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Asymptotic behaviour of Laguerre–Sobolev-type orthogonal polynomials. A nondiagonal case

Herbert Dueñas [a,b], Francisco Marcellán [a,*]

[a] Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911, Leganés, Spain

[b] Universidad Nacional de Colombia, Departamento de Matemáticas, Ciudad Universitaria, Bogotá, Colombia

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ABSTRACT

In this paper we study the asymptotic behaviour of polynomials orthogonal with respect to a Sobolev-type inner product

$$\langle p, q\rangle_S = \int_0^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0), \quad \alpha > -1,$$

where p and q are polynomials with real coefficients,

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}, \qquad \mathbb{P}(0) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix}, \qquad \mathbb{Q}(0) = \begin{pmatrix} q(0) \\ q'(0) \end{pmatrix},$$

and A is a positive semidefinite matrix.

We will focus our attention on their outer relative asymptotics with respect to the standard Laguerre polynomials as well as on an analog of the Mehler–Heine formula for the rescaled polynomials.

1. Introduction

Orthogonal polynomials with respect to a Sobolev-type inner product

$$\langle p, q\rangle = \int_{\mathbb{R}} p(x)q(x)d\mu(x) + \mathbb{P}(c)^t A\mathbb{Q}(c), \tag{1}$$

where $d\mu$ is a nontrivial probability measure supported on the real line, $A \in \mathbb{R}^{(k,k)}$ is a positive semidefinite matrix, p, q are polynomials with real coefficients, and $\mathbb{Q}(c) = \left(q(c), q'(c), \ldots, q^{(k-1)}(c)\right)^t$ have been introduced in [1].

When $A = \text{diag}\,(M_0, M_1, \ldots, M_{k-1})$, the so-called diagonal Sobolev-type case, many researchers were interested in the analytic properties of the polynomials orthogonal with respect to (1). In particular, Koekoek [2] studied the second order linear differential equation satisfied by such orthogonal polynomials when $d\mu = x^{\alpha}e^{-x}dx$, $\alpha \geq 1$, and $c = 0$. They also satisfy a higher order recurrence relation as well as they can be represented as hypergeometric functions.

Later on, when $k = 2$ and $M_0, M_1 > 0$, in [3] the authors focus the attention in the location of the zeros of such orthogonal polynomials that are called Laguerre–Sobolev-type orthogonal polynomials. Finally, the analysis of their asymptotic properties was done in [4] as well in [5].

* Corresponding author. Tel.: +34 916249442; fax: +34 916249151.
E-mail addresses: haduenasr@unal.edu.co (H. Dueñas), pacomarc@ing.uc3m.es (F. Marcellán).

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On the other hand, when $k \geq 2$ if $d\mu = x^{\alpha}e^{-x}dx$, $c = 0$, and $M_0 = M_1 = \cdots = M_{k-2} = 0$, $M_{k-1} > 0$ then the same analog problems were studied in [6] in the framework of the zero distribution. From an algebraic point of view and for more general measures, in [7] the authors deal with representations of Sobolev-type orthogonal polynomials in terms of the polynomials orthogonal with respect to the measure μ assuming the same constraints for the inner product (1) as above.

The first situation of a nondiagonal Sobolev-type inner product like (1) was considered in [8]. Here the authors deal with the measure $d\mu = e^{-x^2}dx$ supported on $\mathbb{R}$, $c = 0$, and $k = 2$. In particular, they analyze scaled asymptotics for the corresponding orthogonal polynomials (Mehler–Heine formulas) and, as a consequence, the asymptotic behaviour of their zeros follows.

Taking into account that generalized Hermite polynomials appear as a consequence of the symmetrization process for Laguerre orthogonal polynomials [9–11] it seems to be very natural to analyze polynomial sequences orthogonal with respect to the inner product (1) when $d\mu = x^{\alpha}e^{-x}dx$, $A \in \mathbb{R}^{(k,k)}$ is a nondiagonal positive semidefinite matrix with $k \geq 2$, and $c = 0$.

In this contribution we focus our attention in the case $k = 2$. Thus we generalize some previous results from the diagonal case (see [4,12,3]). The structure of the manuscript is the following. In Section 2 we present the basic background about the properties of classical Laguerre polynomials which will be needed along the paper. Section 3 deals with the asymptotic properties of the Laguerre–Sobolev-type polynomials, orthogonal with respect to the inner product

$$\langle p, q\rangle = \int_0^{\infty} p(x)q(x)x^{\alpha}e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0), \quad \alpha > -1,$$

where $A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}$ is a positive semidefinite matrix and we denote $\mathbb{Q}(0) = \left(q(0), q'(0)\right)^{t}$. We obtain the outer relative asymptotics of these polynomials in terms of Laguerre polynomials and a Mehler–Heine-type formula as well as the behaviour of the Sobolev norm of the monic Laguerre–Sobolev-type orthogonal polynomials.

2. Preliminaries

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers and let μ be the linear functional defined in the linear space $\mathbb{P}$ of the polynomials with real coefficients, such that

$$\left\langle \mu, x^n\right\rangle = \mu_n, \quad n = 0, 1, 2, \ldots$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Furthermore μ_n is the *n-th moment* of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if

(i) The degree of P_n is n.
(ii) $\langle \mu, P_n(x)P_m(x)\rangle = 0$, $m \neq n$.
(iii) $\left\langle \mu, P_n^2(x)\right\rangle \neq 0$, $n = 0, 1, 2, \ldots$.

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be a sequence of *monic orthogonal polynomials.*

The next theorem, whose proof appears in [10], gives a necessary and sufficient condition for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1 (*[10]*)**.** *Let μ be the moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $\left[\mu_{i+j}\right]_{i,j\in\mathbb{N}}$ are nonsingular.*

A moment functional such that there exists the corresponding sequence of orthogonal polynomials is said to be *regular* or *quasi-definite* [10].

The proof of the next proposition can be founded in [9,10,13,14,11].

Proposition 1 (*The Christoffel–Darboux Formula*)**.** *Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials. If we denote the nth kernel polynomial by*

$$K_n(x, y) = \sum_{j=0}^{n} \frac{P_j(y)P_j(x)}{\left\langle \mu, P_j^2\right\rangle},$$

then, for every $n \in \mathbb{N}$,

$$K_n(x, y) = \frac{1}{\left\langle \mu, P_n^2\right\rangle} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x - y}. \tag{2}$$

Using the following notation for the partial derivatives of the kernel $K_n(x, y)$

$$\frac{\partial^{j+k}\left(K_n(x, y)\right)}{\partial^j x\partial^k y} = K_n^{(j,k)}(x, y),$$

we present some properties about these derivatives. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials. From the Christoffel–Darboux formula (2), we have

$$K_{n-1}(x,y) = \frac{1}{\langle \mu, P_{n-1}^2 \rangle} \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x-y}.$$

The computation of the jth partial derivative with respect to y yields

$$K_{n-1}^{(0,j)}(x,y) = \frac{1}{\langle \mu, P_{n-1}^2 \rangle} \left(P_n(x) \frac{\partial^j}{\partial y^j} \left(\frac{P_{n-1}(y)}{x-y} \right) - P_{n-1}(x) \frac{\partial^j}{\partial y^j} \left(\frac{P_n(y)}{x-y} \right) \right). \tag{3}$$

Using the Leibniz rule

$$\frac{\partial^j}{\partial y^j} \left(\frac{P_n(y)}{x-y} \right) = \sum_{k=0}^{j} \frac{j!}{k!} \frac{P_n^{(k)}(y)}{(x-y)^{j-k+1}},$$

and replacing the last expression in (3), we get

$$\begin{aligned} K_{n-1}^{(0,j)}(x,y) &= \frac{1}{\langle \mu, P_{n-1}^2 \rangle} \left(P_n(x) \sum_{k=0}^{j} \frac{j!}{k!} \frac{P_{n-1}^{(k)}(y)}{(x-y)^{j-k+1}} - P_{n-1}(x) \sum_{k=0}^{j} \frac{j!}{k!} \frac{P_n^{(k)}(y)}{(x-y)^{j-k+1}} \right) \\ &= \frac{j!}{\langle \mu, P_{n-1}^2 \rangle (x-y)^{j+1}} \left(P_n(x) \sum_{k=0}^{j} \frac{1}{k!} P_{n-1}^{(k)}(y)(x-y)^k - P_{n-1}(x) \sum_{k=0}^{j} \frac{1}{k!} P_n^{(k)}(y)(x-y)^k \right). \end{aligned}$$

As a consequence,

Proposition 2 (*[1,7]*). *For every $n \in \mathbb{N}$,*

$$K_{n-1}^{(0,j)}(x,0) = \frac{j!}{\langle \mu, P_{n-1}^2 \rangle x^{j+1}} \left(P_n(x) Q_j(x,0;P_{n-1}) - P_{n-1}(x) Q_j(x,0;P_n) \right) \tag{4}$$

where $Q_j(x,0;P_{n-1})$ and $Q_j(x,0;P_n)$ denote the Taylor polynomials of degree j of the polynomials P_{n-1} and P_n around $x = 0$, respectively.

The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle p, q \rangle_\alpha = \int_0^\infty pqx^\alpha e^{-x} dx, \quad \alpha > -1, p, q \in \mathbb{P}. \tag{5}$$

We will summarize some properties of the Laguerre monic orthogonal polynomials that we will use in what follows. The details of the proof of Proposition 3 and Theorem 2, can be founded in [9,10,13,14,11].

Proposition 3. *Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials.*

(1) *For every $n \in \mathbb{N}$,*

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+1+\alpha) L_n^\alpha(x) + n(n+\alpha) L_{n-1}^\alpha(x), \tag{6}$$

with $L_0^\alpha(x) = 1$, $L_1^\alpha(x) = x - (\alpha+1)$.

(2) *For every $n \in \mathbb{N}$,*

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x). \tag{7}$$

(3) *For every $n \in \mathbb{N}$,*

$$\|L_n^\alpha\|_\alpha^2 = n!\Gamma(n+\alpha+1). \tag{8}$$

(4) *For every $n \in \mathbb{N}$*

$$L_n^\alpha(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \tag{9}$$

(5) *For every $n \in \mathbb{N}$*

$$\left(L_n^\alpha\right)'(x) = nL_{n-1}^{\alpha+1}(x). \tag{10}$$

(6) *For every $n \in \mathbb{N}$,*

$$x\left(L_n^\alpha(x)\right)' = nL_n^\alpha(x) + n(n+\alpha) L_{n-1}^\alpha(x). \tag{11}$$

In particular, for Laguerre polynomials we get

Proposition 4. *For every $n \in \mathbb{N}$*

$$K_{n-1}(x,0) = \frac{L_{n-1}^{\alpha}(0)L_{n-1}^{\alpha+1}(x)}{(n-1)!\Gamma(n+\alpha)}, \tag{12}$$

$$K_{n-1}^{(0,1)}(x,0) = \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)}L_{n-1}^{\alpha+2}(x) + \frac{(-1)^n n}{(n-2)!\Gamma(\alpha+2)}L_{n-2}^{\alpha+2}(x), \tag{13}$$

$$K_{n-1}^{(1,1)}(x,0) = \frac{(-1)^n(n-1)}{(n-2)!\Gamma(\alpha+2)}L_{n-2}^{\alpha+3}(x) + \frac{(-1)^n n}{(n-3)!\Gamma(\alpha+2)}L_{n-3}^{\alpha+3}(x). \tag{14}$$

The proof of (12) is given in [15]. For (13) see [12]. Finally, (14) is a consequence of (13) and (7).
Using (8) and (9) in (12)–(14) we obtain

Proposition 5. *For every $n \in \mathbb{N}$,*

$$K_{n-1}(0,0) = \frac{\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+1)\Gamma(\alpha+2)}, \tag{15}$$

$$K_{n-1}^{(1,0)}(0,0) = -\frac{\Gamma(n+\alpha+1)}{(n-2)!\Gamma(\alpha+1)\Gamma(\alpha+3)} = -\frac{n-1}{\alpha+2}K_{n-1}(0,0), \tag{16}$$

$$K_{n-1}^{(1,1)}(0,0) = \frac{\Gamma(n+\alpha+1)\,(n(\alpha+2)-(\alpha+1))}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)} = \frac{(n(\alpha+2)-(\alpha+1))\,(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)}K_{n-1}(0,0). \tag{17}$$

Theorem 2 (*The Mehler–Heine-Type Formula. See [11]*)**.** *Let J_α be the Bessel function of the first kind defined by*

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j+\alpha}}{j!\Gamma(j+\alpha+1)},$$

then

$$\lim_{n\to\infty} \frac{\widehat{L}_n^{\alpha}(x/(n+j))}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x}), \tag{18}$$

uniformly on compact subsets $\mathbb{C}$ and uniformly in $j \in \mathbb{N} \cup \{0\}$. Here $\widehat{L}_n^{\alpha}(x) = (-1)^n/n! L_n^{\alpha}(x)$.

In what follows, as usual, $a_n \sim b_n$ when $n \to \infty$ means $\lim_{n\to\infty} a_n/b_n = 1$.

3. Asymptotic behaviour

If p is a polynomial with real coefficients, then we will denote

$$\mathbb{P}(x) = \begin{pmatrix} p(x) \\ p'(x) \end{pmatrix}.$$

Let p and q be polynomials with real coefficients. We define the following Sobolev-type inner product

$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0), \quad \alpha > -1, \tag{19}$$

where

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix},$$

$M_0, M_1 \geq 0$, A is a positive semidefinite matrix, i.e $\det A = |A| \geq 0$. Notice that if $M_0 = 0, M_1 > 0$ or $M_1 = 0, M_0 > 0$ it implies that $\lambda = 0$. These situations have been considered in some previous papers by the authors (see [15,12]), as well as in [4,12]. Notice that $\langle p, q\rangle_S$ is an inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients.

Following [1] we will deduce the expression of $\{\widetilde{L}_n^{\alpha}\}_{n\geq 0}$ in terms of $\{L_n^{\alpha}\}_{n\geq 0}$. Let $\{\widetilde{L}_n^{\alpha}\}_{n\geq 0}$ be the sequence of monic polynomials orthogonal with respect to (19). Consider the Fourier expansion of $\widetilde{L}_n^{\alpha}$ in terms of the sequence of Laguerre monic orthogonal polynomials $\{L_n^{\alpha}\}_{n\geq 0}$

$$\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha}(x) + \sum_{k=0}^{n-1} a_{n,k}L_k^{\alpha}(x),$$

where

$$a_{n,k} = \frac{\left\langle \widetilde{L}_n^\alpha(x), L_k^\alpha(x) \right\rangle_\alpha}{\left\| L_k^\alpha \right\|_\alpha^2}, \quad 0 \le k \le n-1.$$

From (19), we get

$$a_{n,k} = -\frac{\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} A \mathbb{L}_k^\alpha(0)}{\left\| L_k^\alpha \right\|_\alpha^2}.$$

As a consequence,

$$\begin{aligned}
\widetilde{L}_n^\alpha(x) &= L_n^\alpha(x) - \sum_{k=0}^{n-1} \frac{\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} A \mathbb{L}_k^\alpha(0)}{\left\| L_k^\alpha \right\|_\alpha^2} L_k^\alpha(x) \\
&= L_n^\alpha(x) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} A \sum_{k=0}^{n-1} \frac{\mathbb{L}_k^\alpha(0) L_k^\alpha(x)}{\left\| L_k^\alpha \right\|_\alpha^2},
\end{aligned}$$

i.e

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} A \begin{pmatrix} K_{n-1}(x,0) \\ K_{n-1}^{(0,1)}(x,0) \end{pmatrix}. \tag{20}$$

From the above expression we obtain

$$\begin{aligned}
\widetilde{L}_n^\alpha(0) &= L_n^\alpha(0) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} A \begin{pmatrix} K_{n-1}(0,0) \\ K_{n-1}^{(0,1)}(0,0) \end{pmatrix}, \\
\left(\widetilde{L}_n^\alpha\right)'(0) &= \left(L_n^\alpha\right)'(0) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} A \begin{pmatrix} K_{n-1}^{(1,0)}(0,0) \\ K_{n-1}^{(1,1)}(0,0) \end{pmatrix}.
\end{aligned}$$

Thus

$$\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} = \left(\mathbb{L}_n^\alpha(0)\right)^{\mathrm{t}} - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} A \mathbb{K}_{n-1}(0,0), \tag{21}$$

where

$$\mathbb{K}_{n-1}(0,0) = \begin{pmatrix} K_{n-1}(0,0) & K_{n-1}^{(1,0)}(0,0) \\ K_{n-1}^{(0,1)}(0,0) & K_{n-1}^{(1,1)}(0,0) \end{pmatrix}.$$

As a consequence, from (21)

$$\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^{\mathrm{t}} \left(I + A\mathbb{K}_{n-1}(0,0)\right) = \left(\mathbb{L}_n^\alpha(0)\right)^{\mathrm{t}}, \tag{22}$$

where I is the 2×2 identity matrix. Notice that

$$\begin{aligned}
I + A\mathbb{K}_{n-1}(0,0) &= K_{n-1}(0,0) \left[\begin{pmatrix} \frac{1}{K_{n-1}(0,0)} & 0 \\ 0 & \frac{1}{K_{n-1}(0,0)} \end{pmatrix} + A \begin{pmatrix} 1 & -\frac{n-1}{\alpha+2} \\ -\frac{n-1}{\alpha+2} & \frac{(n(\alpha+2)-(\alpha+1))(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)} \end{pmatrix} \right] \\
&= K_{n-1}(0,0) \begin{pmatrix} G_n & H_n \\ J_n & K_n \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
G_n &= \frac{1}{K_{n-1}(0,0)} + \left(M_0 + \frac{\lambda}{\alpha+2}\right) - \frac{n\lambda}{\alpha+2} \\
H_n &= \frac{\lambda n^2}{(\alpha+1)(\alpha+3)} - \left(\frac{M_0}{\alpha+2} + \frac{(2\alpha+3)\lambda}{(\alpha+1)(\alpha+2)(\alpha+3)}\right) n + \frac{M_0}{\alpha+2} + \frac{\lambda}{(\alpha+2)(\alpha+3)} \\
J_n &= -\frac{M_1}{\alpha+2} n + \lambda + \frac{M_1}{\alpha+2} \\
K_n &= \frac{M_1 n^2}{(\alpha+1)(\alpha+3)} - \left(\frac{\lambda}{\alpha+2} + \frac{(2\alpha+3)M_1}{(\alpha+1)(\alpha+2)(\alpha+3)}\right) n + \frac{\lambda}{\alpha+2} + \frac{M_1}{(\alpha+2)(\alpha+3)} + \frac{1}{K_{n-1}(0,0)}.
\end{aligned}$$

On the other hand

$$\begin{aligned}|I + A\mathbb{K}_{n-1}(0,0)| &= (K_{n-1}(0,0))^2 \\ &\quad \times \left[\frac{1}{(K_{n-1}(0,0))^2} + \frac{1}{K_{n-1}(0,0)} \operatorname{trace}\left[A \begin{pmatrix} 1 & -\dfrac{n-1}{\alpha+2} \\ -\dfrac{n-1}{\alpha+2} & \dfrac{(n(\alpha+2)-(\alpha+1))\,(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)} \end{pmatrix} \right] \right. \\ &\quad \left. + |A| \left(\frac{(n(\alpha+2)-(\alpha+1))\,(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)} - \frac{(n-1)^2}{(\alpha+2)^2} \right) \right] \\ &= 1 + K_{n-1}(0,0) \left(M_1 \frac{(n(\alpha+2)-(\alpha+1))\,(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)} - \frac{2\lambda}{\alpha+2}(n-1) + M_0 \right) \\ &\quad + (K_{n-1}(0,0))^2\, |A|\, \frac{n-1}{\alpha+2} \left(\frac{n}{(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{1}{(\alpha+2)(\alpha+3)} \right).\end{aligned}$$

As a consequence,

Proposition 6. *Let A defined as in (19), thus*

(1) *If $|A| > 0$, then*

$$|I + A\mathbb{K}_{n-1}(0,0)| \sim \frac{|A|\, n^{2\alpha+4}}{(\alpha+1)(\alpha+2)^2(\alpha+3)\,(\Gamma(\alpha+1)\Gamma(\alpha+2))^2}. \tag{23}$$

(2) *If $|A| = 0$, $M_1 > 0$, then*

$$|I + A\mathbb{K}_{n-1}(0,0)| \sim \frac{n^{\alpha+3} M_1}{(\alpha+1)(\alpha+3)\Gamma(\alpha+1)\Gamma(\alpha+2)}. \tag{24}$$

On the other hand, from (20) and (22)

$$\begin{aligned}\widetilde{L}_n^{\alpha}(x) &= L_n^{\alpha}(x) - \left(\mathbb{L}_n^{\alpha}(0)\right)^{t} (I + A\mathbb{K}_{n-1}(0,0))^{-1} A \begin{pmatrix} \dfrac{(-1)^{n-1}(\alpha+1)}{(n-1)!\Gamma(\alpha+2)} & 0 \\ \dfrac{(-1)^n}{(n-2)!\Gamma(\alpha+2)} & \dfrac{(-1)^n}{(n-2)!\Gamma(\alpha+2)} \end{pmatrix} \begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix} \\ &= L_n^{\alpha}(x) - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2) K_{n-1}(0,0)} \begin{pmatrix} (-1)^n \dfrac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \\ (-1)^{n-1} \dfrac{n\Gamma(n+\alpha+1)}{\Gamma(\alpha+2)} \end{pmatrix}^{t} \begin{pmatrix} G_n & H_n \\ J_n & K_n \end{pmatrix}^{-1} A \\ &\quad \times \begin{pmatrix} -\dfrac{\alpha+1}{n-1} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix}.\end{aligned}$$

Thus

$$\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha}(x) + \begin{pmatrix} -1 \\ \dfrac{n}{\alpha+1} \end{pmatrix}^{t} \begin{pmatrix} G_n & H_n \\ J_n & K_n \end{pmatrix}^{-1} A \begin{pmatrix} -(\alpha+1) & 0 \\ n-1 & n-1 \end{pmatrix} \begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix}. \tag{25}$$

Furthermore, if we denote

$$M_n = \begin{pmatrix} G_n & H_n \\ J_n & K_n \end{pmatrix},$$

then we get

$$M_n^{-1} = \frac{1}{|M_n|} \begin{pmatrix} K_n & -H_n \\ -J_n & G_n \end{pmatrix},$$

where

$$|M_n| = \frac{1}{(K_{n-1}(0,0))^2}\, |I + A\mathbb{K}_{n-1}(0,0)|\,.$$

Therefore, from (25), after some computations we get

$$\widetilde{L}_n^{\alpha}(x) = L_n^{\alpha}(x) + \frac{1}{|M_n|}\left(\widetilde{A}_n n^2 + B_n n + C_n, \widetilde{A}'_n n^2 + B'_n n + C'_n\right)\begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x)\end{pmatrix}, \tag{26}$$

with

$$\begin{aligned}
\widetilde{A}_n &= a_1\,|A| + \frac{b_1}{K_{n-1}(0,0)}, \qquad B_n = a_2\,|A| + \frac{b_2}{K_{n-1}(0,0)} \\
\widetilde{A}'_n &= a_3\,|A| + \frac{b_1}{K_{n-1}(0,0)}, \qquad B'_n = a_4\,|A| + \frac{b_3}{K_{n-1}(0,0)} \\
C_n &= a_5\,|A| + \frac{b_4}{K_{n-1}(0,0)}, \qquad C'_n = a_6\,|A| + \frac{\lambda}{K_{n-1}(0,0)},
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}, \qquad a_2 = \frac{2\alpha}{(\alpha+1)(\alpha+2)(\alpha+3)} \\
a_3 &= \frac{1}{(\alpha+1)(\alpha+2)}, \qquad a_4 = \frac{\alpha}{(\alpha+1)(\alpha+2)} \\
a_5 &= -\frac{2}{(\alpha+2)(\alpha+3)}, \qquad a_6 = -\frac{1}{\alpha+2} \\
b_1 &= \frac{M_1}{\alpha+1}, \qquad b_2 = -2\lambda - \frac{M_1}{\alpha+1}, \qquad b_3 = -\lambda - \frac{M_1}{\alpha+1} \\
b_4 &= \lambda + M_0(\alpha+1).
\end{aligned}$$

Let

$$\begin{aligned}
\widehat{L}_n^{\alpha}(x) &= \frac{(-1)^n}{n!}L_n^{\alpha}(x) \\
Q_n^{\alpha}(x) &= \frac{(-1)^n}{n!}\widetilde{L}_n^{\alpha}(x).
\end{aligned}$$

Then, from (26)

$$Q_n^{\alpha}(x) = \widehat{L}_n^{\alpha}(x) + \varepsilon_n \widehat{L}_{n-1}^{\alpha+1}(x) + \xi_n \widehat{L}_{n-2}^{\alpha+2}(x) \tag{27}$$

with

$$\begin{aligned}
\varepsilon_n &= -\frac{1}{|M_n|}\left(\widetilde{A}_n n + B_n + \frac{C_n}{n}\right) \\
\xi_n &= \frac{1}{(n-1)\,|M_n|}\left(\widetilde{A}'_n n + B'_n + \frac{C'_n}{n}\right)
\end{aligned}$$

where

$$\begin{aligned}
|M_n| = {} & \frac{1}{(K_{n-1}(0,0))^2} + \frac{1}{(K_{n-1}(0,0))}\left(\frac{M_1}{(\alpha+1)(\alpha+3)}n^2 + Rn + T\right) \\
& + |A|\left(\frac{n^2}{(\alpha+1)(\alpha+2)^2(\alpha+3)} + R'n + T'\right),
\end{aligned}$$

R, T, R', and T' depend only on M_0, M_1, λ, and α. As a consequence

(1) If $|A| > 0$,

$$\lim_{n\to\infty} \frac{|M_n|}{n^2} = \frac{|A|}{(\alpha+1)(\alpha+2)^2(\alpha+3)}.$$

(2) If $|A| = 0$,

$$\lim_{n\to\infty} \frac{K_{n-1}(0,0)\,|M_n|}{n^2} = \frac{M_1}{(\alpha+1)(\alpha+3)}.$$

Therefore, for $x \in \mathbb{C} \setminus [0, \infty)$

$$\lim_{n\to\infty} \frac{Q_n^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)} = \lim_{n\to\infty} \left(1 + \varepsilon_n \frac{\widehat{L}_{n-1}^{\alpha+1}(x)}{\widehat{L}_n^{\alpha}(x)} + \xi_n \frac{\widehat{L}_{n-2}^{\alpha+2}(x)}{\widehat{L}_n^{\alpha}(x)}\right),$$

and taking into account

$$\lim_{n\to\infty} n\varepsilon_n = \begin{cases} -2(\alpha+2) & \text{if } |A| > 0 \\ -(\alpha+3) & \text{if } |A| = 0, \end{cases}$$

$$\lim_{n\to\infty} n^2\xi_n = \begin{cases} (\alpha+2)(\alpha+3) & \text{if } |A| > 0 \\ (\alpha+3) & \text{if } |A| = 0, \end{cases}$$

and

$$\lim_{n\to\infty} \frac{n^{(l-j)/2}\widehat{L}_{n+k}^{\alpha+j}(x)}{\widehat{L}_{n+h}^{\alpha+l}(x)} = (-x)^{-(j-l)/2}, \tag{28}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$, where $j, l \in \mathbb{R}$, $h, k \in \mathbb{Z}$, (see [4]) we get

Theorem 3 (*Outer Relative Asymptotics*)**.**

$$\lim_{n\to\infty} \frac{\widetilde{L}_n^{\alpha}(x)}{L_n^{\alpha}(x)} = 1 \tag{29}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

Notice that this outer relative asymptotics does not depend on the matrix A.

We will find the corresponding Mehler–Heine formula for the Laguerre–Sobolev-type orthogonal polynomials $\widetilde{L}_n^{\alpha}(x)$. As mentioned above, in the first case, we will assume that $|A| > 0$. From (27) we get

$$\lim_{n\to\infty} \frac{Q_n^{\alpha}(x/n)}{n^{\alpha}} = \lim_{n\to\infty} \left(\frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}} + \frac{n\varepsilon_n\widehat{L}_{n-1}^{\alpha+1}(x/n)}{n^{\alpha+1}} + \frac{n^2\xi_n\widehat{L}_{n-1}^{\alpha+2}(x/n)}{n^{\alpha+2}}\right).$$

Thus,

$$\lim_{n\to\infty} \frac{Q_n^{\alpha}(x/n)}{n^{\alpha}} = x^{-\alpha/2}J_{\alpha}(2\sqrt{x}) - 2(\alpha+2)x^{-(\alpha+1)/2}J_{\alpha+1}(2\sqrt{x}) + (\alpha+2)(\alpha+3)x^{-(\alpha+2)/2}J_{\alpha+2}(2\sqrt{x})$$

uniformly on compact subsets of $\mathbb{C}$. As a consequence, the second part of the previous expression is

$$x^{-\alpha/2}\left(J_{\alpha}(2\sqrt{x}) - 2(\alpha+2)x^{-1/2}J_{\alpha+1}(2\sqrt{x}) + (\alpha+2)(\alpha+3)x^{-1}J_{\alpha+2}(2\sqrt{x})\right).$$

But, taking into account that

$$J_{\alpha}(2\sqrt{x}) + J_{\alpha+2}(2\sqrt{x}) = \frac{\alpha+1}{\sqrt{x}}J_{\alpha+1}(2\sqrt{x}),$$

then

$$\lim_{n\to\infty} \frac{Q_n^{\alpha}(x/n)}{n^{\alpha}} = x^{-\alpha/2}J_{\alpha+4}(2\sqrt{x}),$$

uniformly on compact subsets of $\mathbb{C}$.

In a similar way if $|A| = 0$ and $M_1 > 0$,

$$\begin{aligned}\lim_{n\to\infty} \frac{Q_n^{\alpha}(x/n)}{n^{\alpha}} &= \lim_{n\to\infty} \left(\frac{\widehat{L}_n^{\alpha}(x/n)}{n^{\alpha}} + \frac{n\varepsilon_n\widehat{L}_{n-1}^{\alpha+1}(x/n)}{n^{\alpha+1}} + \frac{n^2\xi_n\widehat{L}_{n-1}^{\alpha+2}(x/n)}{n^{\alpha+2}}\right) \\ &= x^{-\alpha/2}J_{\alpha}(2\sqrt{x}) - (\alpha+3)x^{-(\alpha+1)/2}J_{\alpha+1}\left(2\sqrt{x}\right) + (\alpha+3)x^{-(\alpha+2)/2}J_{\alpha+2}\left(2\sqrt{x}\right),\end{aligned}$$

uniformly on compact subsets of $\mathbb{C}$. Then we get

Theorem 4. *Let* $\left\{Q_n^{\alpha}\right\}_{n\geq 0}$ *be the sequence of polynomials orthogonal with respect to* (19) *and assume* $|A| = 0$, $M_1 > 0$. *Then*

(1) *If* $|A| > 0$, *then*

$$\lim_{n\to\infty} \frac{Q_n^{\alpha}(x/n)}{n^{\alpha}} = x^{-\alpha/2}J_{\alpha+4}(2\sqrt{x}), \tag{30}$$

uniformly on compact subsets of $\mathbb{C}$.

(2) *If* $|A| = 0$, *then*

$$\lim_{n\to\infty} \frac{Q_n^\alpha(x/n)}{n^\alpha} = x^{-\alpha/2}\left(J_\alpha(2\sqrt{x}) - \frac{\alpha+3}{\sqrt{x}}J_{\alpha+1}\left(2\sqrt{x}\right) + \frac{\alpha+3}{x}J_{\alpha+2}\left(2\sqrt{x}\right)\right), \tag{31}$$

uniformly on compact subsets of $\mathbb{C}$.

Notice that (30) coincides with [5] in the diagonal case, $M_0, M_1 > 0$ and (31) coincides with [4,12], where the cases $M_0 = 0$ and $\lambda = 0$ are studied.

In order to find a scaled outer strong asymptotic formula, we will write the Laguerre–Sobolev-type orthogonal polynomials $Q_n^\alpha(x)$ as a combination of the Laguerre orthogonal polynomials $\widehat{L}_n^{\alpha+2}(x)$, $\widehat{L}_{n-1}^{\alpha+2}(x)$, and $\widehat{L}_{n-2}^{\alpha+2}(x)$. Replacing (7) in (27) we get

$$Q_n^\alpha(x) = \widehat{L}_n^{\alpha+2}(x) + (\varepsilon_n - 2)\widehat{L}_{n-1}^{\alpha+2}(x) + (\xi_n + 1 - \varepsilon_n)\widehat{L}_{n-2}^{\alpha+2}(x). \tag{32}$$

This means that the sequence $\left\{Q_n^\alpha\right\}_{n\geq 0}$ is quasi-orthogonal with respect to the Laguerre weight $d\mu_{\alpha+2} = x^{\alpha+2}e^{-x}dx$. See [16] for more information about quasi-orthogonal families, in particular, the analysis of the zero distribution.

Introducing the change of variable nx in (32), we get

$$Q_n^\alpha(nx) = \widehat{L}_n^{\alpha+2}(nx) + (\varepsilon_n - 2)\widehat{L}_{n-1}^{\alpha+2}(nx) + (\xi_n + 1 - \varepsilon_n)\widehat{L}_{n-2}^{\alpha+2}(nx)$$

thus

$$\frac{Q_n^\alpha(nx)}{\widehat{L}_n^\alpha(nx)} = \frac{\widehat{L}_n^{\alpha+2}(nx)}{\widehat{L}_n^\alpha(nx)} + (\varepsilon_n - 2)\frac{\widehat{L}_{n-1}^{\alpha+2}(nx)}{\widehat{L}_n^\alpha(nx)} + (\xi_n + 1 - \varepsilon_n)\frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_n^\alpha(nx)}.$$

Using that (see [4,11])

$$\lim_{n\to\infty} \frac{\widehat{L}_{n-1}^\alpha(nx)}{\widehat{L}_n^\alpha(nx)} = -\frac{1}{\varphi\left((x-2)/2\right)} \tag{33}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$, where φ is the mapping of $\mathbb{C}\setminus[-1,1]$ onto the exterior of the unit circle given by

$$\varphi(x) = x + \sqrt{x^2-1},$$

Alvarez-Nodarse and Moreno-Balcazar proved in [4] that

$$\lim_{n\to\infty} \frac{\widehat{L}_n^\alpha(nx)}{\widehat{L}_{n-1}^{\alpha+2}(nx)} = -\frac{\left(\varphi\left((x-2)/2\right)+1\right)^2}{\varphi\left(x-2\right)/2}. \tag{34}$$

Then, using (33) and (34) we conclude that

Proposition 7. *For* $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{Q_n^\alpha(nx)}{\widehat{L}_n^\alpha(nx)} = \lim_{n\to\infty} \frac{\widetilde{L}_n^\alpha(nx)}{L_n^\alpha(nx)} = 1 \tag{35}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$.

On the other hand, from (19) we get

$$\left\|\widetilde{L}_n^\alpha\right\|_S^2 = \left\|L_n^\alpha\right\|_\alpha^2 + \mathbb{L}_n^\alpha(0)^t(I + A\mathbb{K}_{n-1}(0,0))^{-1}A\mathbb{L}_n^\alpha(0).$$

If B is a nonsingular matrix, it is straightforward to prove that

$$\begin{vmatrix} 0 & u^t \\ v & B \end{vmatrix} = -|B|\, u^t B^{-1} v$$

where

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Thus

$$\begin{aligned}
\left\|\widetilde{L}_n^\alpha\right\|_S^2 &= \left\|L_n^\alpha\right\|_\alpha^2 - \frac{1}{|I + A\mathbb{K}_{n-1}(0,0)|}\begin{vmatrix} 0 & \mathbb{L}_n^\alpha(0)^t \\ A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix} \\
&= \frac{\left\|L_n^\alpha\right\|_\alpha^2}{|I + A\mathbb{K}_{n-1}(0,0)|}\left(|I + A\mathbb{K}_{n-1}(0,0)| + \begin{vmatrix} 0 & \mathbb{L}_n^\alpha(0)^t/\left\|L_n^\alpha\right\|_\alpha^2 \\ -A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix}\right) \\
&= \frac{\left\|L_n^\alpha\right\|_\alpha^2}{|I + A\mathbb{K}_{n-1}(0,0)|}\begin{vmatrix} 1 & \mathbb{L}_n^\alpha(0)^t/\left\|L_n^\alpha\right\|_\alpha^2 \\ -A\mathbb{L}_n^\alpha(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix}.
\end{aligned}$$

Finally, using the fact that

$$I + A\mathbb{K}_n(0,0) = I + A\mathbb{K}_{n-1}(0,0) + \frac{A}{\left\|L_n^\alpha\right\|_\alpha^2}\mathbb{L}_n^\alpha(0)\mathbb{L}_n^\alpha(0)^{\mathrm{t}},$$

then

$$\frac{\left\|\widetilde{L}_n^\alpha\right\|_S^2}{\left\|L_n^\alpha\right\|_\alpha^2} = \frac{|I + A\mathbb{K}_n(0,0)|}{|I + A\mathbb{K}_{n-1}(0,0)|}. \tag{36}$$

Therefore using (36), (23) and (24) we get

Proposition 8. *Let $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to (19). Then*

$$\lim_{n\to\infty} \frac{\left\|\widetilde{L}_n^\alpha\right\|_S}{\left\|L_n^\alpha\right\|_\alpha} = 1.$$

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