}$, is given by*

$$\psi_{\boldsymbol{f}}(x_1, x_2) = \int_0^L G_{x_2}(x_1, z)\mathrm{curl}\boldsymbol{f}(z, x_2)\,\mathrm{d}z + (a(x_2) + b(x_2)x_1)\mathrm{e}^{\mathrm{i}(kx_1/M(x_2))}, \tag{A.3}$$

where $a(x_2) = \psi_-(x_2)$ and $b(x_2) = \psi_+(x_2)\mathrm{e}^{-\mathrm{i}(kL/M(x_2))} - \psi_-(x_2)/L$.
Moreover, it belongs to $L^2(\Omega)$ and we have

$$\|\psi_{\boldsymbol{f}}\|^2_{L^2(\Omega)} \leqslant C(\|\mathrm{curl}\boldsymbol{f}\|^2_{L^2(\Omega)} + \|\psi_-\|^2_{L^2([0,\ell])} + \|\psi_+\|^2_{L^2([0,\ell])}),$$

where C is a strictly positive constant.

Proof. Obtaining expression (A.3) is straightforward. Concerning the inequality, we use the upper bound L for the quantities $|x_1|, |z|, |L-x_1|$ and $|L-z|$ and deduce easily that $|G_{x_2}(x_1, z)| \leqslant L/M^2(x_2), \forall (x_1, x_2) \in \Omega$ and $\forall z \in [0, L]$. Therefore, for all (x_1, x_2) in Ω, we have

$$|\psi_{\boldsymbol{f}}(x_1, x_2)| \leqslant \frac{L}{M^2(x_2)} \int_0^L |\mathrm{curl}\boldsymbol{f}(z, x_2)|\mathrm{d}z + |\psi_-(x_2)| + |\psi_+(x_2)|.$$

The estimation is then easily deduced from a Cauchy–Schwarz inequality. □

A.3. The field $\mathscr{A}\boldsymbol{u}$

Recall that the field $\mathscr{A}\boldsymbol{u}$ is the solution to problem (13) with $g = 2M'\mathrm{D}(\partial_{x_1} u_1)$ and vanishing on $\Sigma_\pm$. Using the Green function, we get

$$\mathscr{A}\boldsymbol{u}(x_1, x_2) = 2M'(x_2) \int_0^L G_{x_2}(x_1, z)\mathrm{D}(\partial_z u_1)(z, x_2)\,\mathrm{d}z, \quad \forall (x_1, x_2) \in \Omega. \tag{A.4}$$

Still, a more useful expression is given in Lemma 1, which is proved below.

Proof of Lemma 1. Integrating by parts in identity (A.4), we get

$$\mathscr{A}\boldsymbol{u}(x_1, x_2) = -2M'(x_2) \int_0^L \overline{\mathrm{D}}G_{x_2}(x_1, z)\partial_z u_1(z, x_2)\,\mathrm{d}z, \quad \forall (x_1, x_2) \in \Omega,$$

where $\overline{\mathrm{D}} = [\mathrm{i}k + M(x_2)\partial_z]$, the boundary terms vanish due to (A.2). Since we have

$$\overline{\mathrm{D}}G_{x_2}(x_1, z) = \begin{cases} \dfrac{x_1}{M(x_2)L} \mathrm{e}^{\mathrm{i}(k(x_1-z)/M(x_2))} & \text{if } x_1 \leqslant z, \\ -\dfrac{(L-x_1)}{M(x_2)L} \mathrm{e}^{\mathrm{i}(k(x_1-z)/M(x_2))} & \text{if } x_1 > z, \end{cases}$$

we finally obtain

$$\mathscr{A}\boldsymbol{u}(x_1, x_2) = \frac{2M'(x_2)}{M(x_2)L}\left[(L - x_1)\int_0^{x_1} \partial_z u_1(z, x_2)\mathrm{e}^{\mathrm{i}(k(x_1-z)/M(x_2))}\,\mathrm{d}z - x_1\int_{x_1}^{L} \partial_z u_1(z, x_2)\mathrm{e}^{\mathrm{i}(k(x_1-z)/M(x_2))}\,\mathrm{d}z\right].$$

Expression (15) then stems from a last integration by parts and the fact that $u_1(z, x_2) = 0$ for $z = 0$ and $z = L$, $\forall x_2 \in [0, \ell]$. $\square$

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