



Extending the Newton–Kantorovich hypothesis for solving equations

Ioannis K. Argyros^{a,*}, Saïd Hilout^b

^a Cameron University, Department of Mathematics Sciences, Lawton, OK 73505, USA

^b Poitiers University, Laboratoire de Mathématiques et Applications, Bd. Pierre et Marie Curie, Téléport 2, B.P. 30179, 86962 Futuroscope Chasseneuil Cedex, France

ARTICLE INFO

Article history:

Received 13 October 2008

Received in revised form 24 December 2009

MSC:

65H10

65G99

65J15

47H17

49M15

Keywords:

Newton's method

Banach space

Semilocal convergence

Newton–Kantorovich hypothesis

Chandrasekhar-type nonlinear integral equation

Two boundary value problem with Green kernel

ABSTRACT

The famous Newton–Kantorovich hypothesis (Kantorovich and Akilov, 1982 [3], Argyros, 2007 [2], Argyros and Hilout, 2009 [7]) has been used for a long time as a sufficient condition for the convergence of Newton's method to a solution of an equation in connection with the Lipschitz continuity of the Fréchet-derivative of the operator involved. Here, using Lipschitz and center-Lipschitz conditions, and our new idea of recurrent functions, we show that the Newton–Kantorovich hypothesis can be weakened, under the same information. Moreover, the error bounds are tighter than the corresponding ones given by the dominating Newton–Kantorovich theorem (Argyros, 1998 [1]; [2,7]; Ezquerro and Hernández, 2002 [11]; [3]; Proinov 2009, 2010 [16,17]).

Numerical examples including a nonlinear integral equation of Chandrasekhar-type (Chandrasekhar, 1960 [9]), as well as a two boundary value problem with a Green's kernel (Argyros, 2007 [2]) are also provided in this study.

Published by Elsevier B.V.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \quad (1.1)$$

where F is a Fréchet-differentiable operator defined on a convex subset \mathcal{D} of a Banach space \mathcal{X} with values in a Banach space \mathcal{Y} .

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q , where x is the state. Then the equilibrium states are determined by solving Eq. (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative-when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases,

* Corresponding author. Tel.: +1 580 5368754; fax: +1 580 5812616.

E-mail addresses: iargyros@cameron.edu (I.K. Argyros), said.hilout@math.univ-poitiers.fr (S. Hilout).

the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

The famous Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}) \quad (1.2)$$

has long played a central role in approximating solutions x^* of nonlinear equations and systems. Here $F'(x_n)$ denotes the Fréchet-derivative of operator F evaluated at $x = x_n$ ($n \geq 0$) [1–3]. The geometric interpretation of Newton's method is well known, if F is a real function. In such a case x_{n+1} is the point where the tangential line $y - F(x_n) = F'(x_n)(x - x_n)$ of function $F(x_n)$ at the point $(x_n, F(x_n))$ intersects the x -axis. The geometric interpretation of the complex Newton method ($F : \mathbb{C} \rightarrow \mathbb{C}$) is given in [4].

There is much literature concerning the convergence of Newton's method as well as error estimates [1,5,6,2,7–13,3,14–18,4]. Among others, in the real case, Fourier studied the quadratic convergence of Newton's method in 1818, provided that a solution x^* of Eq. (1.1) exists [14]. In 1829, Cauchy first proved a semilocal convergence theorem which does not require any knowledge of the existence of a solution and asserted that the iterates (1.2) converge to a solution x^* if the initial guess x_0 satisfies certain conditions [4]. Ostrowski refined Fourier's and Cauchy's results for the case $\mathcal{X} = \mathbb{R}$ or $\mathcal{X} = \mathbb{C}$ [4].

For the general case when \mathcal{X}, \mathcal{Y} are Banach spaces, Kantorovich established a now famous and dominating semilocal convergence theorem for Newton's method which is called Kantorovich's or Newton–Kantorovich's theorem [3] (see Theorem 3.1 that follows) based on the famous Newton–Kantorovich hypothesis (see Condition 1.1). Three years later, he introduced the majorant principle to present a new proof [3]. His technique is so powerful that many authors have applied it to establish convergence theorems for variants of Newton's method, the so-called Newton-like methods [1,5,6,2,7–13,3,14–18,4].

Despite the fact that many decades have passed the Newton–Kantorovich hypothesis has not been challenged or improved. That is all results have been based or can be reduced to this hypothesis. Our new approach is to use center-Lipschitz (see Condition 1.2) instead of Lipschitz conditions for the bounds on $\|F'(x_n)^{-1} F'(x_0)\|$ (semilocal case) or $\|F'(x_n)^{-1} F'(x^*)\|$ (local case) ($n \geq 0$). This idea arises from the observation that under center-Lipschitz the bounds are more precise and cheaper to compute than in the case of Lipschitz conditions used so far.

Below we state sufficient convergence conditions for Newton's method (1.2).

Condition 1.1 ([1,2,11,3,16,17]). Let $F : \mathcal{D} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$ be a Fréchet-differentiable operator, $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ at point $x_0 \in \mathcal{D}$, and assume:

$$\begin{aligned} \|F'(x_0)^{-1} (F'(x) - F'(y))\| &\leq L \|x - y\| \quad \text{for all } x, y \in \mathcal{D}, \\ \|F'(x_0)^{-1} F(x_0)\| &\leq \eta, \\ 2 h_K = L \eta &\leq 1. \end{aligned}$$

We shall show a semilocal convergence theorem using:

Condition 1.2. Let $F : \mathcal{D} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$ be a Fréchet-differentiable operator, $F'(x_0) \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ at some point $x_0 \in \mathcal{D}$, and assume:

$$\begin{aligned} \|F'(x_0)^{-1} (F'(x) - F'(y))\| &\leq L \|x - y\| \quad \text{for all } x, y \in \mathcal{D}, \\ \|F'(x_0)^{-1} (F'(x) - F'(x_0))\| &\leq K \|x - x_0\| \quad \text{for all } x \in \mathcal{D}, \\ \|F'(x_0)^{-1} F(x_0)\| &\leq \eta, \\ 2 h_A = A \eta &\leq 1, \end{aligned}$$

where,

$$A = \frac{1}{4} \left(L + 4K + \sqrt{L^2 + 8KL} \right).$$

Under Conditions 1.1 or 1.2, we can obtain: error estimates, existence and uniqueness regions of solutions, and also know whether x_0 is a convergent initial point, i.e., Newton's method (1.2) starting at x_0 converges to x^* .

Note that in general

$$K \leq L \quad (1.3)$$

holds, and $\frac{1}{K}$ can be arbitrarily large [5,6,2] (see also Example 3.7).

By comparing h_A and h_K , we deduce

$$h_K \leq \frac{1}{2} \implies h_A \leq \frac{1}{2} \quad (1.4)$$

but not necessarily vice versa unless if $K = L$ (see also Examples in Section 3).

Moreover, if strict inequality holds in (1.3), then our estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ are tighter than the corresponding one given in the Newton–Kantorovich theorem (see also Section 3). Furthermore, note that in Condition 1.2, we use the same information (F, x_0, L) as in Condition 1.1, since in practice the computation of Lipschitz constant L requires that of center-Lipschitz constant K .

The paper is organized as follows: In Section 2 the semilocal convergence of Newton's method is given, whereas the applications, and numerical examples including a nonlinear integral equation of Chandrasekhar-type [9], as well as a two boundary value problem with a Green's kernel [2] are provided at the last Section 3.

2. Semilocal convergence analysis of Newton's method

We need the following result on majorizing sequences for Newton's method (1.2).

Lemma 2.1. Assume there exist parameters $\eta > 0$, $\delta \in (0, 2)$, and positive sequences $\{K_n\}$, $\{L_n\}$ ($n \geq 1$) such that for all $n \geq 1$:

$$(L_1 + \delta K_1)\eta \leq \delta, \quad (2.1)$$

and

$$\left[L_{n+1} \left(\frac{\delta}{2} \right)^n + K_{n+1} \frac{1 - \left(\frac{\delta}{2} \right)^{n+1}}{1 - \frac{\delta}{2}} \delta \right] \eta \leq \delta. \quad (2.2)$$

Then, sequence $\{t_n\}$ ($n \geq 0$) given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{L_{n+1}(t_{n+1} - t_n)^2}{2(1 - K_{n+1}t_{n+1})} \quad (n \geq 0), \quad (2.3)$$

is well defined, and converges to its unique least upper bound $t^* \in [0, t^{**}]$, where

$$t^{**} = \frac{2\eta}{2 - \delta}. \quad (2.4)$$

Moreover the following estimates hold for all $n \geq 0$:

$$0 < t_{n+2} - t_{n+1} \leq \frac{\delta}{2}(t_{n+1} - t_n) \leq \cdots \leq \left(\frac{\delta}{2} \right)^{n+1} \eta. \quad (2.5)$$

Proof. We shall show using induction on k that for all $k \geq 0$:

$$L_{k+1}(t_{k+1} - t_k) + \delta K_{k+1}t_{k+1} < \delta, \quad (2.6)$$

$$0 < t_{k+1} - t_k, \quad (2.7)$$

$$K_{k+1}t_{k+1} < 1, \quad (2.8)$$

and

$$0 < t_{k+2} < t^{**}. \quad (2.9)$$

Estimates (2.6)–(2.8) hold true for $k = 0$ by the initial condition $t_1 = \eta$, and hypothesis (2.1). It then follows from (2.3) that

$$0 < t_2 - t_1 \leq \frac{\delta}{2}(t_1 - t_0) \quad \text{and} \quad t_2 \leq \eta + \frac{\delta}{2}\eta = \frac{2 + \delta}{2}\eta < t^{**}.$$

Let us assume estimates (2.5)–(2.9) hold true for all integer values k : $k \leq n + 1$ ($n \geq 0$).

We have in turn:

$$L_{k+2}(t_{k+2} - t_{k+1}) + \delta K_{k+2}t_{k+2} \leq L_{k+2} \left(\frac{\delta}{2} \right)^{k+1} \eta + K_{k+2} \delta \frac{1 - \left(\frac{\delta}{2} \right)^{k+2}}{1 - \frac{\delta}{2}} \eta \leq \delta, \quad (2.10)$$

(by (2.2)), which shows (2.5)–(2.8) for $k = n + 2$.

We also get

$$\begin{aligned} t_{k+2} &\leq t_{k+1} + \frac{\delta}{2}(t_{k+1} - t_k) \\ &\leq t_k + \frac{\delta}{2}(t_k - t_{k-1}) + \frac{\delta}{2}(t_{k+1} - t_k) \end{aligned}$$

$$\begin{aligned}
&\leq \eta + \left(\frac{\delta}{2}\right)\eta + \cdots + \left(\frac{\delta}{2}\right)^{k+1}\eta \\
&= \frac{1 - \left(\frac{\delta}{2}\right)^{k+2}}{1 - \frac{\delta}{2}}\eta < \frac{2}{2 - \delta}\eta = t^{**}.
\end{aligned} \tag{2.11}$$

The induction is completed.

It then follows from (2.7)–(2.9) that sequences $\{t_n\}$ is well defined, and converges to some $t^* \in [0, t^{**}]$.

That completes the proof of Lemma 2.1. \square

Remark 2.2. Delicate condition (2.2) is not easy to verify in general. However, we wanted to leave Lemma 2.1 as uncluttered as possible. In what follows, we provide some natural choices of sequences $\{K_n\}$ and $\{L_n\}$ as well as a parameter $\delta \in (0, 2)$ for which conditions (2.1) and (2.2) hold true.

Proposition 2.3. Assume there exist parameters $\eta > 0, K > 0, L > 0$, and positive sequences $\{K_n\}, \{L_n\}$ ($n \geq 1$), such that for all $n \geq 1$:

$$K_n \leq K_1 = K, \tag{2.12}$$

$$L_n \leq L_1 = L, \tag{2.13}$$

and

$$2h_0 = b\eta \leq 1, \tag{2.14}$$

where,

$$b = \frac{L + 4K + \sqrt{L^2 + 8KL}}{4}.$$

Then, sequence $\{t_n\}$ ($n \geq 0$) given by (2.3) is well defined, and converges to its unique least upper bound $t^* \in [0, t^{**}]$, where,

$$t^{**} = \frac{2\eta}{2 - \delta_0}, \tag{2.15}$$

and

$$1 \leq \delta_0 = \frac{-a + \sqrt{a^2 + 8a}}{2} < 2, \quad a = \frac{L}{K}. \tag{2.16}$$

Moreover, the following estimates hold for all $n \geq 0$:

$$0 < t_{n+2} - t_{n+1} \leq \frac{\delta_0}{2}(t_{n+1} - t_n) \leq \cdots \leq \left(\frac{\delta_0}{2}\right)^{n+1}\eta. \tag{2.17}$$

Proof. We shall first show that condition (2.1) holds for $\delta = \delta_0$.

Condition (2.14) can be written as

$$\frac{(a + \sqrt{a^2 + 8a})^2 K \eta}{a^2 + 8a - a^2} \leq 1 \tag{2.18}$$

\implies

$$\frac{a + \sqrt{a^2 + 8a}}{-a + \sqrt{a^2 + 8a}} K \eta \leq 1 \tag{2.19}$$

\implies

$$\left(a + \frac{-a + \sqrt{a^2 + 8a}}{2}\right) K \eta \leq \frac{-a + \sqrt{a^2 + 8a}}{2} \tag{2.20}$$

$\implies (L + \delta_0 K)\eta \leq \delta_0 \implies$ (2.1) (by (2.12) and (2.13) for $n = 1$).

We shall next show condition (2.2). It follows from (2.2) and (2.1) that it suffices to show:

$$e_n := L \left(\frac{\delta_0}{2}\right)^n + K \frac{1 - \left(\frac{\delta_0}{2}\right)^{n+1}}{1 - \frac{\delta_0}{2}} \delta \leq L + \delta_0 K \tag{2.21}$$

or

$$\delta_0 K \left\{ \frac{2}{2 - \delta_0} \left(1 - \left(\frac{\delta_0}{2} \right)^{n+1} \right) - 1 \right\} \leq L \left(1 - \left(\frac{\delta_0}{2} \right)^n \right) \quad (2.22)$$

or

$$\left(\frac{K \delta_0^2}{2 - \delta_0} - L \right) \left(1 - \left(\frac{\delta_0}{2} \right)^n \right) \leq 0, \quad (2.23)$$

which is true by the choice of δ_0 .

The result now follows from Lemma 2.1 for $\delta = \delta_0$.

That completes the proof of Proposition 2.3. \square

Under the hypotheses of Proposition 2.3 to obtain (2.2) we showed instead (2.21).

It turns out that to show weaker condition than (2.21):

$$e_n \eta \leq \delta_0 \quad \text{for all } n \geq 0 \quad (2.24)$$

still requires the assumption of condition (2.14):

Proposition 2.4. Under the hypotheses of Proposition 2.3, condition (2.24) holds.

Proof. Estimate (2.24) can be rewritten as:

$$\left\{ L \left(\frac{\delta_0}{2} \right)^n + \delta K \frac{1 - \left(\frac{\delta_0}{2} \right)^{n+1}}{1 - \frac{\delta_0}{2}} \right\} \eta \leq \delta_0. \quad (2.25)$$

Estimate (2.25) motivates us to define for $s = \frac{\delta_0}{2}$, the sequence $\{f_n\}$ of polynomials on $[0, +\infty)$ by

$$f_n(s) = \left(L s^{n-1} + 2 K (1 + s + s^2 + \cdots + s^n) \right) \eta - 2. \quad (2.26)$$

We first find the relationship between two consecutive f_n 's.

$$\begin{aligned} f_{n+1}(s) &= \left(L s^n - L s^{n-1} + L s^{n-1} + 2 K (1 + s + s^2 + \cdots + s^{n+1}) \right) \eta - 2 \\ &= \left(L s^n + L s^{n-1} - L s^{n-1} + 2 K (1 + s + s^2 + \cdots + s^n) + 2 K s^{n+1} \right) \eta - 2 \\ &= \left(L s^n - L s^{n-1} + 2 K s^{n+1} \right) \eta + f_n(s), \end{aligned}$$

so,

$$f_{n+1}(s) = g(s) s^{n-1} \eta + f_n(s) \quad (n \geq 1), \quad (2.27)$$

where,

$$g(s) = 2 K s^2 + L s - L. \quad (2.28)$$

Note that $\frac{\delta_0}{2}$ given by (2.16) is the only positive root of polynomial g .

We shall show

$$f_n \left(\frac{\delta_0}{2} \right) \leq 0 \quad \text{for all } n \geq 1. \quad (2.29)$$

Using (2.16), (2.27) and (2.28), we get

$$f_n \left(\frac{\delta_0}{2} \right) = f_{n-1} \left(\frac{\delta_0}{2} \right) = \cdots = f_1 \left(\frac{\delta_0}{2} \right). \quad (2.30)$$

It follows from (2.29) and (2.30), that we only need to show

$$f_1 \left(\frac{\delta_0}{2} \right) \leq 0, \quad (2.31)$$

which is true by (2.26) (for $n = 1$), and (2.14).

Define:

$$f_{\infty}(s) = \lim_{n \rightarrow \infty} f_n(s) \quad s \in [0, 1). \quad (2.32)$$

Then, we have:

$$f_{\infty}\left(\frac{\delta_0}{2}\right) = \lim_{n \rightarrow \infty} f_n\left(\frac{\delta_0}{2}\right) \leq 0. \quad (2.33)$$

It can easily be seen that (2.1) holds for $\delta = \delta_0$, $L_1 = L$, and $K_1 = K$.

That completes the induction, and the proof of Proposition 2.4. \square

Below is the main semilocal convergence theorem for Newton's method (1.2).

Theorem 2.5. Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet-differentiable operator. Assume the hypotheses of Lemma 2.1 hold, and there exist $x_0 \in \mathcal{D}$, a parameter $\eta \geq 0$, functions $K : \mathcal{D}^2 \longrightarrow [0, +\infty)$, and $L : \mathcal{D}^3 \longrightarrow [0, +\infty)$, such that

$$F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \quad (2.34)$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta, \quad (2.35)$$

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq K(x_0, x)\|x - x_0\| \quad \text{for all } x \in \mathcal{D}, \quad (2.36)$$

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq L(x_0, x, y)\|x - y\| \quad \text{for all } x, y \in \mathcal{D}, \quad (2.37)$$

$$\overline{U}(x_0, t^{**}) = \{x \in \mathcal{X}, \|x - x_0\| \leq t^{**}\} \subseteq \mathcal{D}, \quad (2.38)$$

where, t^{**} is given in Lemma 2.1.

Set

$$K_n = K(x_0, x_n), \quad \text{and} \quad L_n = L(x_0, x_{n-1}, x_n) \quad (n \geq 1).$$

Then sequence $\{x_n\}$ defined by Newton's method (1.2) is well defined, remains in $\overline{U}(x_0, t^*)$ for all $n \geq 0$, and converges to a unique solution $x^* \in \overline{U}(x_0, t^*)$ of equation $F(x) = 0$ in $\overline{U}(x_0, t^*)$.

Moreover the following estimates hold for all $n \geq 0$:

$$\|x_{n+2} - x_{n+1}\| \leq \frac{L_{n+1}\|x_{n+1} - x_n\|^2}{2(1 - K_{n+1}\|x_{n+1} - x^*\|)} \leq t_{n+2} - t_{n+1}, \quad (2.39)$$

$$\|x_n - x^*\| \leq t^* - t_n, \quad (2.40)$$

where, iteration $\{t_n\}$ ($n \geq 0$) is given by (2.3).

Furthermore, if there exists $R > t^*$, such that

$$U(x_0, R) \subseteq \mathcal{D} \quad (2.41)$$

and

$$K(x_0, x)(t^* + R) \leq 2 \quad \text{for all } x \in U(x_0, R), \quad (2.42)$$

then, the solution is unique in $U(x_0, R)$.

Proof. Let us prove:

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad (2.43)$$

and

$$\overline{U}(x_{k+1}, t^* - t_{k+1}) \subseteq \overline{U}(x_k, t^* - t_k) \quad (2.44)$$

hold for all $k \geq 0$.

For every $z \in \overline{U}(x_1, t^* - t_1)$,

$$\begin{aligned} \|z - x_0\| &\leq \|z - x_1\| + \|x_1 - x_0\| \\ &\leq t^* - t_1 + t_1 = t^* - t_0, \end{aligned} \quad (2.45)$$

implies $z \in \overline{U}(x_0, t^* - t_0)$. Since also

$$\|x_1 - x_0\| = \|F'(x_0)^{-1} F(x_0)\| \leq \eta = t_1 - t_0,$$

estimates (2.43) and (2.44) hold for $k = 0$.

Given they hold for $n = 0, 1, \dots, k$, then we have :

$$\begin{aligned} \|x_{k+1} - x_0\| &\leq \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \\ &\leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} \end{aligned}$$

and

$$\|x_k + \theta (x_{k+1} - x_k) - x_0\| \leq t_k + \theta (t_{k+1} - t_k) \leq t^*,$$

for all $\theta \in [0, 1]$.

Using (1.2), we obtain the approximation

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) \\ &= \int_0^1 [F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)](x_{k+1} - x_k) d\theta \end{aligned} \quad (2.46)$$

and by (2.37)

$$\begin{aligned} \|F'(x_0)^{-1} F(x_{k+1})\| &\leq \int_0^1 \|F'(x_0)^{-1} [F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)] (x_{k+1} - x_k)\| d\theta \|x_{k+1} - x_k\| \\ &\leq \frac{K_{k+1}}{2} \|x_{k+1} - x_k\|^2 \leq \frac{K_{k+1}}{2} (t_{k+1} - t_k)^2. \end{aligned} \quad (2.47)$$

It follows from (2.36)

$$\begin{aligned} \|F'(x_0)^{-1} (F'(x_{k+1}) - F'(x_0))\| &\leq K_{k+1} \|x_{k+1} - x_0\| \\ &\leq K_{k+1} \|x_{k+1} - x_k\| \leq K_{k+1} t_{k+1} \end{aligned}$$

and by the Banach lemma on invertible operators [3] that $F'(x_{k+1})^{-1}$ exists, and

$$\begin{aligned} \|F'(x_{k+1})^{-1} F'(x_0)\| &\leq (1 - K_{k+1} \|x_{k+1} - x_0\|)^{-1} \\ &\leq (1 - K_{k+1} t_{k+1})^{-1}. \end{aligned} \quad (2.48)$$

Therefore, by (1.2), (2.47) and (2.48), we obtain in turn

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &= \|F'(x_{k+1})^{-1} F(x_{k+1})\| \\ &\leq \|F'(x_{k+1})^{-1} F'(x_0)\| \|F'(x_0)^{-1} F(x_{k+1})\| \\ &\leq \frac{K_{k+1} \|x_{k+1} - x_k\|^2}{2 (1 - K_{k+1} \|x_{k+1} - x_k\|)} \leq \frac{K_{k+1} (t_{k+1} - t_k)^2}{2 (1 - K_{k+1} t_{k+1})} = t_{k+2} - t_{k+1}. \end{aligned} \quad (2.49)$$

Thus for every $z \in \overline{U}(x_{k+2}, t^* - t_{k+2})$, we have:

$$\begin{aligned} \|z - x_{k+1}\| &\leq \|z - x_{k+2}\| + \|x_{k+2} - x_{k+1}\| \\ &\leq t^* - t_{k+2} + t_{k+2} - t_{k+1} = t^* - t_{k+1}. \end{aligned}$$

That is,

$$z \in \overline{U}(x_{k+1}, t^* - t_{k+1}). \quad (2.50)$$

Estimates (2.49) and (2.50) imply that (2.43) and (2.44) hold for $n = k + 1$. By induction the proof of (2.43) and (2.44) is completed.

Lemma 2.1 implies that sequence $\{t_n\}$ is a Cauchy sequence. From (2.43) and (2.44) $\{x_n\}$ ($n \geq 0$) become a Cauchy sequence too, and as such it converges to some $x^* \in \overline{U}(x_0, t^*)$ (since $\overline{U}(x_0, t^*)$ is a closed set) such that

$$\|x^* - x_k\| \leq t^* - t_k. \quad (2.51)$$

The combination of (2.47) and (2.51) yields $F(x^*) = 0$. Estimate (2.40) follows from (2.39) by using standard majorization techniques [2,7,3]. Finally to show uniqueness: let y^* be a solution of equation $F(x) = 0$ in $U(x_0, R)$. It follows from (2.36), (2.41), (2.42), for $x = y^* + \theta(x^* - y^*)$, $\theta \in [0, 1]$ the estimate

$$\begin{aligned}
& \left\| F'(x_0)^{-1} \int_0^1 (F'(y^* + \theta(x^* - y^*)) - F'(x_0)) d\theta \right\| \leq K(x_0, y^* + \theta(x^* - y^*)) \int_0^1 \|y^* + \theta(x^* - y^*) - x_0\| d\theta \\
& \leq K(x_0, y^* + \theta(x^* - y^*)) \int_0^1 (\theta \|x^* - x_0\| + (1 - \theta) \|y^* - x_0\|) d\theta \\
& \leq \frac{K(x_0, y^* + \theta(x^* - y^*))}{2} (t^* + R) \leq 1,
\end{aligned}$$

and the Banach lemma on invertible operators implies that the linear operator

$$\mathcal{M} = \int_0^1 F'(y^* + \theta(x^* - y^*)) d\theta$$

is invertible.

Using the identity: $0 = F(x^*) - F(y^*) = \mathcal{M}(x^* - y^*)$, we deduce $x^* = y^*$.

The uniqueness in $U(x_0, t^{**})$ follows as above by setting $t^* = R$. That completes the proof of Theorem 2.5. \square

3. Special cases and applications

Case 1: Functions K and L are constants

In order to compare the results of Section 2 with the famous Newton–Kantorovich theorem, we recall it below:

Theorem 3.1 ([3]). Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet-differentiable operator. Assume there exist a point $x_0 \in \mathcal{D}$, and parameters $\eta > 0, L > 0$, such that (2.34), (2.35), (2.37),

$$\begin{aligned}
2h &= L\eta \leq 1, \\
\bar{U}(x_0, s^*) &\subseteq \mathcal{D},
\end{aligned} \tag{3.1}$$

hold, where,

$$s^* = \frac{1 - \sqrt{1 - 2h}}{L}. \tag{3.2}$$

Then, sequence $\{x_n\}$ ($n \geq 0$) generated by Newton's method (1.2) is well defined, remains in $\bar{U}(x_0, s^*)$ for all $n \geq 0$, and converges to a unique solution x^* of equation $F(x) = 0$ in $U(x_0, s^*)$.

Moreover the following estimates hold:

$$\|x_{n+1} - x_n\| \leq \frac{L\|x_n - x_{n-1}\|^2}{2(1 - L\|x_n - x_0\|)} \leq s_n - s_{n-1}, \quad (n \geq 1), \tag{3.3}$$

$$\|x_n - x^*\| \leq s^* - s_n, \quad (n \geq 0), \tag{3.4}$$

$$0 \leq s_{n+1} - s_n = \frac{\frac{1}{2}Ls_n^2 - s_n + \eta}{1 - Ls_n} = \frac{\frac{1}{2}L(s_n - s_{n-1})^2}{1 - Ls_n}, \quad (n \geq 1), \tag{3.5}$$

and

$$s^* - s_{n+1} = \frac{\frac{1}{2}L(s^* - s_n)^2}{1 - Ls_n} \leq \frac{1}{L2^{n+1}} h^{2^{n+1}}, \quad (n \geq 0) \text{ (for } h < 1). \tag{3.6}$$

Remark 3.2. Hypothesis (3.1) is famous for its simplicity and clarity. The Newton–Kantorovich hypothesis is used for solving nonlinear equations using Newton's method. In view of (2.14) and (3.1), we have:

$$2h \leq 1 \implies 2h_0 \leq 1$$

but not necessarily vice versa, unless if $K = L$. Hence, we have extended the applicability of Newton's method under the same computational cost, since in both Theorems 2.5 and 3.1, the same information (F, x_0, L) is used. Note also that the computation of constant L requires that of K . The recent result in [11,16,17] also used (3.1), instead of (2.14).

In the next three results, we compare the error bounds given in Theorems 2.5 and 3.1. In the first result, we provide more estimates on the distances $t_{n+1} - t_n$ and $t^* - t_n$ ($n \geq 0$). The proof can be found in [7,8].

Proposition 3.3. Under the hypotheses of Proposition 2.3, the following estimates hold for all $n \geq 0$:

$$t_{n+1} - t_n \leq \left(\frac{\delta_0}{2}\right)^n (2h_0)^{2^n - 1} \eta$$

and

$$t^* - t_n \leq \left(\frac{\delta_0}{2}\right)^n \frac{(2h_0)^{2^n-1}\eta}{1 - (2h_0)^{2^n}}, \quad (2h_0 < 1). \quad (3.7)$$

If $K = L$, then $h = h_0$. Otherwise, $h_0 < h$, and our error bounds are tighter.

Proposition 3.4. Under hypotheses of [Theorems 2.5](#) (for $K < L$) and [3.1](#), the following estimates hold:

$$t_{n+1} < s_{n+1} \quad (n \geq 1), \quad (3.8)$$

$$t_{n+1} - t_n < s_{n+1} - s_n \quad (n \geq 1), \quad (3.9)$$

$$t^* - t_n < s^* - s_n \quad (n \geq 0), \quad (3.10)$$

and

$$t^* \leq s^*. \quad (3.11)$$

Moreover we have: $t_n = s_n$ ($n \geq 0$) if $L = K$. Furthermore, there exists a finite integer N_0 such that the upper bound in [\(3.7\)](#) is smaller than the upper bound in [\(3.6\)](#) for all $n \geq N_0$, since $h_0 < h$.

Proof. We use induction on the integer k to show [\(3.8\)](#) and [\(3.9\)](#). For $n = 0$ in [\(2.3\)](#), we obtain

$$t_2 - \eta = \frac{L\eta^2}{2(1-K\eta)} \leq \frac{L\eta^2}{2(1-L\eta)} = s_2 - s_1$$

and

$$t_2 < s_2.$$

Assume:

$$t_{k+1} < s_{k+1}, \quad t_{k+1} - t_k < s_{k+1} - s_k \quad (k \leq n+1).$$

Using [\(2.3\)](#) and [\(3.5\)](#), we get

$$t_{k+2} - t_{k+1} = \frac{\frac{L}{2}(t_{k+1} - t_k)^2}{1 - K t_{k+1}} \leq \frac{\frac{L}{2}(s_{k+1} - s_k)^2}{1 - L s_{k+1}} = s_{k+2} - s_{k+1}.$$

Let $m \geq 0$, we can obtain

$$\begin{aligned} t_{k+m} - t_k &< (t_{k+m} - t_{k+m-1}) + (t_{k+m-1} - t_{k+m-2}) + \cdots + (t_{k+1} - t_k) \\ &< (s_{k+m} - s_{k+m-1}) + (s_{k+m-1} - s_{k+m-2}) + \cdots + (s_{k+1} - s_k) \\ &< s_{k+m} - s_k. \end{aligned} \quad (3.12)$$

By letting $m \rightarrow \infty$ in [\(3.12\)](#), we obtain [\(3.10\)](#). For $n = 1$ in [\(3.10\)](#), we get [\(3.11\)](#).

That completes the proof of [Proposition 3.4](#). \square

Remark 3.5. In view of the proof of [Theorem 2.5](#), majorizing sequence $\{t_n\}$ can be replaced by the tighter $\{\bar{t}_n\}$ given by:

$$\bar{t}_0 = 0, \quad \bar{t}_1 = \eta, \quad \bar{t}_{n+2} = \bar{t}_{n+1} + \frac{L^*(\bar{t}_{n+1} - \bar{t}_n)^2}{2(1 - K \bar{t}_{n+1})} \quad (n \geq 0),$$

where,

$$L^* = \begin{cases} K & \text{if } n = 0 \\ L & \text{if } n > 0. \end{cases}$$

Moreover $\frac{L}{K}$ can be arbitrarily large. Indeed:

Remark 3.6. It follows from the above three results that not only [\(2.14\)](#) can always replace stronger [\(3.1\)](#), but our estimates are also tighter.

Example 3.7. Define the scalar function F by $F(x) = c_0 x + c_1 + c_2 \sin e^{c_3 x}$, $x_0 = 0$, where c_i , $i = 0, 1, 2, 3$ are given parameters. Then it can easily be seen that for c_3 large and c_2 sufficiently small, $\frac{L}{K}$ can be arbitrarily large. That is [\(2.14\)](#) may be satisfied but not [\(3.1\)](#).

Example 3.8. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, be equipped with the max-norm, $x_0 = (1, 1)^T$, $U_0 = \{x : \|x - x_0\| \leq 1 - \beta\}$, $\beta \in \left[0, \frac{1}{2}\right)$, and define function F on U_0 by

$$F(x) = (w^3 - \beta, z^3 - \beta)^T, \quad x = (w, z)^T. \quad (3.13)$$

The Fréchet-derivative of operator F is given by

$$F'(x) = \begin{bmatrix} 3w^2 & 0 \\ 0 & 3z^2 \end{bmatrix}. \quad (3.14)$$

Using (2.35)–(2.37), we get:

$$\eta = \frac{1}{3}(1 - \beta), \quad K = 3 - \beta, \quad \text{and} \quad L = 2(2 - \beta).$$

The Kantorovich condition (3.1) is violated, since

$$\frac{4}{3}(1 - \beta)(2 - \beta) > 1 \quad \text{for all } \beta \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton's method (1.2) converges to $x^* = (\sqrt[3]{\beta}, \sqrt[3]{\beta})^T$, starting at x_0 .

However, our condition (2.14) is true for all $\beta \in I = \left[.450339002, \frac{1}{2}\right)$. Hence, the conclusions of our Theorem 2.5 can apply to solve Eq. (3.13) for all $\beta \in I$.

Example 3.9. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be the space of real-valued continuous functions defined on the interval $[0, 1]$ with norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)|.$$

Let $\theta \in [0, 1]$ be a given parameter. Consider the “Cubic” integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s, t) u(t) dt + y(s) - \theta. \quad (3.15)$$

Here the kernel $q(s, t)$ is a continuous function of two variables defined on $[0, 1] \times [0, 1]$; the parameter λ is a real number called the “albedo” for scattering; $y(s)$ is a given continuous function defined on $[0, 1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0, 1]$. Equations of the form (3.15) arise in the kinetic theory of gasses [2,9]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s, t) = \frac{s}{s+t}$, for all $s \in [0, 1]$, and $t \in [0, 1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1 - \theta)$, and define the operator F on \mathcal{D} by

$$F(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 q(s, t) x(t) dt + y(s) - \theta, \quad (3.16)$$

for all $s \in [0, 1]$, then every zero of F satisfies Eq. (3.15).

We have the estimates

$$\max_{0 \leq s \leq 1} \left| \int \frac{s}{s+t} dt \right| = \ln 2.$$

Therefore, if we set $\xi = \|F'(u_0)^{-1}\|$, then it follows from (2.35)–(2.37) that

$$\eta = \xi(|\lambda| \ln 2 + 1 - \theta), \\ L = 2\xi(|\lambda| \ln 2 + 3(2 - \theta)) \quad \text{and} \quad K = \xi(2|\lambda| \ln 2 + 3(3 - \theta)).$$

It follows from Theorem 2.5 that if condition (2.14) holds, then problem (3.15) has a unique solution near u_0 . This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis (3.1).

Note also that $K < L$ for all $\theta \in [0, 1]$.

Example 3.10. Consider the following nonlinear boundary value problem [2]

$$\begin{cases} u'' = -u^3 - \gamma u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 Q(s, t) (u^3(t) + \gamma u^2(t)) dt \quad (3.17)$$

where, Q is the Green function:

$$Q(s, t) = \begin{cases} t(1-s), & t \leq s \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \leq s \leq 1} \int_0^1 |Q(s, t)| dt = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$, with norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)|.$$

Then problem (3.17) is in the form (1.1), where, $F : \mathcal{D} \rightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 Q(s, t)(x^3(t) + \gamma x^2(t)) dt.$$

It is easy to verify that the Fréchet derivative of F is defined in the form

$$[F'(x)v](s) = v(s) - \int_0^1 Q(s, t)(3x^2(t) + 2\gamma x(t)) v(t) dt.$$

If we set $u_0(s) = s$, and $\mathcal{D} = U(u_0, R)$, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R + 1)$. It follows that $2\gamma < 5$, then

$$\|I - F'(u_0)\| \leq \frac{3\|u_0\|^2 + 2\gamma\|u_0\|}{8} = \frac{3 + 2\gamma}{8},$$

$$\|F'(u_0)^{-1}\| \leq \frac{1}{1 - \frac{3+2\gamma}{8}} = \frac{8}{5 - 2\gamma},$$

$$\|F(u_0)\| \leq \frac{\|u_0\|^3 + \gamma\|u_0\|^2}{8} = \frac{1 + \gamma}{8},$$

$$\|F(u_0)^{-1}F(u_0)\| \leq \frac{1 + \gamma}{5 - 2\gamma}.$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v](s) = - \int_0^1 Q(s, t)(3x^2(t) - 3y^2(t) + 2\gamma(x(t) - y(t))) v(t) dt.$$

Consequently,

$$\begin{aligned} \|F'(x) - F'(y)\| &\leq \frac{\|x - y\|(2\gamma + 3(\|x\| + \|y\|))}{8} \\ &\leq \frac{\|x - y\|(2\gamma + 6R + 6\|u_0\|)}{8} \\ &= \frac{\gamma + 6R + 3}{4} \|x - y\|, \\ \|F'(x) - F'(u_0)\| &\leq \frac{\|x - u_0\|(2\gamma + 3(\|x\| + \|u_0\|))}{8} \\ &\leq \frac{\|x - u_0\|(2\gamma + 3R + 6\|u_0\|)}{8} \\ &= \frac{2\gamma + 3R + 6}{8} \|x - u_0\|. \end{aligned}$$

Therefore, conditions of Theorem 2.5 hold with

$$\eta = \frac{1 + \gamma}{5 - 2\gamma}, \quad L = \frac{\gamma + 6R + 3}{4}, \quad K = \frac{2\gamma + 3R + 6}{8}.$$

Note also that $K < L$.

Case 2: Functions K and L are not constants

(i) Let us introduce conditions

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta, \tag{3.18}$$

$$\|F'(x_0)^{-1}F''(x_0)\| \leq \eta_0, \tag{3.19}$$

$$\|F'(x_0)^{-1} (F'(x) - F'(x_0))\| \leq K \|x - x_0\|, \quad (3.20)$$

$$\|F'(x_0)^{-1} (F''(x) - F''(x_0))\| \leq M_0 \|x - x_0\|, \quad (3.21)$$

$$\|F'(x_0)^{-1} (F''(x) - F''(y))\| \leq M \|x - y\|, \quad (3.22)$$

for all $x, y \in \bar{U}(x_0, r)$ ($r > 0$).

Moreover define iteration $\{w_n\}$ as follows

$$w_0 = 0, \quad w_1 = \eta, \quad w_{n+2} = w_{n+1} + \frac{L_{n+1} (w_{n+1} - t_n)^2}{2(1 - K w_{n+1})}, \quad (3.23)$$

where

$$\begin{aligned} \bar{L}_n &= \frac{1}{3} M \|x_n - x_{n-1}\| + M_0 \|x_{n-1} - x_0\| + 2 \eta_0 \\ &\leq L_n = \frac{1}{3} M (w_n - w_{n-1}) + M_0 w_{n-1} + 2 \eta_0 \\ &\leq L(r) = L = \frac{1}{3} M \eta + M_0 r + 2 \eta_0. \end{aligned} \quad (3.24)$$

If for the above choices of K and L , there exists $r > 0$ satisfying condition (2.14) and $\bar{U}(x_0, r) \subseteq \mathcal{D}$, then the conclusions of Theorem 2.5 hold true with r, w_n replacing t^*, t_n , respectively.

Indeed, the proof of Theorem 2.5 goes through if approximation (2.46) is replaced by

$$\begin{aligned} F(x_{k+1}) &= \int_0^1 [F''(x_k + \theta(x_{k+1} - x_k)) - F''(x_k)] (1 - \theta)(x_{k+1} - x_k)^2 d\theta \\ &\quad + \int_0^1 [F''(x_k) - F''(x_0)] (1 - \theta)(x_{k+1} - x_k)^2 d\theta + F''(x_0)(x_{k+1} - x_k)^2, \end{aligned}$$

and (3.18), (3.20), (3.22) are used to arrive at (3.23).

(ii) Using (3.20) instead of (3.19) (see also [2,12,13]), we can replace K in the denominator of (3.23) by

$$\begin{aligned} \bar{K}_n &= \frac{1}{2} M_0 \|x_n - x_0\|^2 + \eta_0 \|x_{n-1} - x_0\| \\ &\leq K_n = \frac{1}{2} M_0 (w_n - w_0)^2 + \eta_0 (w_n - w_0) \\ &\leq K(r) = K = \left(\frac{1}{2} M_0 r + \eta_0 \right) r. \end{aligned} \quad (3.25)$$

Simply use instead of the estimate above (2.48), the approximation

$$F'(x) - F'(x_0) = \int_0^1 [F''(x_0 + \theta(x - x_0)) - F''(x_0)](x - x_0) d\theta + F''(x_0)(x - x_0),$$

in combination with (3.19) and (3.20). The rest follows as in sub-case (i).

It turns out that condition (2.14) can further be weakened using the same information.

Remark 3.11. (a) Let $\mathcal{D} = U_0 = \bar{U}(x_0, R_0)$, where $R_0 \geq \eta$. Set $R_1 = R_0 - \eta$, and define $U_1 = U(x_1, R_1)$, where $x_1 = x_0 - F'(x_0)^{-1} F(x_0)$.

Introduce condition:

$$\|F'(x_0)^{-1} (F'(x) - F'(y))\| \leq M \|x - y\|, \quad \text{for all } x, y \in U_1. \quad (3.26)$$

Then, we have:

$$\begin{aligned} M &= \sup_{\substack{x \neq y \\ x, y \in U_1}} \frac{\|F'(x_0)^{-1} (F'(x) - F'(y))\|}{\|x - y\|} \\ &\leq \sup_{\substack{x \neq y \\ x, y \in U_0}} \frac{\|F'(x_0)^{-1} (F'(x) - F'(y))\|}{\|x - y\|} = L. \end{aligned} \quad (3.27)$$

It follows from the proof of Theorem 2.5, that scalar sequence $\{z_n\}$ given by:

$$z_0 = 0, \quad z_1 = \eta, \quad z_{n+2} = z_{n+1} + \frac{L^{**}(z_{n+1} - z_n)^2}{2(1 - K z_{n+1})} \quad (n \geq 0), \quad (3.28)$$

is more precise majorizing sequence for $\{x_n\}$, than $\{t_n\}$ or $\{\bar{t}_n\}$, where,

$$L^{**} = \begin{cases} K & \text{if } n = 0 \\ M & \text{if } n > 0. \end{cases} \quad (3.29)$$

Note that iterates $\{x_n\}$ ($n \geq 1$) stay U_1 , and consequently (3.26) can replace

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \quad \text{for all } x, y \in \mathcal{D}. \quad (3.30)$$

In this case M, h_1, b_1 can replace L, h_0, b , respectively in the results from Proposition 2.3 until the end of Section 2.

Indeed, we have:

$$2h_1 = b_1\eta \leq 1, \quad (3.31)$$

where,

$$b_1 = \frac{M + 4K + \sqrt{M^2 + 8MK}}{4}. \quad (3.32)$$

Note also that

$$2h_0 \leq 1 \implies 2h_1 \leq 1, \quad (3.33)$$

but not necessarily vice versa unless if $M = K$. The computation of M uses information only at the starting point x_0 .

Returning back to Example 3.8, we have:

Set $R_0 = 1 - \beta$, and $\beta = .41$.

Then, we have

$$\begin{aligned} x_1 &= (.803, .803)^T & \eta &= .196, & R_0 &= .59, & R_1 &= .393, \\ K &= 2.59, & M &= 2.393, & x^* &= (.742895884, .742895884)^T \end{aligned}$$

and (3.31) gives

$$.992720943 < 1.$$

Hence, the applicability of Theorem 2.5 extends for

$$\beta \in I_1 = \left[.41, \frac{1}{2} \right) \supset I = \left[.450339002, \frac{1}{2} \right).$$

Note also that in view of (3.26), there exist $M_0 \in [0, M]$, such that

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq M_0\|x - x_0\| \quad \text{for all } x \in U_1. \quad (3.34)$$

Then M_0 can replace K in the definition of iterate z_2 in (3.28). Note that $M_0 \leq K$.

(b) If: $R \geq 2\eta$, then, we get:

$$\|x_0 - x_1\| \leq \eta \leq R - 2\eta,$$

which show $x_0 \in U_1$.

In this case M_0, M can replace K , and L in the definition of sequence $\{t_n\}$ and condition (2.14).

Following part (a), the sequence $\{t_n\}$ can be further refined, if given by:

$$\begin{aligned} t_0 &= 0, & t_1 &= \eta, & t_2 &= t_1 + \frac{M_0(t_1 - t_0)^2}{2(1 - M_0 t_1)} \\ t_{n+2} &= t_{n+1} + \frac{M(t_{n+1} - t_n)^2}{2(1 - M_0 t_{n+1})}, & (n > 0). \end{aligned}$$

(c) If: $\eta \leq R < 2\eta$, and $M \geq K$, then, we can replace L by M .

Sequence $\{t_n\}$ can be given by:

$$\begin{aligned} t_0 &= 0, & t_1 &= \eta, & t_2 &= t_1 + \frac{K(t_1 - t_0)^2}{2(1 - K t_1)} \\ t_{n+2} &= t_{n+1} + \frac{M(t_{n+1} - t_n)^2}{2(1 - K t_{n+1})}, & (n > 0). \end{aligned}$$

4. Conclusion

We provided a semilocal convergence analysis for Newton's method in order to approximate a locally unique solution of an equation in a Banach space. Using a combination of Lipschitz and center-Lipschitz conditions, instead of only Lipschitz conditions [3], and our new idea of recurrent functions, we provided an analysis with the following advantages over the work in [3]: larger convergence domain, and weaker sufficient convergence conditions. Note that these advantages are obtained under the same computational cost as in [3], since in practice the computation of the Lipschitz constant L requires the computation of K . Numerical examples further validating the results are also provided.

References

- [1] I.K. Argyros, The theory and application of abstract polynomial equations, in: St. Lucie/CRC/Lewis Publ. Mathematics Series, 1998, Boca Raton, FL, USA.
- [2] I.K. Argyros, Computational theory of iterative methods, in: C.K. Chui, L. Wuytack (Eds.), Series: Studies in Computational Mathematics, vol. 15, Elsevier Publ. Co., New York, USA, 2007.
- [3] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.
- [4] M.A. Wolfe, Extended iterative methods for the solution of operator equations, Numer. Math. 31 (1978) 153–174.
- [5] I.K. Argyros, On the Newton–Kantorovich hypothesis for solving equations, J. Comput. Appl. Math. 169 (2004) 315–332.
- [6] I.K. Argyros, A unifying local–semilocal convergence analysis and applications for two-point Newton-like methods in Banach space, J. Math. Anal. Appl. 298 (2004) 374–397.
- [7] I.K. Argyros, S. Hilout, Efficient Methods for Solving Equations and Variational Inequalities, Polimetria Publisher, Milano, Italy, 2009.
- [8] I.K. Argyros, S. Hilout, Enclosing roots of polynomial equations and their applications to iterative processes, Surveys Math. Appl. 4 (2009) 119–132.
- [9] S. Chandrasekhar, Radiative Transfer, Dover Publ., New York, 1960.
- [10] J.E. Dennis, Toward a unified convergence theory for Newton-like methods, in: L.B. Rall (Ed.), Nonlinear Funct. Anal. Appl., Academic Press, New York, 1971, pp. 425–472.
- [11] J.A. Ezquerro, M.A. Hernández, Generalized differentiability conditions for Newton's method, IMA J. Numer. Anal. 22 (2002) 187–205.
- [12] J.M. Gutiérrez, A new semilocal convergence theorem for Newton's method, J. Comput. Appl. Math. 79 (1997) 131–145.
- [13] Z. Huang, A note of Kantorovich theorem for Newton iteration, J. Comput. Appl. Math. 47 (1993) 211–217.
- [14] P. Laasonen, Ein überquadratisch konvergenter iterativer algorithmus, Ann. Acad. Sci. Fenn. Ser. I 450 (1969) 1–10.
- [15] F.A. Potra, Sharp error bounds for a class of Newton-like methods, Libertas Math. 5 (1985) 71–84.
- [16] P.D. Proinov, General local convergence theory for a class of iterative processes and its applications to Newton's method, J. Complexity 25 (2009) 38–62.
- [17] P.D. Proinov, New general convergence theory for iterative processes and its applications to Newton–Kantorovich type theorems, J. Complexity 26 (2010) 3–42.
- [18] J.W. Schmidt, Untere Fehlerschranken für Regula-Falsi Verfahren, Period. Hungar. 9 (1978) 241–247.