

# Finite element approximation of the elasticity spectral problem on curved domains<sup>☆</sup>

Erwin Hernández<sup>\*</sup>

Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile

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## ABSTRACT

We analyze the finite element approximation of the spectral problem for the linear elasticity equation with mixed boundary conditions on a curved non-convex domain. In the framework of the abstract spectral approximation theory, we obtain optimal order error estimates for the approximation of eigenvalues and eigenvectors. Two kinds of problems are considered: the discrete domain does not coincide with the real one and mixed boundary conditions are imposed. Some numerical results are presented.

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## 1. Introduction

The finite element approximation of eigenvalues and eigenfunctions on curved domains is considered in a great number of papers [5–7,11–13,1]. The interest rests on their significant practical relevance. A review can be found in [10], where the author analyzes the convergence for eigenvalue approximation on triangular finite element meshes, including the approximation of curved domains.

The spectral problem for the Laplace equation on curved non-convex domains is considered in [11,7,5,1]. They also prove convergence and optimal order error estimates. In [11] they consider Dirichlet boundary conditions and use the *min–max* characterization theory to prove second order for the eigenvalues; they extended their results to include multiple eigenvalues in [12] and numerical integration in [13]. In [7] they use the spectral approximation theory stated in [2] and assumed Dirichlet boundary conditions and a lumped mass method to prove double order for isoparametric elements. The same theory is used in [5] considering Neumann boundary conditions and using a non-conforming finite element method on the polygonal computational domain. The same authors extended their results in [6] to spectral acoustic problems on curved domains. An extension of the spectral approximation theory is introduced in [1]; this abstract setting can be applied to a variety of eigenvalue problems defined on curved domains. Unfortunately, only one kind of boundary condition can be imposed.

In this paper we extend the theory used in [5] to consider the eigenvalue problem for the elasticity equation with mixed boundary conditions: Dirichlet on part of its boundary and Neumann on the other part. The extension–restriction of functions between the two domains, real and computational,  $\Omega$  and  $\Omega_h$ , respectively, is the main technical difficulty of this paper. In fact, in [7] it is hardly used that continuous and discrete functions vanished on  $\Omega \setminus \overline{\Omega}_h$  and  $\Omega_h \setminus \overline{\Omega}$ , respectively, and in [5] the extension–restriction operator does not retain the Dirichlet conditions.

On the other hand, it is known that convergence results for a linear boundary value problem do not necessarily imply similar results for its associated spectral one. In the framework of the abstract spectral approximation theory as presented in [2], this paper deals with a linear elasticity eigenvalue problem, with mixed boundary conditions, defined on a non-convex domain. Since the techniques in this paper do not rely on the *min–max* characterization of eigenvalues, they can

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<sup>\*</sup> Tel.: +56 32 2 654311.

E-mail address: [erwin.hernandez@usm.cl](mailto:erwin.hernandez@usm.cl).

be used to deal with problems attaining infinite multiplicity eigenvalues, such as those arising in the computation of free fluid-structure vibrations by means of pure displacement formulations.

This paper is organized as follows. In Section 2 the eigenvalue elasticity equation is presented in a variational framework in both continuous and discrete cases. Some basic results and assumptions on the meshes are included. In Section 3 we obtain optimal order error estimates for the approximation using the spectral theory stated in [2]. This is the main result of this work. Special attention is paid to the mixed boundary condition which is the main technical difficulty that is presented. In Section 4 numerical results are shown which confirm the theoretical result.

## 2. Statement of the problem

Let  $\Omega \subset \mathbf{R}^2$  be a bounded open domain, not necessarily convex, with a piecewise smooth (e.g.,  $\mathcal{C}^2$ ) Lipschitz boundary  $\Gamma := \partial\Omega$ . Let  $\Gamma = \Gamma_D \cup \Gamma_N$  and let us consider the eigenvalue problem for the linear elasticity equation in  $\Omega$  with mixed boundary conditions, written in the variational form (see [3], for details):

Find  $\lambda \in \mathbf{R}$  and  $\mathbf{u} \in H_{\Gamma_D}^1(\Omega)^2$ ,  $\mathbf{u} \neq \mathbf{0}$ , such that

$$\underbrace{\int_{\Omega} \sigma(\mathbf{u}) : \epsilon(\mathbf{v}) \, dx}_{a(\mathbf{u}, \mathbf{v})} = \lambda \underbrace{\int_{\Omega} \rho \mathbf{u} \cdot \mathbf{v} \, dx}_{b(\mathbf{u}, \mathbf{v})} \quad \forall \mathbf{v} \in H_{\Gamma_D}^1(\Omega)^2, \quad (2.1)$$

where  $\mathbf{u}$  is the solid displacement,  $\sigma(\mathbf{u})$  is the stress tensor related to the strain tensor,  $\epsilon(\mathbf{v}) = \frac{1}{2}(\nabla(\mathbf{v}) + \nabla(\mathbf{v})^t)$ , by Hooke's law  $\sigma = \lambda_S(\text{tr}\epsilon)\mathbf{I} + 2\mu_S\epsilon$  ( $\lambda_S$  and  $\mu_S$  being the Lamé coefficients). For the sake of simplicity we take the density of the elastic body  $\rho = 1$ .

Because of the compact inclusion  $H_{\Gamma_D}^1(\Omega)^2 \hookrightarrow L^2(\Omega)^2$ , the problem above attains a sequence of finite multiplicity increasing positive eigenvalues  $\{\lambda_k\}_{k=0}^{\infty}$  with corresponding eigenfunctions  $\{\mathbf{u}_k\}_{k=0}^{\infty}$  in  $H_{\Gamma_D}^1(\Omega)^2$ , providing an orthonormal set of  $L^2(\Omega)^2$ . Let  $\mathbf{T}$  be the linear operator defined by

$$\begin{aligned} \mathbf{T} : L^2(\Omega)^2 &\longrightarrow H_{\Gamma_D}^1(\Omega)^2 \hookrightarrow L^2(\Omega)^2 \\ \mathbf{f} &\longmapsto \mathbf{u} \in H_{\Gamma_D}^1(\Omega)^2, \end{aligned}$$

where  $\mathbf{u} \in H_{\Gamma_D}^1(\Omega)^2$  is the solution of:

Find  $\mathbf{u} \in H_{\Gamma_D}^1(\Omega)^2$  such that

$$a(\mathbf{u}, \mathbf{v}) = b(\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_{\Gamma_D}^1(\Omega)^2,$$

where the bilinear forms  $a$  and  $b$  are as in (2.1). By the Lax–Milgram Lemma,  $\mathbf{T}$  is a well defined bounded operator and it holds that  $\|\mathbf{u}\|_{1,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega}$ . Once more, because of the same compact inclusion,  $\mathbf{T}$  is compact, positive, and the eigenvalues of  $\mathbf{T}$  are given by  $\mu_i = 1/\lambda_i$ , with  $\lambda_i$  being the eigenvalues of problem (2.1). Moreover the eigenfunctions coincide. As a consequence of the classical a priori estimates (see [4]), for any  $\mathbf{f} \in L^2(\Omega)^2$ ,  $\mathbf{u} = \mathbf{T}\mathbf{f}$  is known to satisfy some further regularity. In fact,  $\mathbf{u} \in H^{1+r}(\Omega)^2$  for some  $r > 0$  depending on the geometry of  $\Omega$  and boundary conditions, and it holds that

$$\|\mathbf{u}\|_{1+r,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega}. \quad (2.2)$$

Let  $\{\mathcal{T}_h\}$  be a regular family of standard finite element triangulations (cf. [3]) of polygonal domains  $\Omega_h$  approximating  $\Omega$ , such that if  $\Gamma_h := \partial\Omega_h$  and  $\mathcal{N}_h$  is the set of vertices of all the triangles in  $\mathcal{T}_h$ , then  $\Gamma_D \cap \Gamma_N \subset \mathcal{N}_h \cap \Gamma_h$ . The polygonal boundary  $\Gamma_h$  splits into two parts,  $\Gamma_D^h$  and  $\Gamma_N^h$ , approximating  $\Gamma_D$  and  $\Gamma_N$ , respectively. As usual,  $h$  stands for the mesh size and, for a given triangulation  $\mathcal{T}_h$ , we denote by  $\mathcal{T}_h^{\partial}$  the subfamily of the so called *boundary triangles* (those having two vertices on  $\Gamma_h$ ).

Let  $L_h(\Omega_h) := \{q_h \in H_{\Gamma_D^h}^1(\Omega_h)^2 : q_h|_T \in \mathcal{P}_1(T)^2, \forall T \in \mathcal{T}_h\}$  (i.e., standard piecewise linear continuous finite elements on  $\mathcal{T}_h$ ), and let us consider the following classical discrete analogue of (2.1):

Find  $\lambda_h \in \mathbf{R}$  and  $\mathbf{u}_h \in L_h(\Omega_h)$ ,  $\mathbf{u}_h \neq \mathbf{0}$ , such that

$$\underbrace{\int_{\Omega_h} \sigma(\mathbf{u}_h) : \epsilon(\mathbf{v}_h) \, dx}_{a_h(\mathbf{u}_h, \mathbf{v}_h)} = \lambda_h \underbrace{\int_{\Omega_h} \mathbf{u}_h \cdot \mathbf{v}_h \, dx}_{b_h(\mathbf{u}_h, \mathbf{v}_h)} \quad \forall \mathbf{v}_h \in L_h(\Omega_h). \quad (2.3)$$

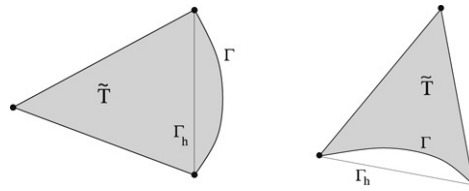
Let  $\bar{\mathbf{T}}_h$  be defined by

$$\begin{aligned} \bar{\mathbf{T}}_h : L^2(\Omega_h)^2 &\longrightarrow L^2(\Omega_h)^2 \\ \mathbf{f} &\longmapsto \mathbf{u}_h \in L_h(\Omega_h), \end{aligned}$$

where  $\mathbf{u}_h \in L_h(\Omega_h)$  is the solution of:

Find  $\mathbf{u}_h \in L_h(\Omega_h)$  such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = b_h(\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in L_h(\Omega_h),$$

Fig. 1. Ideal triangles on  $\Omega$  and  $\Omega_h$ .

with  $a_h$  and  $b_h$  defined as above. It clearly holds that

$$\|\mathbf{u}_h\|_{1,\Omega_h} \leq C\|\mathbf{f}\|_{0,\Omega_h}. \quad (2.4)$$

The eigenvalues of  $\bar{\mathbf{T}}_h$  are given by  $\mu_{hi} = 1/\lambda_{hi}$ , with  $\lambda_{hi}$  being those of the discrete problem. Once again, the corresponding eigenfunctions coincide.

It is important to note that the spectral approximation theory stated in [2] cannot be directly applied to the operators  $\bar{\mathbf{T}}_h$ , since their domains  $L^2(\Omega_h)^2$  do not coincide with that of  $\mathbf{T}$ . In order to overcome this difficulty, we are going to introduce other discrete operators  $\mathbf{T}_h$ , defined on  $L^2(\Omega)^2$ , with the spectrum also related to that of the problem.

For a given triangulation  $\mathcal{T}_h$ , let us denote by  $\tilde{T}$  the curved triangle of edges with two vertices on  $\Gamma_h$  and one edge on  $\Gamma$  (see Fig. 1); we call it the *ideal triangle* associated with  $T$  ( $\tilde{T} \equiv T$ , for inner triangles). We assume that it holds that either  $T \subset \tilde{T}$  or  $T \supset \tilde{T}$ . We denote by  $\tilde{\mathcal{T}}_h := \{\tilde{T}\}_{T \in \mathcal{T}_h}$  the partition of  $\Omega$  provided by the ideal triangles and we call it the *ideal triangulation* of  $\Omega$ . By a slight variant of inequality (5.2-19) in Lemma 5.2-3 of [9] (see also [5]) it holds that

$$\|v\|_{0,\Omega \setminus \bar{\Omega}_h} \leq Ch^s \|v\|_{s,\Omega} \quad \forall v \in H^s(\Omega), \quad (2.5)$$

$$\|v\|_{0,\Omega_h \setminus \bar{\Omega}} \leq Ch^s \|v\|_{s,\Omega_h} \quad \forall v \in H^s(\Omega_h). \quad (2.6)$$

Let  $L_h(\Omega) := \{q \in H_{T_D}^1(\Omega)^2 : q|_{\tilde{T}} \in \mathcal{P}_1(\tilde{T})^2 \ \forall \tilde{T} \in \tilde{\mathcal{T}}_h, \ \tilde{T} \subset \Omega\}$ , and let  $\mathbf{P}$  be the  $L^2(\Omega)^2$ -orthogonal projection onto  $L_h(\Omega)$  satisfying  $\|\mathbf{P}\mathbf{f}\|_{0,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega}$ , and, if  $\mathbf{f} \in H_{T_D}^1(\Omega)^2$ , then

$$\|\mathbf{P}\mathbf{f} - \mathbf{f}\|_{k,\Omega} \leq Ch^{1-k} \|\mathbf{f}\|_{1,\Omega}, \quad k = 0, 1. \quad (2.7)$$

We consider the uniformly bounded linear operators, so called restriction–extension operators,  $\hat{\mathbf{E}} : L_h(\Omega) \longrightarrow L_h(\Omega_h)$  and the inverse  $\check{\mathbf{E}} : L_h(\Omega_h) \longrightarrow L_h(\Omega)$ , defined by

$$\check{\mathbf{E}}v_h = \check{v}_h|_{\tilde{T}} := \begin{cases} v_h|_{\tilde{T}} & \forall \tilde{T} \in \tilde{\mathcal{T}}_h : \tilde{T} \subset \Omega_h, \\ (v_h|_T)^\vee & \forall \tilde{T} \in \tilde{\mathcal{T}}_h : \tilde{T} \not\subset \Omega_h, \end{cases}$$

where  $(v_h|_T)^\vee \in \mathcal{P}_1(\tilde{T})$  denotes the natural extension of the linear function  $v_h|_T \in \mathcal{P}_1(T)$  to the larger set  $\tilde{T}$  (notice that, if  $\tilde{T} \not\subset \Omega_h$ , then  $T \subset \tilde{T}$ ), and  $\hat{\mathbf{E}} := \check{\mathbf{E}}^{-1}$ ; namely,

$$\hat{\mathbf{E}}w_h = \hat{w}_h|_T := \begin{cases} w_h|_T & \forall T \in \mathcal{T}_h : T \subset \Omega, \\ (w_h|_{\tilde{T}})^\wedge & \forall T \in \mathcal{T}_h : T \not\subset \Omega, \end{cases}$$

where  $(w_h|_{\tilde{T}})^\wedge \in \mathcal{P}_1(T)$  denotes the natural extension of the linear function  $w_h|_{\tilde{T}} \in \mathcal{P}_1(\tilde{T})$  to the larger set  $T$  (notice that, now, if  $T \not\subset \Omega$ , then  $\tilde{T} \subset T$ ).

We are able to define the discrete operator  $\mathbf{T}_h$  given by

$$\begin{aligned} \mathbf{T}_h : L^2(\Omega)^2 &\longrightarrow L^2(\Omega)^2 \\ f &\longmapsto \check{\mathbf{u}}_h = \check{\mathbf{E}}\bar{\mathbf{T}}_h\hat{\mathbf{E}}\mathbf{P}\mathbf{f}. \end{aligned}$$

It is simple to show that  $\|\mathbf{T}_h\mathbf{f}\|_{1,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega}$ , the eigenvalues of  $\mathbf{T}_h$  and  $\bar{\mathbf{T}}_h$  coincide, and the eigenfunctions of the former are the restriction–extensions of those of the latter obtained by means of the operator  $\check{\mathbf{E}}$ . More precisely, the eigenfunctions  $\mathbf{u}_h$  and  $\check{\mathbf{u}}_h$  are related by  $\check{\mathbf{u}}_h = \check{\mathbf{E}}\mathbf{u}_h$  and  $\mathbf{u}_h = \hat{\mathbf{E}}\check{\mathbf{u}}_h$ .

### 3. Error estimates

Our next goal is to prove that the operators  $\mathbf{T}_h$  converge to  $\mathbf{T}$  in norm, as  $h$  goes to zero. From now on and throughout the rest of the section, let  $\mathbf{f} \in L^2(\Omega)^2$  be a fixed function,  $\mathbf{u} := \mathbf{T}\mathbf{f}$ ,  $\hat{\mathbf{f}} := \hat{\mathbf{E}}\mathbf{P}\mathbf{f}$ ,  $\mathbf{u}_h := \bar{\mathbf{T}}_h\hat{\mathbf{f}}$ , and  $\check{\mathbf{u}}_h := \check{\mathbf{E}}\mathbf{u}_h = \mathbf{T}_h\mathbf{f}$ . We will also use a bounded extension of  $\mathbf{u}$ , denoted by  $\mathbf{u}^e$ , from  $H^{1+r}(\Omega)^2$  to  $H^{1+r}(\mathbb{R}^2)^2$ , satisfying  $\|\mathbf{u}^e\|_{1+r,\mathbb{R}^2} \leq C\|\mathbf{u}\|_{1+r,\Omega}$ , with  $C$  only depending on  $\Omega$  (see [8]).

The next lemma splits  $\|(\mathbf{T} - \mathbf{T}_h)\mathbf{f}\|_{1,\Omega}$ :

**Lemma 3.1.** *There exists  $C > 0$ , not depending on  $\mathbf{f}$  or  $h$ , such that*

$$\|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega} \leq Ch^r \|\mathbf{f}\|_{0,\Omega} + \sup_{\mathbf{w}_h \in L_h(\Omega_h)} \frac{|b_h(\hat{\mathbf{f}}, \mathbf{w}_h) - a_h(\mathbf{u}^e, \mathbf{w}_h)|}{\|\mathbf{w}_h\|_{1,\Omega_h}}.$$

**Proof.** Let us note that

$$\begin{aligned} \|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega}^2 &= \|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega \cap \Omega_h}^2 + \|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega \setminus \overline{\Omega}_h}^2 \\ &\leq \|\mathbf{u}^e - \mathbf{u}_h\|_{1,\Omega_h}^2 + \|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega \setminus \overline{\Omega}_h}^2 \\ &\leq C \left[ \inf_{\mathbf{v}_h \in L_h(\Omega_h)} \|\mathbf{u}^e - \mathbf{v}_h\|_{1,\Omega_h} + \sup_{\mathbf{w}_h \in L_h(\Omega_h)} \frac{|b_h(\hat{\mathbf{f}}, \mathbf{w}_h) - a_h(\mathbf{u}^e, \mathbf{w}_h)|}{\|\mathbf{w}_h\|_{1,\Omega_h}} \right]^2 + \|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega \setminus \overline{\Omega}_h}^2. \end{aligned}$$

The last inequality is obtained by standard non-conforming techniques (see Strang's Lemma in [3]). The lemma follows by using interpolation results for Sobolev spaces and (2.5).  $\square$

We now focus on the consistency term; this is the main part of the paper, since this term cannot be bounded using the arguments from [5]; in fact, since the Dirichlet boundary condition does not remain with the extension–restriction operator (i.e.  $\check{\mathbf{v}}_h \notin L_h(\Omega)$  for  $\mathbf{v}_h \in L_h(\Omega_h)$ ), we introduce some additional concepts and results from [11] to deal with it.

Let  $T^0$  be the standard reference triangle in the plane with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 1)$ . Let  $F_T$  be an affine map applying the reference triangle onto any  $T \subset \Omega$ . Let  $F_{\tilde{T}} : T^0 \rightarrow \tilde{T}$  be the one-to-one map defined in [14] (Section 22). The properties proved therein show that for all  $\mathbf{v}_h \in L_h(\Omega_h)$ , there exists a function  $\check{\mathbf{v}}_h \in L_h(\Omega)$  (associated with  $\mathbf{v}_h$ ) satisfying:

- $\check{\mathbf{v}}_h \in C^0(\overline{\Omega})$ ,
- $\check{\mathbf{v}}_h(P) = \mathbf{v}_h(P) \quad \forall P_i \in N_h$ ,
- $\check{\mathbf{v}}_h$  is linear on each triangle  $T = \tilde{T}$ ,
- if  $\tilde{T} \subset \tilde{T}$  then  $\check{\mathbf{v}}_h = 0$  on  $\tilde{T} \setminus T$  and  $\check{\mathbf{v}}_h = \mathbf{v}_h$  on  $T$ ,
- if  $\tilde{T} \subset T$  then  $\check{\mathbf{v}}_h = \mathbf{v}_h \circ F_T \circ F_{\tilde{T}}^{-1}$  on  $\tilde{T}$ .

By using these properties we now prove the following approximation result:

**Lemma 3.2.** *Let  $\check{\mathbf{v}}_h \in L_h(\Omega)$  be associated with  $\mathbf{v}_h \in L_h(\Omega_h)$  as above and assume  $T \supset \tilde{T}$ . Then*

$$\|\mathbf{v}_h - \check{\mathbf{v}}_h\|_{i,\tilde{T}} \leq Ch^{2-i} \|\mathbf{v}_h\|_{1,\tilde{T}} = Ch^{2-i} \|\mathbf{v}_h\|_{1,\tilde{T}} \quad (i = 0, 1).$$

**Proof.** It is a direct consequence of Theorem 2 in [15], the definition of  $\check{\mathbf{v}}_h$  and the linearity of  $\mathbf{v}_h$  on  $T$ .  $\square$

Now, we obtain a bound for the consistent terms.

**Lemma 3.3.** *There exists  $C > 0$  such that*

$$\sup_{\mathbf{w}_h \in L_h(\Omega_h)} \left[ \frac{|b_h(\hat{\mathbf{f}}, \mathbf{w}_h) - a_h(\mathbf{u}^e, \mathbf{w}_h)|}{\|\mathbf{w}_h\|_{1,\Omega_h}} \right] \leq Ch^r \|\mathbf{f}\|_{0,\Omega}.$$

**Proof.** Let  $\mathbf{w}_h \in L_h(\Omega_h)$  and  $\check{\mathbf{w}}_h \in L_h(\Omega)$  be associated with  $\mathbf{w}_h$ . Since  $a(\mathbf{u}, \check{\mathbf{w}}_h) = b(\mathbf{f}, \check{\mathbf{w}}_h)$ , it holds that

$$b_h(\hat{\mathbf{f}}, \mathbf{w}_h) - a_h(\mathbf{u}^e, \mathbf{w}_h) = a(\mathbf{u}, \check{\mathbf{w}}_h) - a_h(\mathbf{u}^e, \mathbf{w}_h) + b_h(\hat{\mathbf{f}}, \mathbf{w}_h) - b(\mathbf{f}, \check{\mathbf{w}}_h).$$

From the definition of the bilinear forms and the Cauchy–Schwarz inequality, it is easy to see that

$$a(\mathbf{u}, \check{\mathbf{w}}_h) - a_h(\mathbf{u}^e, \mathbf{w}_h) \leq \left( \sum_{\tilde{T} \subset T \in \mathcal{T}_h^\partial} \|\mathbf{u}\|_{1,\tilde{T}} \|\check{\mathbf{w}}_h - \mathbf{w}_h\|_{1,\tilde{T}} + \|\mathbf{u}^e\|_{0,\tilde{T}} \|\check{\mathbf{w}}_h - \mathbf{w}_h\|_{0,\tilde{T}} \right) + \|\mathbf{u}^e\|_{1,\Omega_h \setminus \overline{\Omega}} \|\mathbf{w}_h\|_{1,\Omega_h \setminus \overline{\Omega}}.$$

On the other hand, using the projection properties we have

$$\begin{aligned} b_h(\hat{\mathbf{f}}, \mathbf{w}_h) - b(\mathbf{f}, \check{\mathbf{w}}_h) &= \int_{\Omega_h} \hat{\mathbf{f}} \cdot \mathbf{w}_h \, dx - \int_{\Omega} \mathbf{f} \cdot \check{\mathbf{w}}_h \, dx \\ &= \int_{\Omega_h \setminus \overline{\Omega}} \hat{\mathbf{f}} \cdot \mathbf{w}_h \, dx + \int_{\Omega \cap \Omega_h} \mathbf{f} \cdot (\mathbf{w}_h - \check{\mathbf{w}}_h) \, dx - \int_{\Omega \setminus \overline{\Omega}_h} (\mathbf{P}\mathbf{f} - \mathbf{f}) \cdot \mathbf{w}_h \, dx, \end{aligned}$$

and then, using the Cauchy–Schwarz inequality,

$$b_h(\hat{\mathbf{f}}, \mathbf{w}_h) - b(\mathbf{f}, \check{\mathbf{w}}_h) \leq \|\hat{\mathbf{f}}\|_{0,\Omega_h \setminus \overline{\Omega}} \|\mathbf{w}_h\|_{0,\Omega_h \setminus \overline{\Omega}} + \|\mathbf{f}\|_{0,\Omega \cap \Omega_h} \|\mathbf{w}_h - \check{\mathbf{w}}_h\|_{0,\Omega \cap \Omega_h} + \|\mathbf{P}\mathbf{f} - \mathbf{f}\|_{0,\Omega \setminus \overline{\Omega}_h} \|\mathbf{w}_h\|_{0,\Omega \setminus \overline{\Omega}_h}. \quad (3.8)$$

So, we conclude the proof by using (2.7), (2.5) and Lemma 3.2.  $\square$

As a consequence of all the previous lemmas we prove the following convergence result:

**Lemma 3.4.** *There exists  $C > 0$  such that for all  $\mathbf{f} \in L^2(\Omega)^2$  it holds that*

$$\|(\mathbf{T} - \mathbf{T}_h)\mathbf{f}\|_{1,\Omega} \leq Ch^r \|\mathbf{f}\|_{0,\Omega}.$$

In order to prove a double order for the convergence of the eigenvalues, we will have to introduce the following lemmas:

**Lemma 3.5.** *Let  $\check{\mathbf{u}}_h \in L_h(\Omega)$  be the function associated with  $\mathbf{u}_h \in L_h(\Omega_h)$ . It holds that*

$$\|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega} \leq Ch^r \|\mathbf{f}\|_{0,\Omega}.$$

**Proof.** Notice that, from (2.5),

$$\|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega}^2 = \|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega \cap \Omega_h}^2 + \|\mathbf{u}\|_{1,\Omega \setminus \overline{\Omega}_h}^2 \leq \|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega \cap \Omega_h}^2 + Ch^{2r} \|\mathbf{u}\|_{1+r,\Omega}^2,$$

and, using Lemma 3.4 and 3.2,

$$\|\mathbf{u} - \check{\mathbf{u}}_h\|_{1,\Omega \cap \Omega_h} \leq \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega \cap \Omega_h} + \|\mathbf{u}_h - \check{\mathbf{u}}_h\|_{1,\Omega \cap \Omega_h} \leq Ch \|\mathbf{f}\|_{0,\Omega} + \sum_{\tilde{T} \subset T \in \mathcal{T}_h^\partial} \|\mathbf{u}_h - \check{\mathbf{u}}_h\|_{1,\tilde{T}}.$$

So, we conclude the proof by using Lemma 3.2, (2.2) and (2.4) and the continuity of the operator  $\widehat{\mathbf{E}}$ .  $\square$

**Lemma 3.6.** *There exists  $C > 0$  such that, for all  $\mathbf{f} \in H_{\Gamma_D}^1(\Omega)^2$ ,*

$$\|(\mathbf{T} - \mathbf{T}_h)\mathbf{f}\|_{0,\Omega} = \sup_{\mathbf{g} \in L^2(\Omega)^2} \frac{b((\mathbf{T} - \mathbf{T}_h)\mathbf{f}, \mathbf{g})}{\|\mathbf{g}\|_{0,\Omega}} \leq Ch^{2r} \|\mathbf{f}\|_{1,\Omega}.$$

**Proof.** It only remains to prove that  $|b((\mathbf{T} - \mathbf{T}_h)\mathbf{f}, \mathbf{g})| \leq Ch^{2r} \|\mathbf{f}\|_{1,\Omega} \|\mathbf{g}\|_{0,\Omega} \forall \mathbf{g} \in L^2(\Omega)^2$ .

Let  $\mathbf{g} \in L^2(\Omega)^2$  and  $\mathbf{v} = \mathbf{T}\mathbf{g}$ . We define  $\mathbf{v}_h = \widehat{\mathbf{T}}_h \mathbf{E} \mathbf{P} \mathbf{g}$  and  $\check{\mathbf{v}}_h = \check{\mathbf{E}} \mathbf{v}_h = \mathbf{T}_h \mathbf{g}$ . Since  $\mathbf{v} \in H_{\Gamma_D}^1(\Omega)^2$ , there exists a bounded extension  $\mathbf{v}^e$  of  $\mathbf{v}$  satisfying  $\|\mathbf{v}^e\|_{1+r,\mathbb{R}^2} \leq C \|\mathbf{v}\|_{1+r,\Omega}$ .

Let  $\check{\mathbf{u}}_h$  and  $\check{\mathbf{v}}_h \in L_h(\Omega)$  be the functions associated with  $\mathbf{u}_h$  and  $\mathbf{v}_h$  in  $L_h(\Omega_h)$ , respectively. By definition of  $\mathbf{T}$  and  $\mathbf{T}_h$ , since  $a$  and  $b$  are symmetric, it holds that

$$\begin{aligned} b(\mathbf{u} - \check{\mathbf{u}}_h, \mathbf{g}) &= b(\mathbf{u} - \check{\mathbf{u}}_h, \mathbf{g}) + b(\check{\mathbf{u}}_h - \mathbf{u}_h, \mathbf{g}) \\ &= a(\mathbf{u} - \check{\mathbf{u}}_h, \mathbf{T}\mathbf{g}) + b(\check{\mathbf{u}}_h - \mathbf{u}_h, \mathbf{g}) \\ &= a(\mathbf{u} - \check{\mathbf{u}}_h, \mathbf{v}) + b(\check{\mathbf{u}}_h - \mathbf{u}_h, \mathbf{g}) \\ &= a(\mathbf{u} - \check{\mathbf{u}}_h, \mathbf{v} - \check{\mathbf{v}}_h) + a(\mathbf{u} - \check{\mathbf{u}}_h, \check{\mathbf{v}}_h) + b(\check{\mathbf{u}}_h - \mathbf{u}_h, \mathbf{g}). \end{aligned}$$

Thus, from the continuity of  $a$  and Lemma 3.5, we only have to estimate the two last terms in the right hand side. For the term with the bilinear form  $b$ , by using Lemma 3.2, (2.5), and the *a priori* estimates, we have

$$b(\check{\mathbf{u}}_h - \mathbf{u}_h, \mathbf{g}) = \int_{\Omega} (\check{\mathbf{u}}_h - \mathbf{u}_h) \cdot \mathbf{g} \, dx \leq \sum_{\tilde{T} \subset T \in \mathcal{T}_h^\partial} \|\check{\mathbf{u}}_h - \mathbf{u}_h\|_{0,\tilde{T}} \|\mathbf{g}\|_{0,\tilde{T}} + \|\mathbf{u}_h\|_{0,\Omega \setminus \overline{\Omega}_h} \|\mathbf{g}\|_{0,\Omega \setminus \overline{\Omega}_h}.$$

For the last term, we note that

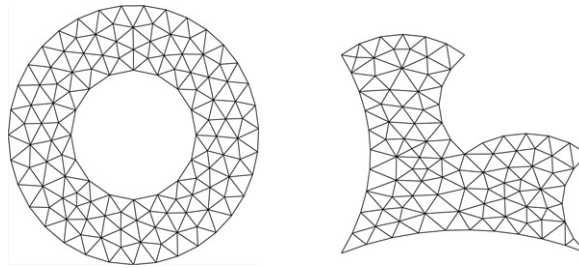
$$a(\mathbf{u} - \check{\mathbf{u}}_h, \check{\mathbf{v}}_h) = \left[ b(\mathbf{f}, \check{\mathbf{v}}_h) - b_h(\hat{\mathbf{f}}, \mathbf{v}_h) \right] + \left[ a_h(\mathbf{u}_h, \mathbf{v}_h) - a(\check{\mathbf{u}}_h, \check{\mathbf{v}}_h) \right],$$

and then, from (3.8) we only need to estimate the last term of this equation. Using the definition of bilinear form  $a$  we have

$$\begin{aligned} [a_h(\mathbf{u}_h, \mathbf{v}_h) - a(\check{\mathbf{u}}_h, \check{\mathbf{v}}_h)] &\leq [a_h(\mathbf{u}_h, \mathbf{v}_h) - a_h(\check{\mathbf{u}}_h, \check{\mathbf{v}}_h)] + \|\check{\mathbf{u}}_h\|_{1,\Omega \setminus \overline{\Omega}_h} \|\check{\mathbf{v}}_h\|_{1,\Omega \setminus \overline{\Omega}_h} \\ &\leq [a_h(\mathbf{u}_h, \mathbf{v}_h - \check{\mathbf{v}}_h) + a_h(\mathbf{u}_h - \check{\mathbf{u}}_h, \mathbf{v}_h) - a_h(\check{\mathbf{u}}_h - \mathbf{u}_h, \check{\mathbf{v}}_h - \mathbf{v}_h)] \\ &\quad + (\|\check{\mathbf{u}}_h - \mathbf{u}_h\|_{1,\Omega \setminus \overline{\Omega}_h} + \|\mathbf{u}_h\|_{1,\Omega \setminus \overline{\Omega}_h}) (\|\check{\mathbf{v}}_h - \mathbf{v}_h\|_{1,\Omega \setminus \overline{\Omega}_h} + \|\mathbf{v}_h\|_{1,\Omega \setminus \overline{\Omega}_h}). \end{aligned}$$

Note that  $a_h(\mathbf{u}_h, \mathbf{v}_h - \check{\mathbf{v}}_h) = a_h(\mathbf{u}_h - \check{\mathbf{u}}_h, \mathbf{u}_h) = 0$ , since  $\mathbf{u}_h$  (resp.  $\mathbf{v}_h$ ) is linear on each element and coincides with  $\check{\mathbf{u}}_h$  (resp.  $\check{\mathbf{v}}_h$ ) on the boundary of  $T$ , for all  $T \subset \Omega_h$ . So, we conclude the proof by using the uniform continuity of the bilinear form  $a_h$ , (2.5), Lemma 3.2 and Lemma 3.5.  $\square$

The main result of this work is stated below. For simplicity, we state the result for a simple eigenvalue; see [2] for the general statement. Let  $\mathcal{E}$  be the eigenspace corresponding to a simple eigenvalue  $\lambda$ .



**Fig. 2.** Computational domains used in the numerical test. A smooth domain (left) and a domain with corners (right).

**Table 1**

Numerical results for a domain with a smooth boundary

d.o.f.	3504	14 176	57 024	Order	Extrapolated
$\hat{\lambda}_1$	6.967603	6.967011	6.966855	1.9	6.966798
$\hat{\lambda}_2$	7.030892	7.031004	7.031026	2.34	7.031032
$\hat{\lambda}_3$	8.283649	8.282053	8.281645	1.95	8.281504
$\hat{\lambda}_4$	8.746009	8.736489	8.734077	1.96	8.733248
$\hat{\lambda}_5$	8.980803	8.971444	8.969049	1.95	8.968222
$\hat{\lambda}_6$	10.20135	10.17582	10.16923	1.94	10.16694

**Table 2**

Numerical results for a domain with a re-entrant corner

d.o.f.	4044	15 992	63 600	Order	Extrapolated
$\hat{\lambda}_1$	2.381894	2.378122	2.377061	1.84	2.376646
$\hat{\lambda}_2$	3.717265	3.696976	3.689189	1.39	3.684348
$\hat{\lambda}_3$	4.602764	4.590986	4.587292	1.69	4.585632
$\hat{\lambda}_4$	5.277851	5.257181	5.249713	1.48	5.245516
$\hat{\lambda}_5$	5.961521	5.946962	5.942464	1.71	5.940478
$\hat{\lambda}_6$	7.040018	7.014931	7.007612	1.79	7.004615

**Theorem 3.1.** Let  $\mu_i$  be the  $i$ -th (simple) eigenvalue of  $\mathbf{T}$  and  $\mu_{hi}$  the  $i$ -th eigenvalue of  $\mathbf{T}_h$ . Let  $\mathbf{u}_i$  and  $\mathbf{u}_{hi}$  be the corresponding eigenfunctions, normalized in the same way. Then there exists  $C > 0$  such that

$$\|\mathbf{u}_i - \mathbf{u}_{hi}\|_{j,\Omega} \leq Ch^{(2-j)r}, \quad j = 0, 1,$$

$$|\mu_i - \mu_{hi}| \leq Ch^{2r}.$$

#### 4. Numerical results

In this section we show the results obtained for two numerical tests that confirm our theoretical statements. We consider two different domains: one with smooth boundary (an annular domain) and the other with corners (L-shape domain).

In both cases, we have considered different mixed boundary conditions. For the annular region, we have considered Dirichlet conditions on all of the exterior boundary and Neumann conditions on the interior one. For the domain with corners we have considered Dirichlet conditions on the boundaries above and below and Neumann condition on the rest (cf. Fig. 2).

For the numerical examples, we have considered refined meshes like those shown in Fig. 2. Each mesh is identified by its total number of degrees of freedom (d.o.f.). The eigenfrequencies computed are scaled by  $\hat{\lambda} = \lambda \frac{2\rho(1+\nu)}{E}$ , where  $\nu$  and  $E$  are the physical properties of the body: Poisson's coefficient and Young's modulus, respectively.

Table 1 shows the corresponding results obtained for the annular domain. We also include the estimated order of convergence for each of the eigenfrequencies and an extrapolated more accurate approximation, both obtained by means of a least square fitting of the computed values. In this case  $r = 1$  and then all the eigenvalues have double order of convergence; in fact, the rate of convergence shown in Table 1 (in powers of  $h$ ) is clearly 2.

Table 2 shows the results for the curved domain with re-entrant corners. The orders of convergence obtained are once again in good agreement with those predicted by the theory; indeed, the domain has re-entrant corners with angle  $3\pi/2$  and then, following [4], 1.36 was an expected order; it can be seen that this order of convergence is almost attained.

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