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A Chebyshev spectral collocation method for solving Burgers' -type equations

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Abstract

In this paper, we elaborated a spectral collocation method based on differentiated Chebyshev polynomials to obtain numerical solutions for some different kinds of nonlinear partial differential equations. The problem is reduced to a system of ordinary differential equations that are solved by Runge–Kutta method of order four. Numerical results for the nonlinear evolution equations such as 1D Burgers', KdV–Burgers', coupled Burgers', 2D Burgers' and system of 2D Burgers' equations are obtained. The numerical results are found to be in good agreement with the exact solutions. Numerical computations for a wide range of values of Reynolds' number, show that the present method offers better accuracy in comparison with other previous methods. Moreover the method can be applied to a wide class of nonlinear partial differential equations.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) arise in many fields of science, particularly in physics, engineering, chemistry and finance, and are fundamental for the mathematical formulation of continuum models. Systems of NLPDEs have attracted much attention in studying evolution equations describing wave propagation, in investigating the shallow water waves [7,37], and in examining the chemical reaction-diffusion model of Brusselator [2]. While the Burgers' equation has been found to describe various kinds of phenomena such as a mathematical model of turbulence [4] and the approximate theory of flow through a shock wave traveling in a viscous fluid [6]. Fletcher using the Hopf–Cole transformation [10], gave an analytical solution for the system of 2D Burgers' equations. Several numerical methods to solve this system have been given such as algorithms based on cubic spline function technique [14], the explicit–implicit method [49], Adomian's decomposition method [10].

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High-order accurate schemes for solving the 2D Burgers' equations have been used [39], several multilevel schemes with alternating direction implicit (ADI) method [11] and implicit finite-difference scheme [3].

Many researchers have used various methods to seek exact solutions to NLPDEs [17–35,41–43]. The variational iteration method was used to solve the 1D Burgers' and coupled Burgers' equations [1]; the solution was obtained under a series of initial conditions and was transformed into a closed form one.

The KdV–Burgers' equation is a 1D generalization of the model description of the density and velocity fields that takes into account pressure forces as well as the viscosity and the dispersion. It may be a more flexible tool for physicists than the Burgers' equation. Several numerical methods to solve this equation and NLPDEs have been given such as algorithms based on Adomian's decomposition method [12,15,16,44–48], finite-difference method [12], Galerkin quadratic B-spline finite element method [38] and spectral collocation method have been used to obtain numerical solutions of some nonlinear evolution equations [36].

The purpose of this paper is to present a numerical method for the solution of the following NLPDEs defined on a bounded domain as:

(P1) 1D Burgers' equation [1,41]

$$u_t + \alpha uu_x - \nu u_{xx} = 0, \quad (1)$$

where α and ν are arbitrary constants.

(P2) KdV–Burgers' equation [1,16,41]

$$u_t + \alpha uu_x - \nu u_{xx} + \mu u_{xxx} = 0, \quad (2)$$

where α , ν and μ are arbitrary constants.

(P3) Coupled Burgers' equations [16,41]

$$\begin{aligned} u_t - u_{xx} + 2uu_x + \alpha(uv)_x &= 0, \\ v_t - v_{xx} + 2vv_x + \beta(uv)_x &= 0, \end{aligned} \quad (3)$$

where α and β are arbitrary constants.

(P4) The 2D Burgers' equation [39,41]

$$u_t + uu_x + uu_y = \nu(u_{xx} + u_{yy}), \quad (4)$$

where ν is an arbitrary constant.

(P5) The system of 2D Burgers' equations [3,9,41]

$$\begin{aligned} u_t + uu_x + vv_y &= \nu(u_{xx} + u_{yy}), \\ v_t + uv_x + vv_y &= \nu(v_{xx} + v_{yy}), \end{aligned} \quad (5)$$

where ν is an arbitrary constant.

This method will be referred to as the Chebyshev spectral collocation (ChSC) method. The ChSC method is accomplished through, starting with Chebyshev approximation for the approximate solution and generating approximations for the higher-order derivatives through successive differentiation of the approximate solution.

Chebyshev polynomials [40] are well-known family of orthogonal polynomials on the interval $[-1, 1]$ of the real line. These polynomials present, among others, very good properties in the approximation of functions. Therefore, Chebyshev polynomials appear frequently in several fields of Mathematics, Physics and Engineering. Spectral methods based on Chebyshev polynomials as basis functions for solving numerically differential equations have been used by many authors, (see for example [8,13,36]).

The paper is organized as follows: In Section 2, the Chebyshev spectral collocation method is used to obtain the numerical solutions of the problems (P1)–(P5), the fourth-order Runge–Kutta method is used for solving the obtained system of ordinary differential equations. In Section 3, numerical experiments are performed to test the accuracy and efficiency of the ChSC method and compared with the finite-difference method and Galerkin quadratic B-spline finite element method, the numerical results show that the ChSC method is more accurate in comparison with [3,12,38]. Conclusions are given in Section 4.

2. Numerical solutions by the Chebyshev spectral collocation method

To illustrate the procedure, five examples related to the 1D Burgers', KdV–Burgers', coupled Burgers', 2D Burgers and system of 2D Burgers' equations are given in the following.

2.1. 1D Burgers' equation

Let us first consider the 1D Burgers' equation which has the form [1,41]

$$u_t + \alpha u u_x - \nu u_{xx} = 0, \quad (x, t) \in D \times [0, T], \quad (6)$$

with the initial condition

$$u(x, 0) = f(x), \quad x \in D \quad (7)$$

and the boundary conditions

$$u(x, t) = g(t), \quad (x, t) \in \delta D \times [0, T] \quad (8)$$

where $D = \{x : a < x < b\}$ and δD is its boundary; α and ν are arbitrary constants.

The solution u is approximated as [5,8]

$$u(x, t) = \sum_{j=0}^N {}'' a_j T_j^*(x) \quad (9)$$

and the collocation points are given by

$$x_n = \frac{1}{2} \left((a+b) - (b-a) \cos \left(\frac{\pi n}{N} \right) \right), \quad n = 0, 1, \dots, N, \quad (10)$$

where $T_j^*(x_n) = T_j((2x_n - (b+a))/(b-a))$ is the j th Chebyshev polynomial of the first kind. A summation symbol with double primes denotes a sum with the first and last term halved.

Using the discrete orthogonality relation

$$\sum_{n=0}^N {}'' T_i^*(x_n) T_j^*(x_n) = \alpha_i \delta_{ij}$$

with

$$\alpha_i = \begin{cases} N/2, & i \neq 0, N; \\ N, & i = 0, N. \end{cases}$$

We can invert (9) and find

$$a_j = \frac{2}{N} \sum_{n=0}^N {}'' T_j^*(x_n) u(x_n, t). \quad (11)$$

Derivatives $u_x(x, t)$ and $u_{xx}(x, t)$ can be computed at the collocation points (10) by using the expansion (9) and the Chebyshev coefficients (11). The derivative $u_x(x, t)$ is approximated as

$$\begin{aligned} u_x(x_i, t) &= \sum_{j=0}^N {}'' a_j T_j'^*(x_i) \\ &= \sum_{n=0}^N {}'' \left(\frac{2}{N} \sum_{j=0}^N {}'' T_j'^*(x_i) T_j^*(x_n) \right) u(x_n, t) \\ &= \sum_{n=0}^N [A_x]_{in} u(x_n, t), \end{aligned} \quad (12)$$

where

$$[A_x]_{in} = \frac{2c_n}{N} \sum_{j=0}^N {}''T_j^*(x_i) T_j^*(x_n), \quad i, n = 0, 1, \dots, N,$$

$$c_0 = c_N = 1/2 \quad \text{and} \quad c_n = 1 \quad \text{for } n = 1, 2, \dots, N-1.$$

The first derivative of the Chebyshev functions is formed as follows [8]:

$$T_j^*(x_i) = 2j\lambda \sum_{\substack{n=0, n+j \\ \text{odd}}}^{j-1} c_n T_n^*(x_i), \quad (13)$$

where $\lambda = \frac{2}{b-a}$, $c_0 = c_N = 1/2$ and $c_n = 1$ for $n = 1, 2, \dots, N-1$.

Similarly, the derivative $u_{xx}(x, t)$ is approximated as

$$\begin{aligned} u_{xx}(x_i, t) &= \sum_{n=0}^N [A_x]_{in} u_{xx}(x_n, t) \\ &= \sum_{j=0}^N \left(\sum_{n=0}^N [A_x]_{in} [A_x]_{nj} \right) u(x_j, t) \\ &= \sum_{j=0}^N [B_x]_{ij} u(x_j, t), \end{aligned} \quad (14)$$

where $B_x = A_x^2$ and the elements of the matrix B_x are given by

$$[B_x]_{ij} = \sum_{n=0}^N [A_x]_{in} [A_x]_{nj}, \quad i, j = 0, 1, \dots, N.$$

If we denote $u(x, t)$ and $u_t(x, t)$ at the point x_n by $u_n(t)$ and $\dot{u}_n(t)$ respectively, and using the boundary conditions (8), then it is not difficult to show that

$$u_x(x_i, t) = d_i(t) + \sum_{n=1}^{N-1} [A_x]_{in} u_n(t), \quad (15)$$

$$u_{xx}(x_i, t) = \bar{d}_i(t) + \sum_{n=1}^{N-1} [B_x]_{in} u_n(t), \quad (16)$$

where $d_i(t) = [A_x]_{i0} u_0(t) + [A_x]_{iN} u_N(t)$ and $\bar{d}_i(t) = [B_x]_{i0} u_0(t) + [B_x]_{iN} u_N(t)$.

Substituting (15) and (16) into (6), we obtain

$$\begin{aligned} \dot{u}_i(t) + \alpha u_i(t) \sum_{n=1}^{N-1} [A_x]_{in} u_n(t) - \nu \sum_{n=1}^{N-1} [B_x]_{in} u_n(t) + \alpha u_i(t) d_i(t) - \nu \bar{d}_i(t) &= 0, \\ u_i(0) &= f(x_i). \end{aligned} \quad (17)$$

Then system (17) can be written in the following form

$$\begin{aligned} \dot{u}(t) &= F(t, u(t)), \\ u(0) &= u_0, \end{aligned} \quad (18)$$

where

$$\begin{aligned} u(t) &= [u_1(t), u_2(t), \dots, u_{N-1}(t)]^T, \quad \dot{u}(t) = [\dot{u}_1(t), \dot{u}_2(t), \dots, \dot{u}_{N-1}(t)]^T, \\ u_0 &= [u_1(0), u_2(0), \dots, u_{N-1}(0)]^T, \\ F(t, u(t)) &= [F_1(t, u(t)), F_2(t, u(t)), \dots, F_{N-1}(t, u(t))]^T \end{aligned}$$

and

$$F_i(t, u(t)) = -\alpha u_i(t) \sum_{n=1}^{N-1} [A_x]_{in} u_n(t) + \nu \sum_{n=1}^{N-1} [B_x]_{in} u_n(t) - \alpha u_i(t) d_i(t) + \nu \bar{d}_i(t).$$

Eq. (18) forms a system of ordinary differential equations (ODEs) in time. Therefore to advance the solution in time, we use ODE solver such as the Runge–Kutta method of order four because it is an explicit method which gives a good accuracy and extends trivially to nonlinear. The Runge–Kutta method of order four is given by

$$\begin{aligned} u^{(1)} &= u(t_n) + \frac{1}{2} F(t_n, u(t_n)) \\ u^{(2)} &= u(t_n) + \frac{1}{2} F(t_{n+\Delta t/2}, u^{(1)}) \\ u^{(3)} &= u(t_n) + F(t_{n+\Delta t/2}, u^{(2)}) \\ u(t_{n+1}) &= u(t_n) + \frac{\Delta t}{6} [F(t_n, u(t_n)) + 2F(t_{n+\Delta t/2}, u^{(1)}) + 2F(t_{n+\Delta t/2}, u^{(2)}) + F(t_{n+\Delta t}, u^{(3)})]. \end{aligned}$$

2.2. KdV–Burgers' equation

A second important example is the KdV–Burgers' equation [1,15,41]:

$$u_t + \alpha u u_x - \nu u_{xx} + \mu u_{xxx} = 0, \quad (x, t) \in D \times [0, T], \quad (19)$$

with the initial condition

$$u(x, 0) = \bar{f}(x), \quad x \in D \quad (20)$$

and the boundary conditions

$$\begin{aligned} u(x, t) &= \bar{g}(t), \\ u_x(x, t) &= \bar{h}(t), \quad (x, t) \in \delta D \times [0, T], \end{aligned} \quad (21)$$

where $D = \{x : a < x < b\}$ and δD is its boundary ; α, ν and μ are arbitrary constants.

Applying the Chebyshev spectral collocation method to the problem (19)–(21), we get

$$\begin{aligned} \dot{u}_i(t) + \alpha u_i(t) \sum_{n=1}^{N-1} [A_x]_{in} u_n(t) - \nu \sum_{n=1}^{N-1} [\bar{B}_x]_{in} u_n(t) + \mu \sum_{n=1}^{N-1} [C_x]_{in} u_n(t) \\ + \alpha u_i(t) d_i(t) - \nu \hat{d}_i(t) + \mu \bar{\bar{d}}_i(t) = 0, \\ u_i(0) = \bar{f}(x_i), \end{aligned} \quad (22)$$

where

$$\begin{aligned} [\bar{B}_x]_{ij} &= \sum_{n=1}^{N-1} [A_x]_{in} [A_x]_{nj}, \quad [C_x]_{ij} = \sum_{n=0}^N [A_x]_{in} [\bar{B}_x]_{nj}, \\ \hat{d}_i(t) &= [A_x]_{i0} u_0^*(t) + [A_x]_{iN} u_N^*(t) + [\bar{B}_x]_{i0} u_0(t) + [\bar{B}_x]_{iN} u_N(t), \\ \bar{\bar{d}}_i(t) &= [B_x]_{i0} u_0^*(t) + [B_x]_{iN} u_N^*(t) + [C_x]_{i0} u_0(t) + [C_x]_{iN} u_N(t), \end{aligned}$$

and $u_n^*(t)$ denotes $u_x(x, t)$ at the point x_n .

Then the system (22) can be written in the following form

$$\begin{aligned} \dot{u}(t) &= \bar{F}(t, u(t)), \\ u(0) &= u_0, \end{aligned} \quad (23)$$

where

$$\bar{F}(t, u(t)) = [\bar{F}_1(t, u(t)), \bar{F}_2(t, u(t)), \dots, \bar{F}_{N-1}(t, u(t))]^T$$

and

$$\begin{aligned} \bar{F}_i(t, u(t)) = & -\alpha u_i(t) \sum_{n=1}^{N-1} [A_x]_{in} u_n(t) + v \sum_{n=1}^{N-1} [\bar{B}_x]_{in} u_n(t) - \mu \sum_{n=1}^{N-1} [C_x]_{in} u_n(t) \\ & - \alpha u_i(t) d_i(t) + v \hat{d}_i(t) - \mu \bar{\bar{d}}_i(t), \quad i = 1, 2, \dots, N-1. \end{aligned}$$

Again the system (23) is solved by using the Runge–Kutta method of order four.

2.3. Coupled Burgers' equations

The third instructive example to illustrate the Chebyshev spectral collocation method is the homogeneous form of a coupled Burgers equation [16]. We will consider the following system of equations:

$$\begin{aligned} u_t - u_{xx} + 2uu_x + \alpha(uv)_x &= 0, \\ v_t - v_{xx} + 2vv_x + \beta(uv)_x &= 0, \end{aligned} \quad (24)$$

with the initial conditions

$$u(x, 0) = f_1(x), \quad v(x, 0) = f_2(x), \quad x \in D \quad (25)$$

and the boundary conditions

$$u(x, t) = g_1(t), \quad v(x, t) = g_2(t), \quad (x, t) \in \delta D \times [0, T], \quad (26)$$

where $D = \{x : a < x < b\}$ and δD is its boundary; α and β are arbitrary constants.

Applying the Chebyshev spectral collocation method to the problem (24)–(26), we get

$$\begin{aligned} \dot{u}_i(t) = & \sum_{n=1}^{N-1} [B_x]_{in} u_n(t) - 2u_i(t) \sum_{n=1}^{N-1} [A_x]_{in} u_n(t) - \alpha u_i(t) \sum_{n=1}^{N-1} [A_x]_{in} v_n(t) \\ & - \alpha v_i(t) \sum_{n=1}^{N-1} [A_x]_{in} u_n(t) + \bar{d}_i(t) - 2u_i(t) d_i(t) - \alpha u_i(t) r_i(t) - \alpha v_i(t) d_i(t), \end{aligned} \quad (27)$$

$$u_i(0) = f_1(x_i)$$

and

$$\begin{aligned} \dot{v}_i(t) = & \sum_{n=1}^{N-1} [B_x]_{in} v_n(t) - 2v_i(t) \sum_{n=1}^{N-1} [A_x]_{in} v_n(t) - \beta u_i(t) \sum_{n=1}^{N-1} [A_x]_{in} v_n(t) \\ & - \beta v_i(t) \sum_{n=1}^{N-1} [A_x]_{in} u_n(t) + \bar{r}_i(t) - 2v_i(t) r_i(t) - \beta u_i(t) r_i(t) - \beta v_i(t) d_i(t), \end{aligned} \quad (28)$$

$$v_i(0) = f_2(x_i),$$

where

$$\begin{aligned} r_i(t) &= [A_x]_{i0} v_0(t) + [A_x]_{iN} v_N(t), \quad \bar{r}_i(t) = [B_x]_{i0} v_0(t) + [B_x]_{iN} v_N(t), \\ v_i(t) &= v(x_i, t) \quad \text{and} \quad \dot{v}_i(t) = v_t(x_i, t). \end{aligned}$$

Put $w_i(t) = u_i(t)$, $\dot{w}_i(t) = \dot{u}_i(t)$, $w_{i+N+1}(t) = v_i(t)$, $i = 0(1)N$ and $\dot{w}_{i+N+1}(t) = \dot{v}_i(t)$, $i = 0(1)N$. Then the systems (27) and (28) can be written in the following form

$$\begin{aligned} \dot{w}(t) &= L(t, w(t)), \\ w(0) &= w_0, \end{aligned} \quad (29)$$

where

$$\begin{aligned} w(t) &= [w_1(t), \dots, w_{N-1}(t), w_{N+2}(t), \dots, w_{2N}(t)]^T, \\ \dot{w}(t) &= [\dot{w}_1(t), \dots, \dot{w}_{N-1}(t), \dot{w}_{N+2}(t), \dots, \dot{w}_{2N}(t)]^T, \\ w_0 &= [w_1(0), \dots, w_{N-1}(0), w_{N+2}(0), \dots, w_{2N}(0)]^T, \\ L(t, w(t)) &= [L_1(t, w(t)), \dots, L_{N-1}(t, w(t)), L_{N+2}(t, w(t)), \dots, L_{2N}(t, w(t))]^T, \end{aligned}$$

and

$$\begin{aligned} L_i(t, w(t)) &= \sum_{n=1}^{N-1} [B_x]_{in} w_n(t) - 2w_i(t) \sum_{n=1}^{N-1} [A_x]_{in} w_n(t) \\ &\quad - \alpha w_i(t) \sum_{n=1}^{N-1} [A_x]_{in} w_{n+N+1}(t) - \alpha w_{i+N+1}(t) \sum_{n=1}^{N-1} [A_x]_{in} w_n(t) \\ &\quad + \bar{d}_i(t) - 2w_i(t) d_i(t) - \alpha w_i(t) r_i(t) - \alpha w_{i+N+1}(t) d_i(t); \quad i = 1, 2, \dots, N-1, \\ L_{i+N+1}(t, w(t)) &= \sum_{n=1}^{N-1} [B_x]_{in} w_{n+N+1}(t) - 2w_{i+N+1}(t) \sum_{n=1}^{N-1} [A_x]_{in} w_{n+N+1}(t) \\ &\quad - \beta w_i(t) \sum_{n=1}^{N-1} [A_x]_{in} w_{n+N+1}(t) - \beta w_{i+N+1}(t) \sum_{n=1}^{N-1} [A_x]_{in} w_n(t) \\ &\quad + \bar{r}_i(t) - 2w_{i+N+1}(t) r_i(t) - \beta w_i(t) r_i(t) - \beta w_{i+N+1}(t) d_i(t); \quad i = 1, 2, \dots, N-1. \end{aligned}$$

Again the system (29) is solved by using the Runge–Kutta method of order four.

2.4. The 2D Burgers' equation

A fourth example is the 2D Burgers' equation [39,41]:

$$u_t + uu_x + uu_y = \nu(u_{xx} + u_{yy}), \quad (30)$$

where ν is an arbitrary constant.

With the initial condition

$$u(x, y, 0) = h(x, y), \quad (x, y) \in D, \quad (31)$$

and the boundary conditions

$$u(x, y, t) = p(x, y, t), \quad (x, y) \in \delta D, t \in [0, T], \quad (32)$$

where $D = \{(x, y) : a < x, y < b\}$ and δD is its boundary; $\nu = 1/R$ and R is the Reynolds' number.

The solution u is sought to be in the form

$$u(x, y, t) = \sum_{m=0}^M \sum_{n=0}^N T_n^*(x) T_m^*(y) u_{nm}(t), \quad (33)$$

where $u_{nm}(t)$ is the approximate solution at the point (x_n, y_m) and the collocation points (x_n, y_m) are given by (10) and

$$y_m = \frac{1}{2} \left((a+b) - (b-a) \cos\left(\frac{\pi m}{M}\right) \right), \quad m = 0, 1, \dots, M. \quad (34)$$

If we denote u_t at the point (x_n, y_m) by $\dot{u}_{nm}(t)$ and using boundary conditions (32), then it is not difficult to show that

$$u_x(x_i, y_j, t) = z_{ij}(t) + \sum_{n=1}^{N-1} [A_x]_{in} u_{nj}(t), \quad (35)$$

$$u_{xx}(x_i, y_j, t) = \bar{z}_{ij}(t) + \sum_{n=1}^{N-1} [B_x]_{in} u_{nj}(t), \quad (36)$$

$$u_y(x_i, y_j, t) = q_{ij}(t) + \sum_{m=1}^{M-1} [A_y]_{jm} u_{im}(t), \quad (37)$$

and

$$u_{yy}(x_i, y_j, t) = \bar{q}_{ij}(t) + \sum_{m=1}^{M-1} [B_y]_{jm} u_{im}(t), \quad (38)$$

where A_y and B_y are matrices obtained in the same way as A_x and B_x respectively but using the points y_m instead of the points x_n ,

$$\begin{aligned} z_{ij}(t) &= [A_x]_{i0} u_{0j}(t) + [A_x]_{iN} u_{Nj}(t), & \bar{z}_{ij}(t) &= [B_x]_{i0} u_{0j}(t) + [B_x]_{iN} u_{Nj}(t), \\ q_{ij}(t) &= [A_y]_{j0} u_{i0}(t) + [A_y]_{jM} u_{iM}(t) & \text{and} & \quad \bar{q}_{ij}(t) = [B_y]_{j0} u_{i0}(t) + [B_y]_{jM} u_{iM}(t). \end{aligned}$$

Substituting (35)–(38) into (30), we obtain

$$\begin{aligned} \dot{u}_{ij}(t) &= -u_{ij}(t) \left(z_{ij}(t) + \sum_{n=1}^{N-1} [A_x]_{in} u_{nj}(t) \right) - u_{ij}(t) \left(q_{ij}(t) + \sum_{m=1}^{M-1} [A_y]_{jm} u_{im}(t) \right) \\ &\quad + v \left(\bar{z}_{ij}(t) + \sum_{n=1}^{N-1} [B_x]_{in} u_{nj}(t) \right) + v \left(\bar{q}_{ij}(t) + \sum_{m=1}^{M-1} [B_y]_{jm} u_{im}(t) \right), \\ u_{ij}(0) &= h(x_i, y_j). \end{aligned} \quad (39)$$

Then system (39) can be written in the following form

$$\begin{aligned} \dot{\phi}(t) &= E(t, \phi(t)), \\ \phi(0) &= \phi_0, \end{aligned} \quad (40)$$

where

$$\begin{aligned} \phi(t) &= [a_1(t), \dots, a_{N-1}(t)]^T, & \dot{\phi}(t) &= [\dot{a}_1(t), \dots, \dot{a}_{N-1}(t)]^T, \\ \phi_0 &= [a_1(0), \dots, a_{N-1}(0)]^T, & a_i(t) &= [u_{i1}(t), \dots, u_{iM-1}(t)]^T, \\ \dot{a}_i(t) &= [\dot{u}_{i1}(t), \dots, \dot{u}_{iM-1}(t)]^T, & E(t, \phi(t)) &= [E_1(t, \phi(t)), \dots, E_{N-1}(t, \phi(t))]^T, \\ E_i(t, \phi(t)) &= [\eta_{i1}, \dots, \eta_{iM-1}]^T; & i &= 1, 2, \dots, N-1 \end{aligned}$$

and

$$\begin{aligned} \eta_{ij} &= -u_{ij}(t) \left(z_{ij}(t) + \sum_{n=1}^{N-1} [A_x]_{in} u_{nj}(t) \right) - u_{ij}(t) \left(q_{ij}(t) + \sum_{m=1}^{M-1} [A_y]_{jm} u_{im}(t) \right) \\ &\quad + v \left(\bar{z}_{ij}(t) + \sum_{n=1}^{N-1} [B_x]_{in} u_{nj}(t) \right) + v \left(\bar{q}_{ij}(t) + \sum_{m=1}^{M-1} [B_y]_{jm} u_{im}(t) \right); \\ i &= 1, 2, \dots, N-1, j = 1, 2, \dots, M-1. \end{aligned}$$

Again system (40) is solved by using the Runge–Kutta method of order four.

2.5. The system of 2D Burgers' equations

Our last example is a system of 2D Burgers' equations [3,9,41]:

$$\begin{aligned} u_t + uu_x + vu_y &= v(u_{xx} + u_{yy}), \\ v_t + uv_x + vv_y &= v(v_{xx} + v_{yy}), \end{aligned} \quad (41)$$

with the initial conditions

$$u(x, y, 0) = h_1(x, y), \quad v(x, y, 0) = h_2(x, y), \quad (x, y) \in D \quad (42)$$

and the boundary conditions

$$u(x, y, t) = p_1(x, y, t), \quad v(x, y, t) = p_2(x, y, t), \quad (x, y) \in \delta D, t \in [0, T], \quad (43)$$

where $D = \{(x, y) : a < x, y < b\}$ and δD is its boundary; $\nu = 1/R$ and R is the Reynolds' number.

Applying the Chebyshev spectral collocation method to problems (41)–(43), we get

$$\begin{aligned} \dot{u}_{ij}(t) = & -u_{ij}(t) \left(z_{ij}(t) + \sum_{n=1}^{N-1} [A_x]_{in} u_{nj}(t) \right) - v_{ij}(t) \left(q_{ij}(t) + \sum_{m=1}^{M-1} [A_y]_{jm} u_{im}(t) \right) \\ & + v \left(\bar{z}_{ij}(t) + \sum_{n=1}^{N-1} [B_x]_{in} u_{nj}(t) \right) + v \left(\bar{q}_{ij}(t) + \sum_{m=1}^{M-1} [B_y]_{jm} u_{im}(t) \right), \end{aligned} \quad (44)$$

$$u_{ij}(0) = h_1(x_i, y_j)$$

and

$$\begin{aligned} \dot{v}_{ij}(t) = & -u_{ij}(t) \left(w_{ij}(t) + \sum_{n=1}^{N-1} [A_x]_{in} v_{nj}(t) \right) - v_{ij}(t) \left(k_{ij}(t) + \sum_{m=1}^{M-1} [A_y]_{jm} v_{im}(t) \right) \\ & + v \left(\bar{w}_{ij}(t) + \sum_{n=1}^{N-1} [B_x]_{in} v_{nj}(t) \right) + v \left(\bar{k}_{ij}(t) + \sum_{m=1}^{M-1} [B_y]_{jm} v_{im}(t) \right), \end{aligned} \quad (45)$$

$$v_{ij}(0) = h_2(x_i, y_j),$$

where

$$\begin{aligned} w_{ij}(t) &= [A_x]_{i0} v_{0j}(t) + [A_x]_{iN} v_{Nj}(t), & \bar{w}_{ij}(t) &= [B_x]_{i0} v_{0j}(t) + [B_x]_{iN} v_{Nj}(t), \\ k_{ij}(t) &= [A_y]_{j0} v_{i0}(t) + [A_y]_{jM} v_{iM}(t) & \text{and} & \quad \bar{k}_{ij}(t) = [B_y]_{j0} v_{i0}(t) + [B_y]_{jM} v_{iM}(t). \end{aligned}$$

Put

$$\begin{aligned} a_i(t) &= [u_{i1}(t), \dots, u_{iM-1}(t)]^T, & b_i(t) &= [v_{i1}(t), \dots, v_{iM-1}(t)]^T, \\ \dot{a}_i(t) &= [\dot{u}_{i1}(t), \dots, \dot{u}_{iM-1}(t)]^T, & \dot{b}_i(t) &= [\dot{v}_{i1}(t), \dots, \dot{v}_{iM-1}(t)]^T, \\ \psi_i(t) &= a_i(t), & \dot{\psi}_i(t) &= \dot{a}_i(t), & \psi_{i+N+1}(t) &= b_i(t), \quad i = 1(1)N-1 \end{aligned}$$

and $\dot{\psi}_{i+N+1}(t) = \dot{b}_i(t), i = 1(1)N-1$.

Then systems (44) and (45) can be written in the following form

$$\begin{aligned} \dot{\psi}(t) &= K(t, \psi(t)), \\ \psi(0) &= \psi_0, \end{aligned} \quad (46)$$

where

$$\begin{aligned} \psi(t) &= [\psi_1(t), \dots, \psi_{N-1}(t), \psi_{N+2}(t), \dots, \psi_{2N}(t)]^T, \\ \dot{\psi}(t) &= [\dot{\psi}_1(t), \dots, \dot{\psi}_{N-1}(t), \dot{\psi}_{N+2}(t), \dots, \dot{\psi}_{2N}(t)]^T, \\ \Psi_0 &= [\psi_1(0), \dots, \psi_{N-1}(0), \psi_{N+2}(0), \dots, \psi_{2N}(0)]^T, \\ K(t, \Psi(t)) &= [\psi_1(t, \Psi(t)), \dots, \psi_{N-1}(t, \Psi(t)), \psi_{N+2}(t, \Psi(t)), \dots, \psi_{2N}(t, \Psi(t))]^T, \\ \psi_i(t, \psi(t)) &= [\eta_{i1}, \dots, \eta_{iM-1}]^T, \quad i = 1(1)N-1, \\ \psi_{i+N+1}(t, \psi(t)) &= [\xi_{i1}, \dots, \xi_{iM-1}]^T, \quad i = 1(1)N-1, \end{aligned}$$

$$\begin{aligned} \eta_{ij} = & -u_{ij}(t) \left(z_{ij}(t) + \sum_{n=1}^{N-1} [A_x]_{in} u_{nj}(t) \right) \\ & - v_{ij}(t) \left(q_{ij}(t) + \sum_{m=1}^{M-1} [A_y]_{jm} u_{im}(t) \right) + v \left(\bar{z}_{ij}(t) + \sum_{n=1}^{N-1} [B_x]_{in} u_{nj}(t) \right) \\ & + v \left(\bar{q}_{ij}(t) + \sum_{m=1}^{M-1} [B_y]_{jm} u_{im}(t) \right); \quad i = 1, 2, \dots, N-1, j = 1, 2, \dots, M-1, \end{aligned}$$

and

$$\begin{aligned} \xi_{ij} = & -u_{ij}(t) \left(w_{ij}(t) + \sum_{n=1}^{N-1} [A_x]_{in} v_{nj}(t) \right) - v_{ij}(t) \left(k_{ij}(t) + \sum_{m=1}^{M-1} [A_y]_{jm} v_{im}(t) \right) \\ & + v \left(\bar{w}_{ij}(t) + \sum_{n=1}^{N-1} [B_x]_{in} v_{nj}(t) \right) \\ & + v \left(\bar{k}_{ij}(t) + \sum_{m=1}^{M-1} [B_y]_{jm} v_{im}(t) \right); \quad i = 1, 2, \dots, N-1, j = 1, 2, \dots, M-1. \end{aligned}$$

Again system (46) is solved by using the Runge–Kutta method of order four.

3. Numerical results

In this section we apply the Chebyshev spectral collocation (ChSC) method to different examples. To show the efficiency of the present method for our problems in comparison with the exact solution, we report norm infinity and the norm of relative errors of the solution which are defined by

$$\|E\|_{\infty} = \max_{1 \leq i \leq n_{ip}} |u_i - \bar{u}_i| \quad (47)$$

and

$$\|E\| = \sqrt{\frac{\sum_{i=1}^{n_{ip}} (u_i - \bar{u}_i)^2}{\sum_{i=1}^{n_{ip}} (\bar{u}_i)^2}}, \quad (48)$$

where n_{ip} is the number of interior points, \bar{u}_i and u_i are the exact and computed values of the solution u at point i . In all the examples the initial and boundary conditions are taken from the exact solutions.

Example 1. Consider a 1D Burgers' equation:

$$u_t + \alpha u u_x - \nu u_{xx} = 0, \quad x \in D \times [0, T], \quad (49)$$

with the solitary wave solutions [41],

$$u(x, t) = c/\alpha + (2\nu/\alpha) \tanh(x - ct), \quad (50)$$

in the region $D = \{x : a < x < b\}$; α and ν are arbitrary constants.

Table 1, shows norm infinity and norm relative of errors for various values of α and ν with $N = 10$, $c = 0.1$, $\Delta t = 0.01$, $a = 0$ and $b = 1$.

Example 2. Consider a 1D Burgers' equation [38]:

$$u_t + u u_x - \nu u_{xx} = 0, \quad 0 \leq x \leq 1, t > 0, \quad (51)$$

Table 1
Norm infinity and norm relative of errors for various values of α and ν

α	ν	$\ E\ _\infty$			$\ E\ $		
		$t = 0.1$	$t = 0.2$	$t = 0.25$	$t = 0.1$	$t = 0.2$	$t = 0.25$
1	0.01	3.06, −5	6.11, −5	7.62, −5	2.08, −4	4.07, −4	5.04, −4
	0.001	3.06, −7	6.18, −7	7.82, −7	2.37, −6	4.69, −6	5.85, −6
	0.0001	2.24, −8	5.22, −8	8.94, −8	1.31, −7	3.60, −7	4.31, −7
0.1	0.01	3.06, −4	6.11, −4	7.62, −4	2.08, −4	4.06, −4	5.04, −4
	0.001	3.10, −6	6.32, −6	7.99, −6	2.32, −6	4.70, −6	5.88, −6
	0.0001	7.15, −7	1.31, −6	1.67, −6	3.68, −7	6.99, −7	8.88, −7

Table 2
Comparison of numerical solutions of Example 2 with results from [38]

x	$\nu = 1.0$				$\nu = 0.1$			
	t	[38]	ChSC	Exact	t	[38]	ChSC	Exact
0.25	0.40	0.31760	0.31752	0.31752	0.10	0.26149	0.26148	0.26148
	0.60	0.24618	0.24614	0.24614	0.15	0.16149	0.16148	0.16148
	0.80	0.19959	0.19956	0.19956	0.20	0.09948	0.09947	0.09947
	1.00	0.16562	0.16560	0.16560	0.25	0.06109	0.06108	0.06108
	3.00	0.02776	0.02776	0.02775				
0.75	0.40	0.64558	0.64561	0.64562	0.10	0.28159	0.28157	0.28157
	0.60	0.50269	0.50268	0.50268	0.15	0.16976	0.16974	0.16974
	0.80	0.38536	0.38534	0.38534	0.20	0.10267	0.10266	0.10266
	1.00	0.29589	0.29586	0.29586	0.25	0.06230	0.06229	0.06229
	3.00	0.03044	0.03044	0.03044				

with the initial condition

$$u(x, 0) = 4x(1 - x), \quad 0 \leq x \leq 1 \quad (52)$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t > 0. \quad (53)$$

The numerical solutions of Example 2 are obtained by the ChSC method ($N = 15$, $\Delta t = 0.001$ for $\nu = 0.1$ and $\Delta t = 0.0001$ for $\nu = 1.0$) have been compared with results from Galerkin quadratic B-spline finite element method [38] ($N = 80$, $\Delta t = 0.00001$) and the analytic solution in Table 2 for $\nu = 1.0$ and 0.1 at different times. It is clearly seen that the present solutions are in very good agreement with the exact. The ChSC method produced slightly better results in little computer time than the Galerkin quadratic B-spline finite element method [38]. Fig. 1, illustrates the graph of the numerical results of Example 2 at different times $t = 0.1, 0.3$ and 0.5 for $\nu = 0.01$, $N = 15$ and $\Delta t = 0.001$.

Example 3. Consider the KdV–Burgers' equation:

$$u_t + \epsilon uu_x - \nu u_{xx} + \mu u_{xxx} = 0 \quad (54)$$

with the solitary wave solutions [41],

$$u(x, t) = A[9 - 6\tanh(B(x - Ct)) - 3\tanh^2(B(x - Ct))], \quad (55)$$

in the region $D = \{x : a < x < b\}$; $A = \nu^2/(25\epsilon\mu)$, $B = \nu/(10\mu)$, $C = 6\nu^2/(25\mu)$, α , ν and μ are arbitrary constants.

Table 3, shows norm infinity and norm relative of errors for various values of ϵ , ν and μ with $N = 10$, $a = -10$, $b = 10$ and $\Delta t = 0.1$.

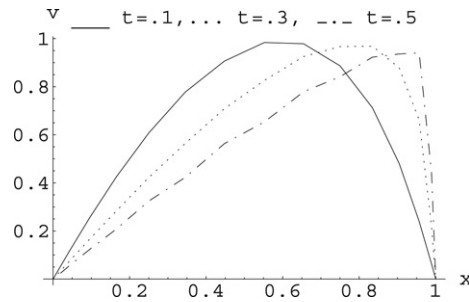


Fig. 1. Solutions of Example 2 at different times for $N = 15$, $\nu = 0.01$, $\Delta t = 0.001$.

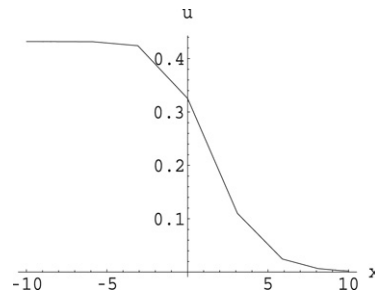


Fig. 2. The numerical and the exact solutions of Example 3 for $N = 10$, $\epsilon = 0.1$, $\nu = 0.03$, $\mu = 0.01$, $\Delta t = 0.1$ and $t = 1$ (— Exact, ... Numerical).

Table 3

Norm infinity and norm relative of errors for various values of ϵ , ν and μ

ϵ	ν	μ	$\ E\ _{\infty}$		$\ E\ $	
			$t = 0.3$	$t = 0.9$	$t = 0.3$	$t = 0.9$
1	0.1	0.1	8.20, -8	2.26, -7	8.60, -7	2.36, -6
	0.1	1.0	1.07, -8	6.31, -8	1.39, -6	6.82, -6
0.1	0.1	0.1	8.27, -7	2.27, -6	8.61, -7	2.36, -6
	0.1	1.0	1.01, -7	5.92, -7	1.34, -6	6.42, -6

In Fig. 2, we display the numerical and exact solutions of Example 3 for $N = 10$, $\epsilon = 0.1$, $\nu = 0.03$, $\mu = 0.01$, $\Delta t = 0.1$ and $t = 1$.

Example 4. Consider the modified KdV–Burgers' equation:

$$u_t + 2(u^3)_x + u_{xx} - u_{xxx} = 0 \quad (56)$$

with the traveling wave solutions [12],

$$u(x, t) = A[1 + \tanh(A(x - Ct))], \quad (57)$$

in the region $D = \{x : a < x < b\}$; $A = 1/6$ and $C = 2/9$.

In Table 4, we give the absolute errors between the exact and numerical results obtained by the present method (ChSC method) ($N = 15$) compared with the results given by the finite-difference method ($h = 0.1$) [12] given between brackets, at different values of x and t . We can observe from the table, that the ChSC method is more accurate as compared with the finite-difference method [12].

In Fig. 3, we display the numerical and exact solutions of Example 4 for $N = 10$, $\Delta t = 0.001$ and $t = 0.1$.

Table 4

Absolute errors with $N = 15$, $\Delta t = 0.0005$, $a = -10$ and $b = 10$

x	t						
	0.001	0.002	0.006	0.02	0.03	0.04	0.05
1.04529	3.58, -7	9.98, -7	2.76, -6	1.97, -6	1.65, -6	5.16, -6	7.05, -6
3.09017	3.58, -7	1.01, -6	2.83, -6	1.73, -6	4.19, -6	1.37, -5	2.74, -5
5.00000	3.58, -7 (1.78, -6)	1.07, -6	2.80, -6	2.89, -6	1.94, -5	4.97, -5	9.97, -5

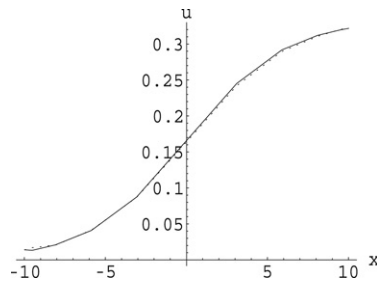
Fig. 3. The numerical and the exact solutions of Example 4 for $N = 10$, $\Delta t = 0.001$ and $t = 0.1$ (— Exact, ... Numerical).

Table 5

Norm infinity and norm relative of errors of the solution u with $a_0 = 0.05$

N	a	b	Δt	α	β	$\ E\ _\infty$		$\ E\ $	
						$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$
10	0	1	0.001	1.0	0.3	8.81, -6	8.82, -6	4.84, -5	4.92, -5
				0.3	0.03	4.46, -5	4.52, -5	1.46, -4	1.46, -4
10	-10	10	0.1	0.1	0.3	4.16, -5	8.23, -5	6.45, -4	1.25, -3
				0.3	0.03	4.59, -5	9.16, -5	6.90, -4	1.34, -3
20	-10	10	0.01	0.1	0.3	4.38, -5	8.66, -5	1.44, -3	1.27, -3
				0.3	0.03	4.58, -5	9.16, -5	6.68, -4	1.30, -3

Example 5. Consider the coupled Burgers' equations:

$$\begin{aligned} u_t - u_{xx} + 2uu_x + \alpha(uv)_x &= 0, \\ v_t - v_{xx} + 2vv_x + \beta(uv)_x &= 0 \end{aligned} \quad (58)$$

with the solitary wave solutions [41],

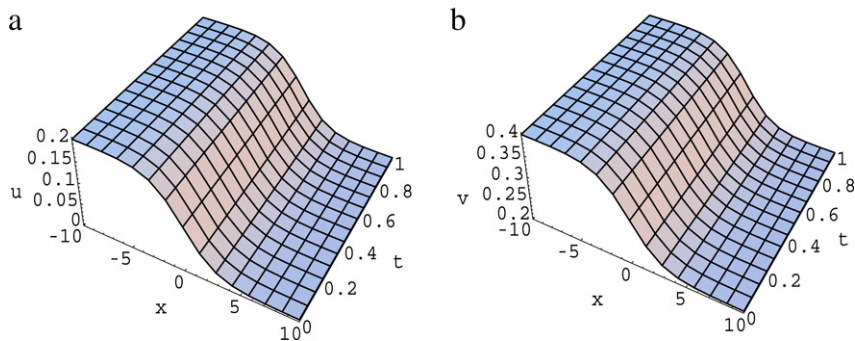
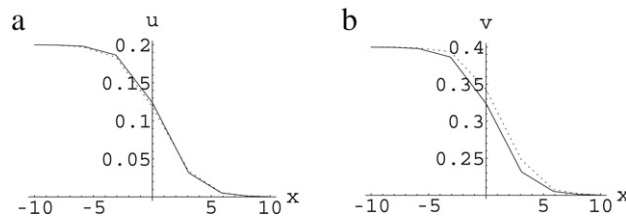
$$\begin{aligned} u(x, t) &= a_0 - 2A \left(\frac{2\alpha - 1}{4\alpha\beta - 1} \right) \tanh[A(x - 2At)], \\ v(x, t) &= a_0 \left(\frac{2\beta - 1}{2\alpha - 1} \right) - 2A \left(\frac{2\alpha - 1}{4\alpha\beta - 1} \right) \tanh[A(x - 2At)] \end{aligned} \quad (59)$$

in the region $D = \{x : a < x < b\}$ with $A = (1/2)(a_0(4\alpha\beta - 1))/(2\alpha - 1)$, a_0 , α and β are arbitrary constants.Tables 5 and 6, show norm infinity and norm relative of errors with $a_0 = 0.05$ at time levels $t = 0.5$ and 1.0 . The results from the present study are in good agreement with the exact solutions.In Fig. 4, we display the exact solutions of Example 5: (a), (b) when $\alpha = 1$, $\beta = 2$ and $a_0 = 0.1$.In Fig. 5, we display the numerical and the exact solutions of Example 5: (a), (b) for $N = 10$, $\Delta t = 0.1$, $t = 1$, $\alpha = 1$, $\beta = 2$, and $a_0 = 0.1$.

Table 6

Norm infinity and norm relative of errors of the solution v with $a_0 = 0.05$

N	a	b	Δt	α	β	$\ E\ _\infty$		$\ E\ $	
						$t = 0.5$	$t = 1.0$	$t = 0.5$	$t = 1.0$
10	0	1	0.001	1.0	0.3	2.86, -6	2.86, -6	4.22, -5	4.25, -5
				0.3	0.03	1.11, -5	1.13, -5	2.48, -4	2.51, -4
10	-10	10	0.1	0.1	0.3	2.19, -5	4.10, -5	4.90, -4	9.36, -4
				0.3	0.03	1.80, -4	3.59, -4	1.24, -3	2.42, -3
20	-10	10	0.01	0.1	0.3	4.99, -5	9.92, -5	5.42, -4	1.29, -3
				0.3	0.03	1.81, -4	3.62, -4	1.20, -3	2.35, -3

Fig. 4. The exact solutions of Example 5: (a), (b) when $\alpha = 1$, $\beta = 2$ and $a_0 = 0.1$.Fig. 5. The numerical and the exact solutions of Example 5: (a), (b) for $N = 10$, $\Delta t = 0.1$, $t = 1$, $\alpha = 1$, $\beta = 2$, and $a_0 = 0.1$ (— Exact, ... Numerical).

Example 6. Consider the 2D unsteady Burgers' equation:

$$u_t + uu_x + uu_y = v(u_{xx} + u_{yy}) \quad (60)$$

with the exact solutions [11],

$$u(x, y, t) = \frac{1}{1 + e^{(x+y-t)/2v}} \quad (61)$$

in the region $D = \{x : a < x < b\}$, $v = 1/R$ and R is the Reynolds' number.

Table 7, shows norm infinity and norm relative of errors for various values of N , M , Δt and v with $a = 0$, $b = 1$.

In Fig. 6, we display the numerical and the exact solutions of Example 6 for $N = M = 30$, $v = 0.01$, $\Delta t = 0.0005$, $t = 0.25$ and $y = 0.25$.

In Fig. 7, we display Error ($|\text{Numerical} - \text{Exact}|$) of Example 6 at time $t = 0.1$ for $N = M = 30$, $v = 0.005$, $y = 0.25$, $\Delta t = 0.0005$.

Table 7
Norm infinity and norm relative of errors for various values of N , M , Δt and ν

N	M	Δt	ν	$\ E\ _\infty$		$\ E\ $	
				$t = 0.05$	$t = 0.25$	$t = 0.05$	$t = 0.25$
5	5	0.0050	1.00	8.94, -8	1.19, -7	1.50, -7	1.70, -7
10	10	0.0005	1.00	7.45, -7	8.05, -7	8.50, -7	9.82, -7
10	10	0.0050	0.10	1.28, -6	5.84, -6	2.49, -6	4.02, -6
10	10	0.0010	0.10	1.37, -6	2.09, -6	2.15, -6	2.63, -6
30	30	0.0005	0.01	4.14, -5	4.32, -3	4.65, -5	2.26, -3

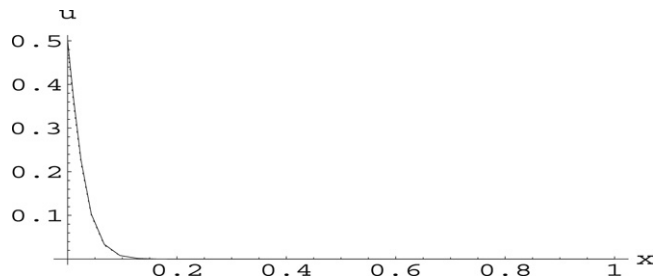


Fig. 6. The numerical and the exact solutions of Example 6 for $N = M = 30$, $\nu = 0.01$, $\Delta t = 0.0005$, $t = 0.25$ and $y = 0.25$ (— Exact, ... Numerical).

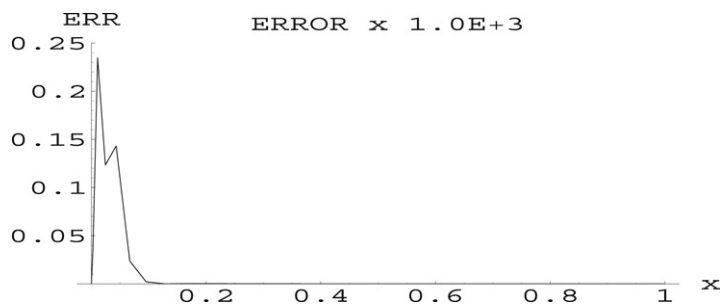


Fig. 7. Error(|Numerical - Exact|) of Example 6 at time $t = 0.1$ for $N = M = 30$, $\nu = 0.005$, $y = 0.25$, $\Delta t = 0.0005$.

Example 7. Consider the system of 2D Burgers' equations:

$$\begin{aligned} u_t + uu_x + vv_y &= \nu(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y &= \nu(v_{xx} + v_{yy}) \end{aligned} \quad (62)$$

with the exact solutions [3,10],

$$\begin{aligned} u(x, t) &= \frac{3}{4} - \frac{1}{4[1 + \exp((-4x + 4y - t)/32\nu)]} \\ v(x, t) &= \frac{3}{4} + \frac{1}{4[1 + \exp((-4x + 4y - t)/32\nu)]} \end{aligned} \quad (63)$$

in the region $D = \{(x, y) : a < x, y < b\}$, $\nu = 1/R$ and R is the Reynolds' number.

Tables 8 and 9, show norm infinity and norm relative of errors for various values of N , M , Δt and ν with $a = 0$, $b = 1$ at time levels $t = 0.01$, 0.5 and 2.0. The results from the present study are in good agreement with the exact solutions.

In Table 10, we give the exact and numerical values of u and v at the point $(0.5, 0.5)$ for $\nu = 0.01$, $N = 20$ and $\Delta t = 0.001$ at time levels $t = 0.01$, 0.5 and 2.0 compared with the results given by [3] with $\nu = 0.01$, $N = 20$ and $\Delta t = 0.0001$. It is seen that the present method offers better accuracy in comparison with [3].

Table 8

Norm infinity and norm relative of errors of the solution u for various values of N , M , Δt and ν

N	M	Δt	ν	$\ E\ _\infty$		$\ E\ $	
				$t = 0.01$	$t = 2.0$	$t = 0.01$	$t = 2.0$
10	10	0.0005	1.000	1.31, -6	1.61, -6	7.97, -7	1.18, -6
10	10	0.0050	0.100	4.77, -7	1.13, -6	3.10, -7	7.89, -7
20	20	0.0010	0.010	3.22, -6	1.49, -5	1.71, -6	8.23, -6
30	30	0.0010	0.005	2.25, -5	9.99, -5	7.34, -6	3.54, -5

Table 9

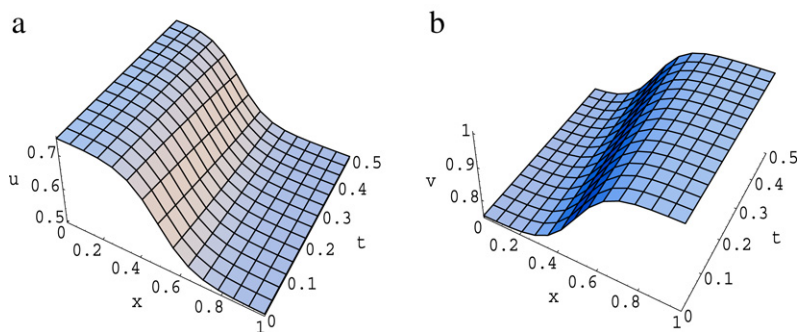
Norm infinity and norm relative of errors of the solution v for various values of N , M , Δt and ν

N	M	Δt	ν	$\ E\ _\infty$		$\ E\ $	
				$t = 0.01$	$t = 2.0$	$t = 0.01$	$t = 2.0$
10	10	0.0005	1.000	1.91, -6	1.91, -6	7.92, -7	8.94, -7
10	10	0.0050	0.100	5.96, -6	1.97, -6	2.96, -7	8.42, -7
20	20	0.0010	0.010	4.53, -6	1.03, -5	1.55, -6	4.35, -6
30	30	0.0010	0.005	2.19, -5	1.06, -4	5.48, -6	2.05, -5

Table 10

Comparison of numerical values of u and v with results from [3] for $\nu = 0.01$

t	Values of u			Values of v		
	ChSC	Exact	[3]	ChSC	Exact	[3]
0.01	0.62305	0.62305	0.62311	0.87695	0.87695	0.87689
0.50	0.54332	0.54332	0.54222	0.95668	0.95668	0.95685
2.00	0.50049	0.50048	0.49931	0.99952	0.99952	0.99821

Fig. 8. The exact solutions of Example 7: (a), (b) when $\nu = 0.01$ and $y = 0.5$.

In Fig. 8, we display the exact solutions of Example 7: (a), (b) when $\nu = 0.01$ and $y = 0.5$.

In Fig. 9, we display the numerical and the exact solutions of Example 7: (a), (b) for $N = M = 20$, $\nu = 0.01$, $\Delta t = 0.001$, $t = 0.5$ and $y = 0.5$.

In Fig. 10, we display Error ($|Numerical - Exact|$) of Example 7 at time $t = 0.5$: (a), (b) for $N = M = 20$, $\nu = 0.005$, $y = 0.5$ and $\Delta t = 0.001$.

4. Conclusions

In this paper the Chebyshev spectral collocation methods are elaborated to obtain numerical solutions of some NLPDEs. The problem is reduced to a system of ODES that are solved by the Runge–Kutta method of order four. Numerical results of 1D Burgers', KdV–Burgers', coupled Burgers', 2D Burgers' and system of 2D Burgers' equations are obtained. The obtained approximate numerical solution maintains a good accuracy in little computer

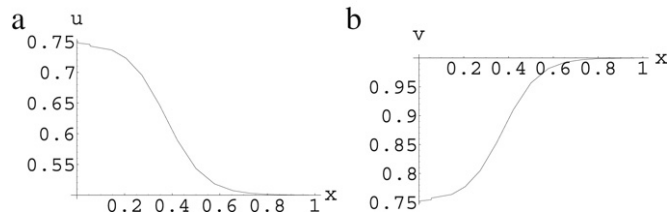


Fig. 9. The numerical and the exact solutions of Example 7: (a), (b) for $N = M = 20$, $\nu = 0.01$, $\Delta t = 0.001$, $t = 0.5$ and $y = 0.5$ (— Exact, ... Numerical).

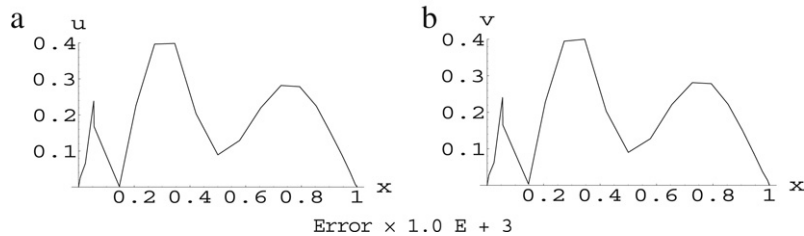


Fig. 10. Error(|Numerical – Exact|) of Example 7 at time $t = 0.5$: (a), (b) for $N = M = 20$, $\nu = 0.005$, $y = 0.5$, $\Delta t = 0.001$.

time compared with exact solution and the finite-difference method [3,12] and Galerkin quadratic B-spline finite element method [38]. In addition the results are very useful not only for its numerical value but also for its physical value in fluid mechanics, plasma physics, field theory, solid state physics and engineering science. Moreover the method is applicable to a wide class of NLPDEs.

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