

# A priori error estimation for the dual mixed finite element method of the elastodynamic problem in a polygonal domain, II

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## ABSTRACT

In this paper we analyze a new dual mixed formulation of the elastodynamic system in polygonal domains by using an implicit scheme for the time discretization. After the analysis of stability of the fully discrete scheme,  $L^\infty$  in time,  $L^2$  in space a priori error estimates for the approximation of the displacement, the strain, the pressure and the rotational are derived. Numerical tests are presented which confirm our theoretical results.

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## 1. Introduction

The purpose of this paper is the analysis of a finite element method for approximating the linear elastodynamic system using a new dual mixed formulation for discretization in the spatial variables and an implicit Newmark scheme for discretization in time. We have already presented the analysis of this method by using an explicit Newmark scheme in [1].

The analysis of a priori error estimates for the mixed finite element method of a second order hyperbolic system in regular domains using symmetric approximations of the stress was initiated in [2,3]; see also [4]. While the analysis for the dual mixed formulation of the linear elastodynamic system in nonregular domains, introducing as a new unknown the strain tensor, was done in [1]. However, in [1] we have used an explicit Newmark scheme and we have proved that this scheme is stable under an appropriate CFL condition. Therefore the goal of this paper is to make this analysis by using an implicit Newmark scheme for discretization in time. The main reason that we investigate the implicit Newmark scheme is that the scheme is unconditionally stable as will be shown in Section 6. While the explicit in time scheme presented in [1] requires a very small time step because of the CFL stability condition.  $L^\infty$  in time,  $L^2$  in space a priori error estimates for the approximation of the displacement, the strain, the pressure and the rotational are derived. Numerical tests are presented which confirm our theoretical results.

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Let us point out that the results concerning the analysis of the continuous problem and the semi-discretized problem in the spatial variables are proved in [1]. Hence we recall the main results and we focus on the analysis of the fully discretized problem by the implicit Newmark scheme.

## 2. The model problem

Let us fix a bounded plane domain  $\Omega$  with a polygonal boundary. More precisely, we assume that  $\Omega$  is a simply connected domain and that its boundary  $\Gamma$  is the union of a finite number of linear segments  $\bar{\Gamma}_j$ ,  $1 \leq j \leq n_e$  ( $\Gamma_j$  is assumed to be an open segment). We also fix a partition of  $\{1, 2, \dots, n_e\}$  into two subsets  $I_N$  and  $I_D$ . The union  $\Gamma_D$  of the  $\Gamma_j$ ,  $j$  running over  $I_D$ , is the part of the boundary  $\Gamma$ , where we assume zero displacement field. The union  $\Gamma_N$ , of the  $\Gamma_j$ ,  $j \in I_N$ , is the part of the boundary  $\Gamma$  where we assume zero traction field.

In this domain  $\Omega$ , we consider isotropic elastic homogeneous materials. Let  $u = (u_1, u_2)$  be the displacement field and  $f = (f_1, f_2) \in [L^2(\Omega)]^2$  the body force per unit of mass. Thus the displacement field  $u = (u_1, u_2)$  satisfies the following equations:

$$\begin{cases} u_{tt} - \operatorname{div} \sigma_s(u) = f & \text{in } [0, T] \times \Omega, \\ u = 0 & \text{on } [0, T] \times \Gamma_D, \\ \sigma_s(u) \cdot n = 0 & \text{on } [0, T] \times \Gamma_N, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \\ u_t(0, \cdot) = u_1 & \text{in } \Omega, \end{cases} \quad (2.1)$$

where  $u_0$  and  $u_1$  are the initial conditions on displacements and velocities.  $n$  denotes the unit outward normal field along  $\Gamma$ . The stress tensor  $\sigma_s(u)$  is defined by

$$\sigma_s(u) := 2\mu \epsilon(u) + \lambda \operatorname{tr} \epsilon(u) \delta. \quad (2.2)$$

The positive constants  $\mu$  and  $\lambda$  are called the Lamé coefficients. We assume that

$$(\lambda, \mu) \in [\lambda_0, \lambda_1] \times [\mu_1, \mu_2] \quad (2.3)$$

where

$$0 < \mu_1 < \mu_2 \quad \text{and} \quad 0 < \lambda_0 < \lambda_1.$$

As usual,  $\epsilon(u)$  denotes the linearized strain tensor (i.e.,  $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ ) and  $\delta$  the identity tensor.

For reasons of simplicity in our theoretical analysis, we have chosen homogeneous boundary conditions on both Dirichlet and Neumann boundaries. The extension to nonhomogeneous boundary conditions is done without difficulty. Let us note that numerical tests (see Section 7) are made under the nonhomogeneous surface traction. In what follows, we will use the following notation. For  $\tau = (\tau_{ij}) \in [H(\operatorname{div}; \Omega)]^2$ , we denote

$$\operatorname{div}(\tau) = \left( \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2}, \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} \right),$$

$$\operatorname{as}(\tau) = \tau_{21} - \tau_{12}.$$

For  $v = (v_1, v_2) \in [H^1(\Omega)]^2$ , we recall that

$$\operatorname{rot} v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

As usual, we denote by  $L^2(\cdot)$  the Lebesgue space of square integrable functions and by  $H^s(\cdot)$ ,  $s \geq 0$ , the standard Sobolev spaces. The usual norm and seminorm of  $H^s(D)$  are denoted by  $\|\cdot\|_{s,D}$  and  $|\cdot|_{s,D}$ . The inner product in  $[L^2(\Omega)]^2$  will be written  $(\cdot, \cdot)$ . If  $\sigma = (\sigma_{ij})$ ,  $\tau = (\tau_{ij}) \in [L^2(\Omega)]^{2 \times 2}$ , then we denote

$$\sigma : \tau = \sum_{i,j} \sigma_{ij} \tau_{ij} \quad \text{and} \quad (\sigma, \tau) = \int_{\Omega} \sigma : \tau \, dx.$$

We now introduce the Hilbert space

$$[H_{\Gamma_D}^1(\Omega)]^2 := \{v \in [H^1(\Omega)]^2; v|_{\Gamma_D} = 0\}.$$

Finally, in order to avoid excessive use of constants, we use the following notation:  $a \lesssim b$  stands for  $a \leq cb$ , with positive constants  $c$  independent of  $a$ ,  $b$ ,  $h$  and  $\Delta t$ .

## 3. The dual mixed formulation

Introducing as new unknowns:

$$\sigma := 2\mu \epsilon(u), \quad p := -\lambda \operatorname{div}(u) \quad \text{and} \quad \omega := \frac{1}{2} \operatorname{rot}(u),$$

and the spaces:

$$\Sigma_0 := \{(\tau, q) \in [L^2(\Omega)]^{2 \times 2} \times L^2(\Omega); \operatorname{div}(\tau - q\delta) \in [L^2(\Omega)]^2, (\tau - q\delta) \cdot n = 0 \text{ on } \Gamma_N\}, \quad (3.1)$$

$$M := \{(v, \theta) \in [L^2(\Omega)]^2 \times L^2(\Omega)\}, \quad (3.2)$$

we state the dual mixed formulation for our model hyperbolic equation (2.1): find  $(\sigma(\cdot), p(\cdot)) \in L^2([0, T]; \Sigma_0)$ ,  $u(\cdot) \in H^2([0, T]; [L^2(\Omega)]^2)$  and  $\omega(\cdot) \in L^2([0, T]; L^2(\Omega))$  such that for all  $(\tau, q) \in \Sigma_0$ , for all  $(v, \theta) \in M$  and for a.e.  $t \in [0, T]$ , we have

$$\begin{cases} \frac{1}{2\mu}(\sigma(t), \tau) + \frac{1}{\lambda}(p(t), q) + (\operatorname{div}(\tau - q\delta), u(t)) + (\operatorname{as}(\tau), \omega(t)) = 0, \\ (u_{tt}(t), v) - (\operatorname{div}(\sigma(t) - p(t)\delta), v) - (\operatorname{as}(\sigma(t)), \theta) - (f(t), v) = 0, \\ u(0) = u_0, \quad u_t(0) = u_1. \end{cases} \quad (3.3)$$

We conclude this section by introducing some notations. We set

$$\underset{\sim}{\sigma} = (\sigma, p), \quad \underset{\sim}{\tau} = (\tau, q), \quad \underset{\sim}{u} = (u, \omega), \quad \underset{\sim}{v} = (v, \theta),$$

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) := \frac{1}{2\mu}(\sigma, \tau) + \frac{1}{\lambda}(p, q), \quad \forall \underset{\sim}{\sigma}, \underset{\sim}{\tau} \in \Sigma_0, \quad (3.4)$$

$$b(\underset{\sim}{\tau}, \underset{\sim}{v}) := (\operatorname{div}(\tau - q\delta), v) + (\operatorname{as}(\tau), \theta), \quad \forall \underset{\sim}{\tau} \in \Sigma_0, \quad \forall \underset{\sim}{v} \in [L^2(\Omega)]^2 \times L^2(\Omega). \quad (3.5)$$

With these notations, the mixed formulation (3.3) may be rewritten: find  $\underset{\sim}{\sigma}(\cdot) = (\sigma(\cdot), p(\cdot)) \in L^2([0, T]; \Sigma_0)$  and  $\underset{\sim}{u}(\cdot) = (u(\cdot), \omega(\cdot)) \in H^2([0, T]; [L^2(\Omega)]^2) \times L^2([0, T]; L^2(\Omega))$  such that  $u(0) = u_0$ ,  $u_t(0) = u_1$  and for a.e.  $t \in [0, T]$ :

$$\begin{cases} a(\underset{\sim}{\sigma}(t), \underset{\sim}{\tau}) + b(\underset{\sim}{\tau}, \underset{\sim}{u}(t)) = 0, \quad \forall \underset{\sim}{\tau} \in \Sigma_0, \\ b(\underset{\sim}{\sigma}(t), \underset{\sim}{v}) + (\mathcal{F}(t), \underset{\sim}{v}) = (u_{tt}(t), v), \quad \forall \underset{\sim}{v} \in [L^2(\Omega)]^2 \times L^2(\Omega), \end{cases} \quad (3.6)$$

where  $(\mathcal{F}(t), \underset{\sim}{v}) := (f(t), v)$ .

**Remark 3.1.** From the second equation of (3.3), we have  $(\operatorname{as}(\sigma(t)), \theta) = 0, \forall \theta \in L^2(\Omega)$ . This is nothing else than the relaxation of the symmetry of  $\sigma(t)$  by a Lagrange multiplier. This technique is already used for mixed finite element discretizations of the corresponding stationary problem, i.e. the system of linear elasticity (see, e.g., [5–8]).

#### 4. Regularity of the solutions

Let  $u \in L^2(0, T; [H_{\Gamma_D}^1(\Omega)]^2)$  such that  $\frac{du}{dt} \in L^2(0, T; [L^2(\Omega)]^2)$ , be the solution of (2.1). We consider the Lamé operator defined by

$$L := -\mu \Delta - (\lambda + \mu) \nabla \operatorname{div}.$$

Thus, equivalently  $u$  is the weak solution of the problem

$$\begin{cases} u_{tt} + Lu = f & \text{in } [0, T] \times \Omega, \\ u = 0 & \text{on } [0, T] \times \Gamma_D, \\ \sigma_s(u) \cdot n = 0 & \text{on } [0, T] \times \Gamma_N, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \\ u_t(0, \cdot) = u_1 & \text{in } \Omega. \end{cases} \quad (4.1)$$

It is well known (see [8] or [9–11]) that the weak solution of the corresponding Lamé system of (4.1) presents vertex singularities. To describe them, we need to introduce the following notations:

**Definition 4.1.** Let  $S_j$  ( $1 \leq j \leq n_e$ ) be the vertex of our polygonal domain  $\Omega$  at the intersection of the sides  $\Gamma_j$  and  $\Gamma_{j+1}$  ( $\Gamma_{n_e+1} := \Gamma_1$ ). Let us denote by  $\omega_j$  the measure of the angle at the vertex  $S_j$ . By the characteristic equation associated to the vertex  $S_j$ , we mean the transcendental equation in the complex variable  $\alpha$ :

$$\sin^2(\alpha \omega_j) = \left[ \frac{\lambda + \mu}{\lambda + 3\mu} \right]^2 \alpha^2 \sin^2 \omega_j, \quad (4.2)$$

if  $S_j$  is a vertex of Dirichlet type i.e.  $j, j+1 \in I_D$ ,

$$\sin^2(\alpha\omega_j) = \alpha^2 \sin^2 \omega_j, \quad (4.3)$$

if  $S_j$  is a vertex of Neumann type i.e.  $j, j+1 \in I_N$ ,

$$\sin^2(\alpha\omega_j) = \frac{(\lambda + 2\mu)^2 - (\lambda + \mu)^2 \alpha^2 \sin^2 \omega_j}{(\lambda + \mu)(\lambda + 3\mu)}, \quad (4.4)$$

if  $S_j$  is a vertex of mixed type i.e.  $j \in I_D, j+1 \in I_N$  or  $j \in I_N, j+1 \in I_D$ .

**Definition 4.2.** For any scalar function  $\phi \in C^0(\overline{\Omega})$  such that  $\phi(x) > 0$  for every  $x \in \bar{\Omega} \setminus \{S_1, S_2, \dots, S_{n_e}\}$  and any  $m, k \in \mathbb{N}$ , we define

$$H_\phi^{m,k}(\Omega) = \{v \in H^m(\Omega) \cap H_{\text{loc}}^{m+k}(\Omega); \phi D^\beta v \in L^2(\Omega), \forall \beta \in \mathbb{N}^2 \text{ such that } m < |\beta| \leq m+k\}.$$

$H_\phi^{m,k}(\Omega)$  is a Hilbert space equipped with the norm:

$$\|v\|_{m,k;\phi,\Omega} = \left( \|v\|_{m,\Omega}^2 + \sum_{m < |\beta| \leq m+k} \|\phi D^\beta v\|_{0,\Omega}^2 \right)^{1/2}.$$

On this space, we also define the seminorm:

$$|v|_{m,k;\phi,\Omega} = \left( \sum_{|\beta|=m+k} \|\phi D^\beta v\|_{0,\Omega}^2 \right)^{1/2}.$$

We consider also the spaces  $L^2(0, T; H_\phi^{m,k}(\Omega))$  endowed with the norm:

$$\|v\|_{L^2(0,T;H_\phi^{m,k})} = \left( \int_0^T \|v\|_{m,k;\phi,\Omega}^2 dt \right)^{1/2},$$

and  $L^\infty(0, T; H_\phi^{m,k}(\Omega))$  endowed with the norm  $\|v\|_{L^\infty(0,T;H_\phi^{m,k})} = \text{ess sup}_{0 \leq t \leq T} \|v(t)\|_{m,k;\phi,\Omega}$ .

Let us set  $\xi = \min_{j=1,\dots,n_e} \xi_j$  where

$$\xi_j = \inf_k \{\text{Re } \alpha_{j,k}; \text{Re } \alpha_{j,k} > 0\},$$

where  $\alpha_{j,k}$  is the solution of the appropriate transcendental equation appearing in Definition 4.1. By [8, Lemma 2.2],  $\xi > \frac{1}{2}$ . Let us pick some  $\alpha \in ]1 - \xi, 1/2[$  if  $\xi \leq 1$ , and let us take  $\alpha = 0$  if  $\xi > 1$ .

Now we recall the following regularity results (see [1]):

**Proposition 4.3.** Let us suppose that the appropriate characteristic equation among (4.2)–(4.4) for each vertex of  $\Omega$  has no root on the vertical line  $\text{Re } \alpha = 1$  in the complex plane. Let  $\phi \in C^0(\overline{\Omega})$ , as above in Definition 4.2, such that  $\phi(x) = r_j(x)^\alpha$  in a neighborhood of the vertex  $S_j$  of the polygonal domain  $\Omega$  for every  $j = 1, \dots, n_e$  where  $r_j(x) = |x - S_j|$  ( $|\cdot|$  means Euclidean norm).

Let us suppose that:

$$\begin{cases} f \in H^3(0, T; [L^2(\Omega)]^2), \\ u_0, u_1, f(0) - Lu_0, f_t(0) - Lu_1 \in [H_{\Gamma_D}^1(\Omega)]^2, \\ f_{tt}(0) - Lf(0) + L^2u_0 \in [L^2(\Omega)]^2. \end{cases} \quad (4.5)$$

Then  $u \in C(0, T; [H_\phi^{1,1}(\Omega)]^2 \cap [H_{\Gamma_D}^1(\Omega)]^2)$  and  $u_{tt} \in L^2(0, T; [H_\phi^{1,1}(\Omega)]^2 \cap [H_{\Gamma_D}^1(\Omega)]^2)$ .

Consequently  $\sigma \in L^\infty(0, T; [H_\phi^{0,1}(\Omega)]^{2 \times 2})$ ,  $p \in L^\infty(0, T; H_\phi^{0,1}(\Omega))$  and  $\omega \in L^\infty(0, T; H_\phi^{0,1}(\Omega))$ . Moreover  $\sigma_{tt} \in L^2(0, T; [H_\phi^{0,1}(\Omega)]^{2 \times 2})$ ,  $p_{tt} \in L^2(0, T; H_\phi^{0,1}(\Omega))$  and  $\omega_{tt} \in L^2(0, T; H_\phi^{0,1}(\Omega))$ .

**Proposition 4.4.** Let us suppose that the appropriate characteristic equation among (4.2)–(4.4) for each vertex of  $\Omega$  has no root on the vertical line  $\operatorname{Re} \alpha = 2$  in the complex plane. Let  $\phi \in C^0(\overline{\Omega})$  as in Proposition 4.3. Let us suppose that:

$$\begin{cases} f \in H^6(0, T; [L^2(\Omega)]^2) \\ f^{(4)} \in L^2(0, T; [H^1(\Omega)]^2) \\ u_0, u_1, f(0) - Lu_0 \in [H_{\Gamma_D}^1(\Omega)]^2 \\ f^{(1)}(0) - Lu_1 \in [H_{\Gamma_D}^1(\Omega)]^2 \\ f^{(2)}(0) - Lf(0) + L^2u_0 \in [H_{\Gamma_D}^1(\Omega)]^2 \\ f^{(3)}(0) - Lf^{(1)}(0) + L^2u_1 \in [H_{\Gamma_D}^1(\Omega)]^2 \\ f^{(4)}(0) - Lf^{(2)}(0) + L^2f(0) - L^3u_0 \in [H_{\Gamma_D}^1(\Omega)]^2 \\ f^{(5)}(0) - Lf^{(3)}(0) + L^2f^{(1)}(0) - L^3u_1 \in [L^2(\Omega)]^2. \end{cases} \quad (4.6)$$

Then  $\sigma_{tttt} \in L^2(0, T; [H_\phi^{0,1}(\Omega)]^{2 \times 2})$ ,  $p_{tttt} \in L^2(0, T; H_\phi^{0,1}(\Omega))$ ,  $\omega_{tttt} \in L^2(0, T; H_\phi^{0,1}(\Omega))$  and  $u_{tttt} \in L^2(0, T; [H_\phi^{1,2}(\Omega)]^2) \cap C(0, T; [H_{\Gamma_D}^1(\Omega)]^2)$ .

## 5. The elliptic projection error estimates

We assume that  $\Omega$  is discretized by a regular family of triangulations  $(\mathcal{T}_h)_{h>0}$  in the sense of [12]. If  $T \in \mathcal{T}_h$ , then we denote by  $h_T$  its diameter. By abuse of notation [12, Remark 17.1, p. 131]  $h$  denotes also  $\max_{T \in \mathcal{T}_h} h_T$  (the real meaning of  $h$  is indicated by the context). We introduce the finite dimensional subspaces  $\Sigma_{0,h}$  and  $V_h \times W_h$  of  $\Sigma_0$  and  $M$  respectively defined by

$$\Sigma_{0,h} := \{(\tau_h, q_h) \in \Sigma_0; \forall T \in \mathcal{T}_h: q_{h|T} \in \mathbb{P}_1(T) \text{ and } \tau_{h|T} \in [\mathbb{P}_1(T)]^{2 \times 2} \oplus [\mathbb{R} \operatorname{Curl} b_T]^2\} \quad (5.1)$$

$$V_h \times W_h := \{(v_h, \theta_h) \in M; \forall T \in \mathcal{T}_h: v_{h|T} \in [\mathbb{P}_0(T)]^2 \text{ and } \theta_{h|T} \in \mathbb{P}_1(T)\}. \quad (5.2)$$

Note that by  $\tau_{h|T} \in [\mathbb{P}_1(T)]^{2 \times 2} \oplus [\mathbb{R} \operatorname{Curl} b_T]^2$ , we mean that there exist polynomials on  $T$  of degree  $\leq 1$ :  $p_{11} \in \mathbb{P}_1(T)$ ,  $p_{12} \in \mathbb{P}_1(T)$ ,  $p_{21} \in \mathbb{P}_1(T)$ ,  $p_{22} \in \mathbb{P}_1(T)$  and two real numbers  $\alpha_1, \alpha_2$  such that

$$\tau_{h|T} = \begin{bmatrix} p_{11} + \alpha_1 \frac{\partial b_T}{\partial x_2} & p_{12} - \alpha_1 \frac{\partial b_T}{\partial x_1} \\ p_{21} + \alpha_2 \frac{\partial b_T}{\partial x_2} & p_{22} - \alpha_2 \frac{\partial b_T}{\partial x_1} \end{bmatrix},$$

where  $b_T$  denotes the bubble function for the actual triangular element  $T$  defined by

$$b_T = 27\lambda_1\lambda_2\lambda_3.$$

$\lambda_1, \lambda_2, \lambda_3$  denote the barycentric coordinates on  $T$ . Now we introduce the following discrete elliptic projection problem:

Find  $\widehat{\sigma}_{\sim h}(\cdot) = (\widehat{\sigma}_h(\cdot), \widehat{p}_h(\cdot)) \in L^2([0, T]; \Sigma_{0,h})$ ,  $\widehat{u}_{\sim h}(\cdot) = (\widehat{u}_h(\cdot), \widehat{\omega}_h(\cdot)) \in L^2([0, T]; V_h \times W_h)$  such that for all  $(\tau_h, q_h) \in \Sigma_{0,h}$ , for all  $(v_h, \theta_h) \in V_h \times W_h$  and for a.e.  $t \in [0, T]$ , we have:

$$\begin{cases} \frac{1}{2\mu}(\widehat{\sigma}_h(t), \tau_h) + \frac{1}{\lambda}(\widehat{p}_h(t), q_h) + (\operatorname{div}(\tau_h - q_h\delta), \widehat{u}_h(t)) + (\operatorname{as}(\tau_h), \widehat{\omega}_h(t)) = 0, \\ (u_{tt}(t), v_h) - (\operatorname{div}(\widehat{\sigma}_h(t) - \widehat{p}_h(t)\delta), v_h) - (\operatorname{as}(\widehat{\sigma}_h(t)), \theta_h) - (f(t), v_h) = 0. \end{cases} \quad (5.3)$$

With the notations (3.4) and (3.5), the discrete elliptic projection formulation (5.3) may be rewritten: find  $\widehat{\sigma}_{\sim h}(\cdot) = (\widehat{\sigma}_h(\cdot), \widehat{p}_h(\cdot)) \in L^2([0, T]; \Sigma_{0,h})$ ,  $\widehat{u}_{\sim h}(\cdot) = (\widehat{u}_h(\cdot), \widehat{\omega}_h(\cdot)) \in L^2([0, T]; V_h \times W_h)$  such that, for a.e.  $t \in [0, T]$ , we have

$$\begin{cases} a\left(\widehat{\sigma}_{\sim h}(t), \tau_{\sim h}\right) + b\left(\tau_{\sim h}, \widehat{u}_{\sim h}(t)\right) = 0, \quad \forall \tau_{\sim h} := (\tau_h, q_h) \in \Sigma_{0,h}, \\ b\left(\widehat{\sigma}_{\sim h}(t), v_{\sim h}\right) + \left(\mathcal{F}(t), v_{\sim h}\right) = (u_{tt}(t), v_h), \quad \forall v_{\sim h} := (v_h, \theta_h) \in V_h \times W_h. \end{cases} \quad (5.4)$$

Before giving some error estimates between the exact solution and its elliptic projection, let us recall from [8] three adequate refinement rules of grids imposing constraints on the diameters of the triangles of the triangulations according to their geometrical situation in order to recapture the optimal order of convergence of the interpolates.

Let  $(\mathcal{T}_h)_{h>0}$  be a regular family of triangulations on  $\Omega$ . In the following, we will suppose that  $(\mathcal{T}_h)_{h>0}$  satisfies some of the following refinement rules:

R<sub>1</sub>: if  $T$  is a triangle of  $\mathcal{T}_h$  admitting  $S_j$  as a vertex, then

$$h_T \lesssim h^{1/(1-\alpha)},$$

( $\alpha$  has been defined just before Proposition 4.3); as usual  $h := \max_{T \in \mathcal{T}_h} h_T$ ;

R<sub>2</sub>: if  $T$  is a triangle of  $\mathcal{T}_h$  admitting no  $S_j$  ( $j = 1, \dots, n_e$ ) as a vertex, then

$$h_T \lesssim h \inf_{x \in T} \phi(x),$$

( $\phi$  has been defined in Proposition 4.3);

R<sub>3</sub>: for all  $T \in \mathcal{T}_h$

$$h_T \gtrsim h^\beta,$$

where  $\beta \geq 1/(1-\alpha)$ .

**Remark 5.1.** Regular families of meshes satisfying the refinement conditions R<sub>1</sub>–R<sub>3</sub> are easily built; see for instance [13].

Now, we give optimal error estimates between  $((\sigma(\cdot), p(\cdot)), (u(\cdot), \omega(\cdot)))$  the exact solution of the mixed problem (3.3) or equivalently (3.6) and  $((\hat{\sigma}_h(\cdot), \hat{p}_h(\cdot)), (\hat{u}_h(\cdot), \hat{\omega}_h(\cdot)))$  the solution of the discrete elliptic projection problem (5.3) or equivalently (5.4) (see [1] for the proofs of the following results).

**Proposition 5.2.** Let  $(\mathcal{T}_h)_{h>0}$  be a regular family of triangulations on  $\Omega$ . We suppose that  $(\mathcal{T}_h)_{h>0}$  satisfies conditions R<sub>1</sub> and R<sub>2</sub>. Under the hypotheses of Proposition 4.3, the following error estimate holds for a.e.  $t \in [0, T]$ :

$$\|\sigma(t) - \hat{\sigma}_h(t)\|_{0,\Omega} + \|p(t) - \hat{p}_h(t)\|_{0,\Omega} \lesssim h [|u(t)|_{1,1;\phi,\Omega} + |p(t)|_{0,1;\phi,\Omega}]. \quad (5.5)$$

**Proposition 5.3.** Under the hypotheses of Proposition 5.2, the following error estimates hold for a.e.  $t \in [0, T]$ :

$$\|\sigma_{tt}(t) - \hat{\sigma}_{h,tt}(t)\|_{0,\Omega} + \|p_{tt}(t) - \hat{p}_{h,tt}(t)\|_{0,\Omega} \lesssim h [|u_{tt}(t)|_{1,1;\phi,\Omega} + |p_{tt}(t)|_{0,1;\phi,\Omega}], \quad (5.6)$$

$$\|\omega_{tt}(t) - \hat{\omega}_{h,tt}(t)\|_{0,\Omega} + \|P_h^0 u_{tt}(t) - \hat{u}_{h,tt}(t)\|_{0,\Omega} \lesssim h [|u_{tt}(t)|_{1,1;\phi,\Omega} + |p_{tt}(t)|_{0,1;\phi,\Omega}], \quad (5.7)$$

$$\|u_{tt}(t) - \hat{u}_{h,tt}(t)\|_{0,\Omega} \lesssim h [|u_{tt}(t)|_{1,1;\phi,\Omega} + |u_{tt}(t)|_{1,\Omega} + |p_{tt}(t)|_{0,1;\phi,\Omega}], \quad (5.8)$$

where  $P_h^0$  is the  $L^2$ -orthogonal projection from  $[L^2(\Omega)]^2$  onto  $V_h$ .

**Remark 5.4.** Under the hypotheses of Proposition 4.4, we have (see [1]) the following estimates for a.e.  $t \in [0, T]$ :

$$\|u_{ttt}(t) - \hat{u}_{h,ttt}(t)\|_{0,\Omega} \lesssim h [|u_{ttt}(t)|_{1,1;\phi,\Omega} + |u_{ttt}(t)|_{1,\Omega} + |p_{ttt}(t)|_{0,1;\phi,\Omega}], \quad (5.9)$$

and

$$\|u_{tt}(t) - \hat{u}_{h,tt}(t)\|_{0,\Omega} \lesssim h [|u_{tt}(t)|_{1,1;\phi,\Omega} + |u_{tt}(t)|_{1,\Omega} + |p_{tt}(t)|_{0,1;\phi,\Omega}]. \quad (5.10)$$

## 6. The fully discrete mixed finite element scheme

### 6.1. Notation

Let  $\Delta t := \frac{T}{N} > 0$  denote the time step size and define  $t_i = i\Delta t$  ( $i = 0, 1, \dots, N$ ),  $t_N = T$  and  $t_0 = 0$ . For any function  $\phi$  of time, let  $\phi^n$  denote  $\phi(t_n)$ . We denote  $t^{n+\frac{1}{2}} := \frac{t^n + t^{n+1}}{2}$ ,  $\phi^{n+\frac{1}{2}} := \frac{\phi^n + \phi^{n+1}}{2}$ ,  $\phi^{n;\frac{1}{4}} := \frac{\phi^{n+1} + 2\phi^n + \phi^{n-1}}{4}$ , and we define the following discrete temporal derivatives:

$$\Delta_t \phi^n := \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t}, \quad \Delta_t \phi^{n+\frac{1}{2}} := \frac{\phi^{n+1} - \phi^n}{\Delta t}, \quad \Delta_t^2 \phi^n := \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{(\Delta t)^2}.$$

We can easily see that we have

$$\Delta_t^2 \phi^n := \frac{\Delta_t \phi^{n+\frac{1}{2}} - \Delta_t \phi^{n-\frac{1}{2}}}{\Delta t} \quad \text{and} \quad \Delta_t \phi^n := \frac{\Delta_t \phi^{n+\frac{1}{2}} + \Delta_t \phi^{n-\frac{1}{2}}}{2}. \quad (6.1)$$

## 6.2. The implicit Newmark scheme

The implicit-in-time discrete mixed formulation is as follows: Find  $(\sigma_h^{n+1}, p_h^{n+1}) \in \Sigma_{0,h}$ , and  $(u_h^{n+1}, \omega_h^{n+1}) \in V_h \times W_h$  such that

$$u_h^0 = \hat{u}_h(0), \quad u_h^1 = \hat{u}_h(\Delta t), \quad (6.2)$$

and

$$\begin{cases} \frac{1}{2\mu}(\sigma_h^{n+1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1}, q_h) + (\operatorname{div}(\tau_h - q_h \delta), u_h^{n+1}) + (\operatorname{as}(\tau_h), \omega_h^{n+1}) = 0, & \forall n \geq -1, \\ (\operatorname{as}(\sigma_h^{n+1}), \theta_h) = 0, & \forall n \geq -1, \\ (\Delta_t^2 u_h^n, v_h) - \left( \operatorname{div} \left( \sigma_h^{n; \frac{1}{4}} - p_h^{n; \frac{1}{4}} \delta \right), v_h \right) - \left( f^{n; \frac{1}{4}}, v_h \right) = 0, & \forall n \geq 1, \end{cases} \quad (6.3)$$

$$\forall (\tau_h, q_h) \in \Sigma_{0,h} \text{ and } \forall (v_h, \theta_h) \in V_h \times W_h.$$

This implicit scheme is inspired by the implicit method introduced in [14] for mixed finite element approximations of a second order hyperbolic equation.

Before the statement of existence and uniqueness of a solution to problem (6.3), let us recall the result concerning the uniform inf-sup condition. Adapting the proof of Proposition 4.2 of [8] we obtain

**Lemma 6.1** ([8]). *There exists a strictly positive constant  $\beta^*$ , independent of  $h$ , such that*

$$\sup_{\tau_h = (\tau_h, q_h) \in \Sigma_{0,h}} \frac{b \left( \begin{smallmatrix} \tau \\ \sim_h \end{smallmatrix}, \begin{smallmatrix} v \\ \sim_h \end{smallmatrix} \right)}{\left\| \begin{smallmatrix} \tau \\ \sim_h \end{smallmatrix} \right\|_{0,\Omega}} \geq \beta^* \left\| \begin{smallmatrix} v \\ \sim_h \end{smallmatrix} \right\|_{0,\Omega}, \quad \forall v = (v_h, \theta_h) \in V_h \times W_h. \quad (6.4)$$

**Lemma 6.2.** *A solution  $((\sigma_h^{n+1}, p_h^{n+1}), (u_h^{n+1}, \omega_h^{n+1}))$  of (6.3) exists and is unique.*

**Proof.** With every  $((\sigma_h^{n+1}, p_h^{n+1}), (u_h^{n+1}, \omega_h^{n+1})) \in \Sigma_{0,h} \times (V_h \times W_h)$ , we associate the element of its dual  $\Sigma'_{0,h} \times V'_h \times W'_h$ :

$$\left( \begin{array}{l} (\tau_h, q_h) \mapsto \frac{1}{2\mu}(\sigma_h^{n+1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1}, q_h) + (\operatorname{div}(\tau_h - q_h \delta), u_h^{n+1}) + (\operatorname{as}(\tau_h), \omega_h^{n+1}) \\ \theta_h \mapsto (\operatorname{as}(\sigma_h^{n+1}), \theta_h) \\ v_h \mapsto \frac{1}{(\Delta t)^2}(u_h^{n+1}, v_h) - \frac{1}{4}(\operatorname{div}(\sigma_h^{n+1} - p_h^{n+1} \delta), v_h) \end{array} \right).$$

Let us call this mapping  $T_h^{(n+1)}$ ; it is a linear mapping from  $\Sigma_{0,h} \times V_h \times W_h$  into its dual. We have to prove that  $T_h^{(n+1)}$  is bijective. But the arrival and departure spaces have the same dimension. Thus by a well known theorem of linear algebra, it suffices to prove that  $T_h^{(n+1)}$  is injective. Thus let  $((\sigma_h^{n+1}, p_h^{n+1}), (u_h^{n+1}, \omega_h^{n+1})) \in \Sigma_{0,h} \times (V_h \times W_h)$  be such that:

$$\frac{1}{2\mu}(\sigma_h^{n+1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1}, q_h) + (\operatorname{div}(\tau_h - q_h \delta), u_h^{n+1}) + (\operatorname{as}(\tau_h), \omega_h^{n+1}) = 0, \quad \forall (\tau_h, q_h) \in \Sigma_{0,h}, \quad (6.5)$$

$$(\operatorname{as}(\sigma_h^{n+1}), \theta_h) = 0, \quad \forall \theta_h \in W_h, \quad (6.6)$$

$$\frac{1}{(\Delta t)^2}(u_h^{n+1}, v_h) - \frac{1}{4}(\operatorname{div}(\sigma_h^{n+1} - p_h^{n+1} \delta), v_h) = 0, \quad \forall v_h \in V_h. \quad (6.7)$$

Then, by taking  $(\tau_h, q_h) = (\sigma_h^{n+1}, p_h^{n+1})$  in (6.5) and  $v_h = u_h^{n+1}$  in (6.7) and using the fact that  $(\operatorname{as}(\sigma_h^{n+1}), \theta_h) = 0, \forall \theta_h \in W_h$ , we get

$$\frac{1}{2\mu} \|\sigma_h^{n+1}\|_{0,\Omega}^2 + \frac{1}{\lambda} \|p_h^{n+1}\|_{0,\Omega}^2 + \frac{4}{(\Delta t)^2} \|u_h^{n+1}\|_{0,\Omega}^2 = 0.$$

It follows that

$$\sigma_h^{n+1} = 0, \quad p_h^{n+1} = 0 \quad \text{and} \quad u_h^{n+1} = 0.$$

Thus (6.5) reduces to:

$$(\operatorname{as}(\tau_h), \omega_h^{n+1}) = 0, \quad \forall (\tau_h, q_h) \in \Sigma_{0,h}.$$

By the inf-sup inequality (6.4) applied with  $(v_h, \theta_h) = (0, \omega_h^{n+1})$ , we get  $\omega_h^{n+1} = 0$ .  $\square$

**Remark 6.3.** Note that by the first two equations of system (6.3) with  $n = -1$ , and similar arguments as in the previous proof,  $\sigma_h^0$ ,  $p_h^0$  and  $\omega_h^0$  are defined. Similarly by taking  $n = 0$ ,  $\sigma_h^1$ ,  $p_h^1$  and  $\omega_h^1$  are defined. Moreover we have

$$\sigma_h^0 = \hat{\sigma}_h(0), \quad p_h^0 = \hat{p}_h(0) \quad \text{and} \quad \omega_h^0 = \hat{\omega}_h(0), \quad (6.8)$$

and

$$\sigma_h^1 = \hat{\sigma}_h(\Delta t), \quad p_h^1 = \hat{p}_h(\Delta t) \quad \text{and} \quad \omega_h^1 = \hat{\omega}_h(\Delta t). \quad (6.9)$$

### 6.3. The stability of the fully discrete implicit scheme

As expected for such an implicit scheme this method is unconditionally stable. Before the statement of this result, we begin by the proof of the following lemma:

**Lemma 6.4.** We have

$$\begin{aligned} & \left\| \Delta_t u_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2\mu} \left\| \sigma_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 \\ &= 2\Delta t \sum_{n=1}^N \left( f^{n;\frac{1}{4}}, \Delta_t u_h^n \right) + \left\| \Delta_t u_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2\mu} \left\| \sigma_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{\frac{1}{2}} \right\|_{0,\Omega}^2. \end{aligned} \quad (6.10)$$

**Proof.** Subtracting the first equation of (6.3) from itself in step  $n - 1$ , we get for all  $(\tau_h, q_h) \in \Sigma_{0,h}$

$$\frac{1}{2\mu} (\sigma_h^{n+1} - \sigma_h^{n-1}, \tau_h) + \frac{1}{\lambda} (p_h^{n+1} - p_h^{n-1}, q_h) + (\operatorname{div}(\tau_h - q_h \delta), u_h^{n+1} - u_h^{n-1}) + (\operatorname{as}(\tau_h), \omega_h^{n+1} - \omega_h^{n-1}) = 0. \quad (6.11)$$

Taking  $(\tau_h, q_h) = \frac{1}{2\Delta t} \left( \sigma_h^{n;\frac{1}{4}}, p_h^{n;\frac{1}{4}} \right)$  in (6.11) and the fact that  $\left( \operatorname{as} \left( \sigma_h^{n;\frac{1}{4}} \right), \theta_h \right) = 0, \forall \theta_h \in W_h$ , we get

$$\frac{1}{2\mu} \left( \Delta_t \sigma_h^n, \sigma_h^{n;\frac{1}{4}} \right) + \frac{1}{\lambda} \left( \Delta_t p_h^n, p_h^{n;\frac{1}{4}} \right) + \left( \operatorname{div} \left( \sigma_h^{n;\frac{1}{4}} - p_h^{n;\frac{1}{4}} \delta \right), \Delta_t u_h^n \right) = 0. \quad (6.12)$$

The third equation of (6.3) with  $v_h = \Delta_t u_h^n$  becomes

$$(\Delta_t^2 u_h^n, \Delta_t u_h^n) - \left( \operatorname{div} \left( \sigma_h^{n;\frac{1}{4}} - p_h^{n;\frac{1}{4}} \delta \right), \Delta_t u_h^n \right) = \left( f^{n;\frac{1}{4}}, \Delta_t u_h^n \right). \quad (6.13)$$

Adding (6.13) and (6.12), we obtain

$$(\Delta_t^2 u_h^n, \Delta_t u_h^n) + \frac{1}{2\mu} \left( \Delta_t \sigma_h^n, \sigma_h^{n;\frac{1}{4}} \right) + \frac{1}{\lambda} \left( \Delta_t p_h^n, p_h^{n;\frac{1}{4}} \right) = \left( f^{n;\frac{1}{4}}, \Delta_t u_h^n \right). \quad (6.14)$$

Now, we examine separately the three terms on the left-hand side of this last equation. We have from (6.1)

$$\begin{aligned} (\Delta_t^2 u_h^n, \Delta_t u_h^n) &= \frac{1}{2\Delta t} \left( \Delta_t u_h^{n+\frac{1}{2}} - \Delta_t u_h^{n-\frac{1}{2}}, \Delta_t u_h^{n+\frac{1}{2}} + \Delta_t u_h^{n-\frac{1}{2}} \right) \\ &= \frac{1}{2\Delta t} \left( \left\| \Delta_t u_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| \Delta_t u_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right) \end{aligned}$$

and

$$\begin{aligned} \Delta_t \sigma_h^n &= \frac{\sigma_h^{n+1} - \sigma_h^{n-1}}{2\Delta t} = \frac{(\sigma_h^{n+1} + \sigma_h^n) - (\sigma_h^n + \sigma_h^{n-1})}{2\Delta t} = \frac{\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}}{\Delta t}, \\ \sigma_h^{n;\frac{1}{4}} &= \frac{\sigma_h^{n+1} + \sigma_h^n}{4} + \frac{\sigma_h^{n-1} + \sigma_h^n}{4} = \frac{1}{2} \left( \sigma_h^{n+\frac{1}{2}} + \sigma_h^{n-\frac{1}{2}} \right). \end{aligned}$$

Using these two last equalities, we can write

$$\begin{aligned} \frac{1}{2\mu} \left( \Delta_t \sigma_h^n, \sigma_h^{n;\frac{1}{4}} \right) &= \frac{1}{2\mu} \left( \frac{\sigma_h^{n+\frac{1}{2}} - \sigma_h^{n-\frac{1}{2}}}{\Delta t}, \frac{\sigma_h^{n+\frac{1}{2}} + \sigma_h^{n-\frac{1}{2}}}{2} \right) \\ &= \frac{1}{4\mu \Delta t} \left( \left\| \sigma_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| \sigma_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right). \end{aligned}$$

In the same way we get

$$\frac{1}{\lambda} \left( \Delta_t p_h^n, p_h^{n;\frac{1}{4}} \right) = \frac{1}{2\lambda \Delta t} \left( \left\| p_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| p_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right).$$

Eq. (6.14) becomes

$$\begin{aligned} \frac{1}{2\Delta t} \left( \left\| \Delta_t u_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| \Delta_t u_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right) &+ \frac{1}{4\mu \Delta t} \left( \left\| \sigma_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| \sigma_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right) \\ &+ \frac{1}{2\lambda \Delta t} \left( \left\| p_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| p_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right) = (f^{n;\frac{1}{4}}, \Delta_t u_h^n). \end{aligned} \quad (6.15)$$

We then sum (6.15) from  $n = 1, \dots, N$ , and we get (6.10).  $\square$

**Theorem 6.5.** *Supposing  $\Delta t < 1$ , the implicit-in-time scheme defined by (6.3) is unconditionally stable i.e. without any condition linking the time step  $\Delta t$  and  $h$ ; more precisely:*

$$\begin{aligned} \frac{1}{2} \left\| \Delta_t u_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 &+ \frac{1}{2\mu} \left\| \sigma_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 \\ &\leq \left[ \left\| \Delta_t u_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2\mu} \left\| \sigma_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + T \left( \max_{t \in [0,T]} \|f(t)\|_{0,\Omega} \right)^2 \right] \times \exp(2T), \end{aligned} \quad (6.16)$$

$$\beta^* \left( \left\| u_h^{N+\frac{1}{2}} \right\|_{0,\Omega} + \left\| \omega_h^{N+\frac{1}{2}} \right\|_{0,\Omega} \right) \leq \frac{1}{2\mu} \left\| \sigma_h^{N+\frac{1}{2}} \right\|_{0,\Omega} + \frac{1}{\lambda} \left\| p_h^{N+\frac{1}{2}} \right\|_{0,\Omega}, \quad (6.17)$$

where  $\beta^*$  is the constant appearing in the inf-sup inequality (6.4).

**Proof.** By (6.10) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \left\| \Delta_t u_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 &+ \frac{1}{2\mu} \left\| \sigma_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 \leq \Delta t \sum_{n=1}^N \left\| f^{n;\frac{1}{4}} \right\|_{0,\Omega}^2 + \Delta t \sum_{n=1}^N \left\| \Delta_t u_h^n \right\|_{0,\Omega}^2 \\ &+ \left\| \Delta_t u_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2\mu} \left\| \sigma_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{\frac{1}{2}} \right\|_{0,\Omega}^2. \end{aligned} \quad (6.18)$$

Moreover

$$\begin{aligned} \sum_{n=1}^N \left\| \Delta_t u_h^n \right\|_{0,\Omega}^2 &= \sum_{n=1}^N \left\| \frac{\Delta_t u_h^{n+\frac{1}{2}} + \Delta_t u_h^{n-\frac{1}{2}}}{2} \right\|_{0,\Omega}^2 \\ &\leq \frac{1}{2} \left( \sum_{n=1}^N \left\| \Delta_t u_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \sum_{n=0}^{N-1} \left\| \Delta_t u_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 \right) \\ &= \frac{1}{2} \left( 2 \sum_{n=1}^{N-1} \left\| \Delta_t u_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \left\| \Delta_t u_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \left\| \Delta_t u_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 \right) \\ &\leq \sum_{n=0}^{N-1} \left\| \Delta_t u_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2} \left\| \Delta_t u_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2. \end{aligned}$$

Thus using our hypothesis  $\Delta t < 1$ , we obtain

$$\begin{aligned} \frac{1}{2} \left\| \Delta_t u_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2\mu} \left\| \sigma_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 &\leq \Delta t \sum_{n=1}^N \left\| f^{n;\frac{1}{4}} \right\|_{0,\Omega}^2 + \Delta t \sum_{n=0}^{N-1} \left\| \Delta_t u_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 \\ &+ \left\| \Delta_t u_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2\mu} \left\| \sigma_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{\frac{1}{2}} \right\|_{0,\Omega}^2. \end{aligned} \quad (6.19)$$

Discrete Gronwall's inequality (see [1,15]) applied to (6.19) yields

$$\begin{aligned} \frac{1}{2} \left\| \Delta_t u_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2\mu} \left\| \sigma_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{N+\frac{1}{2}} \right\|_{0,\Omega}^2 \\ \leq \left[ \left\| \Delta_t u_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| p_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{2\mu} \left\| \sigma_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 + T \left( \max_{t \in [0,T]} \|f(t)\|_{0,\Omega} \right)^2 \right] \times \exp(2T), \end{aligned}$$

which proves inequality (6.16). Finally (6.17) comes from the inf-sup condition (6.4) and the first equation of the implicit scheme (6.3) which implies  $\forall (\tau_h, q_h) \in \Sigma_{0,h}$ :

$$\left( \operatorname{div}(\tau_h - q_h \delta), u_h^{N+\frac{1}{2}} \right) + \left( \operatorname{as}(\tau_h), \omega_h^{N+\frac{1}{2}} \right) = -\frac{1}{2\mu} \left( \sigma_h^{N+\frac{1}{2}}, \tau_h \right) - \frac{1}{\lambda} \left( p_h^{N+\frac{1}{2}}, q_h \right). \quad (6.20)$$

Inequalities (6.16) and (6.17) imply that the quantities  $\left\| \sigma_h^{N+\frac{1}{2}} \right\|_{0,\Omega}$ ,  $\left\| p_h^{N+\frac{1}{2}} \right\|_{0,\Omega}$ ,  $\left\| u_h^{N+\frac{1}{2}} \right\|_{0,\Omega}$ ,  $\left\| \omega_h^{N+\frac{1}{2}} \right\|_{0,\Omega}$  are bounded independently of  $N$ , therefore proving the stability of the implicit scheme defined by (6.3).  $\square$

#### 6.4. A priori error estimates for the fully discrete implicit scheme

We shall prove the optimal error estimates between the solution of the fully discrete implicit-in-time mixed finite element problem and the solution of the continuous problem. To this end, we start with the proofs of the following lemmas:

**Lemma 6.6.** Let  $(\mathcal{T}_h)_{h>0}$  be a regular family of triangulations on  $\Omega$ . We suppose that  $(\mathcal{T}_h)_{h>0}$  satisfies the refinement rules R<sub>1</sub>–R<sub>3</sub>. Let  $((\widehat{\sigma}_h(t_n), \widehat{p}_h(t_n)), (\widehat{\omega}_h(t_n), \widehat{u}_h(t_n)))$  be the elliptic projection of  $((\sigma(t_n), p(t_n)), (\omega(t_n), u(t_n)))$  and set

$$\varepsilon_h^n = \sigma_h^n - \widehat{\sigma}_h(t_n), \quad \chi_h^n = u_h^n - \widehat{u}_h(t_n), \quad \psi_h^n = \omega_h^n - \widehat{\omega}_h(t_n) \quad \text{and} \quad r_h^n = p_h^n - \widehat{p}_h(t_n).$$

Under the hypotheses of Proposition 4.4, we have

$$\left\| \varepsilon_h^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)} + \left\| r_h^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)} \lesssim \Delta t \sum_{j=1}^N \left\| (u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_j) \right\|_{0,\Omega}, \quad (6.21)$$

where  $\varepsilon_h^{\bullet+\frac{1}{2}}$  denotes the mapping  $t_n \mapsto \varepsilon_h^{n+\frac{1}{2}}$  and  $r_h^{\bullet+\frac{1}{2}}$  the mapping  $t_n \mapsto r_h^{n+\frac{1}{2}}$ .

**Proof.** From (5.3) and (6.3), we may write the error system in the form

$$\begin{cases} \frac{1}{2\mu} (\varepsilon_h^{n+1}, \tau_h) + \frac{1}{\lambda} (r_h^{n+1}, q_h) + (\operatorname{div}(\tau_h - q_h \delta), \chi_h^{n+1}) + (\operatorname{as}(\tau_h), \psi_h^{n+1}) = 0, \\ (\operatorname{as}(\varepsilon_h^{n+1}), \theta_h) = 0, \\ \left( \operatorname{div} \left( \varepsilon_h^{n;\frac{1}{4}} - r_h^{n;\frac{1}{4}} \delta \right), v_h \right) = \left( -(u_{tt})^{n;\frac{1}{4}} + \Delta_t^2 \widehat{u}_h(t_n) + \Delta_t^2 \chi_h^n, v_h \right), \end{cases} \quad (6.22)$$

where  $\varepsilon_h^{n;\frac{1}{4}} := \frac{\varepsilon_h^{n+1} + 2\varepsilon_h^n + \varepsilon_h^{n-1}}{4}$ ,  $r_h^{n;\frac{1}{4}} := \frac{r_h^{n+1} + 2r_h^n + r_h^{n-1}}{4}$  and  $(u_{tt})^{n;\frac{1}{4}} := \frac{u_{tt}^{n+1} + 2u_{tt}^n + u_{tt}^{n-1}}{4}$ .

From (6.2) it follows that  $\chi_h^0 = 0$  and  $\chi_h^1 = 0$ . Thus arguing as in Remark 6.3, it follows from the system formed by the first two equations of the error system (6.22) for  $n = -1$  and for  $n = 0$ , that  $\varepsilon_h^0 = 0$ ,  $r_h^0 = 0$  and  $\varepsilon_h^1 = 0$ ,  $r_h^1 = 0$ , respectively, implying  $\varepsilon_h^{\frac{1}{2}} = 0$  and  $r_h^{\frac{1}{2}} = 0$ . Furthermore  $\Delta_t \chi_h^{\frac{1}{2}} = 0$ .

Applying the finite difference operator to the first equation of (6.22), we obtain

$$\frac{1}{2\mu} (\Delta_t \varepsilon_h^n, \tau_h) + \frac{1}{\lambda} (\Delta_t r_h^n, q_h) + (\operatorname{div}(\tau_h - q_h \delta), \Delta_t \chi_h^n) + (\operatorname{as}(\tau_h), \Delta_t \psi_h^n) = 0. \quad (6.23)$$

Now taking  $(\tau_h, q_h) = \left( \varepsilon_h^{n;\frac{1}{4}}, r_h^{n;\frac{1}{4}} \right)$  in this last equality and using the second equation of (6.22), we obtain

$$\frac{1}{2\mu} \left( \Delta_t \varepsilon_h^n, \varepsilon_h^{n;\frac{1}{4}} \right) + \frac{1}{\lambda} \left( \Delta_t r_h^n, r_h^{n;\frac{1}{4}} \right) + \left( \operatorname{div} (\varepsilon_h^{n;\frac{1}{4}} - r_h^{n;\frac{1}{4}} \delta), \Delta_t \chi_h^n \right) = 0. \quad (6.24)$$

The last equation of (6.22) with  $v_h = \Delta_t \chi_h^n$  gives

$$\left( \operatorname{div} \left( \varepsilon_h^{n;\frac{1}{4}} - r_h^{n;\frac{1}{4}} \delta \right), \Delta_t \chi_h^n \right) = \left( -(u_{tt})^{n;\frac{1}{4}} + \Delta_t^2 \widehat{u}_h(t_n) + \Delta_t^2 \chi_h^n, \Delta_t \chi_h^n \right). \quad (6.25)$$

Subtracting (6.25) from (6.24), we get

$$\frac{1}{2\mu} \left( \Delta_t \varepsilon_h^n, \varepsilon_h^{n;\frac{1}{4}} \right) + \frac{1}{\lambda} \left( \Delta_t r_h^n, r_h^{n;\frac{1}{4}} \right) = \left( (u_{tt})^{n;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_n) - \Delta_t^2 \chi_h^n, \Delta_t \chi_h^n \right).$$

Thus

$$\frac{1}{2\mu} \left( \Delta_t \varepsilon_h^n, \varepsilon_h^{n;\frac{1}{4}} \right) + \frac{1}{\lambda} \left( \Delta_t r_h^n, r_h^{n;\frac{1}{4}} \right) + \left( \Delta_t^2 \chi_h^n, \Delta_t \chi_h^n \right) = \left( (u_{tt})^{n;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_n), \Delta_t \chi_h^n \right). \quad (6.26)$$

We expand (6.26) to get

$$\begin{aligned} & \frac{1}{2\mu} \left( \frac{\varepsilon_h^{n+\frac{1}{2}} - \varepsilon_h^{n-\frac{1}{2}}}{\Delta t}, \frac{\varepsilon_h^{n+\frac{1}{2}} + \varepsilon_h^{n-\frac{1}{2}}}{2} \right) + \frac{1}{\lambda} \left( \frac{r_h^{n+\frac{1}{2}} - r_h^{n-\frac{1}{2}}}{\Delta t}, \frac{r_h^{n+\frac{1}{2}} + r_h^{n-\frac{1}{2}}}{2} \right) \\ & + \left( \frac{\Delta_t \chi_h^{n+\frac{1}{2}} - \Delta_t \chi_h^{n-\frac{1}{2}}}{\Delta t}, \frac{\Delta_t \chi_h^{n+\frac{1}{2}} + \Delta_t \chi_h^{n-\frac{1}{2}}}{2} \right) = \left( (u_{tt})^{n;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_n), \frac{\Delta_t \chi_h^{n+\frac{1}{2}} + \Delta_t \chi_h^{n-\frac{1}{2}}}{2} \right), \end{aligned}$$

so that

$$\begin{aligned} & \frac{1}{2\Delta t} \left( \frac{1}{2\mu} \left( \left\| \varepsilon_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| \varepsilon_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right) \right) + \frac{1}{2\Delta t} \left( \frac{1}{\lambda} \left( \left\| r_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| r_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right) \right) \\ & + \frac{1}{2\Delta t} \left( \left\| \Delta_t \chi_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| \Delta_t \chi_h^{n-\frac{1}{2}} \right\|_{0,\Omega}^2 \right) = \left( (u_{tt})^{n;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_n), \frac{\Delta_t \chi_h^{n+\frac{1}{2}} + \Delta_t \chi_h^{n-\frac{1}{2}}}{2} \right). \end{aligned} \quad (6.27)$$

Replacing  $n$  by  $j$  in Eq. (6.27), then summing these equations from  $j = 1$  to  $j = n$  and multiplying by  $2\Delta t$  both sides, we get

$$\begin{aligned} & \frac{1}{2\mu} \left( \left\| \varepsilon_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| \varepsilon_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 \right) + \frac{1}{\lambda} \left( \left\| r_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| r_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 \right) + \left\| \Delta_t \chi_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 - \left\| \Delta_t \chi_h^{\frac{1}{2}} \right\|_{0,\Omega}^2 \\ & = \Delta t \sum_{j=1}^n \left( (u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_j), \Delta_t \chi_h^{j+\frac{1}{2}} + \Delta_t \chi_h^{j-\frac{1}{2}} \right). \end{aligned} \quad (6.28)$$

Recalling that  $\varepsilon_h^{\frac{1}{2}} = 0$ ,  $r_h^{\frac{1}{2}} = 0$  and  $\Delta_t \chi_h^{\frac{1}{2}} = 0$ , (6.28) becomes

$$\frac{1}{2\mu} \left\| \varepsilon_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| r_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \left\| \Delta_t \chi_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 = \Delta t \sum_{j=1}^n \left( (u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_j), \Delta_t \chi_h^{j+\frac{1}{2}} + \Delta_t \chi_h^{j-\frac{1}{2}} \right). \quad (6.29)$$

Thus we have

$$\begin{aligned} & \frac{1}{2\mu} \left\| \varepsilon_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| r_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \left\| \Delta_t \chi_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 \leq \Delta t \sum_{j=1}^n \left\| (u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_j) \right\|_{0,\Omega} \left\| \Delta_t \chi_h^{j+\frac{1}{2}} + \Delta_t \chi_h^{j-\frac{1}{2}} \right\|_{0,\Omega} \\ & \leq 2\Delta t \left\| \Delta_t^{\bullet+\frac{1}{2}} \chi_h \right\|_{L^\infty(L^2)} \sum_{j=1}^n \left\| (u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_j) \right\|_{0,\Omega}, \end{aligned}$$

where  $\left\| \Delta_t^{\bullet+\frac{1}{2}} \chi_h \right\|_{L^\infty(L^2)} := \sup_{0 \leq j \leq N-1} \left\| \Delta_t \chi_h^{j+\frac{1}{2}} \right\|_{0,\Omega}$ . Then

$$\frac{1}{2\mu} \left\| \varepsilon_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \frac{1}{\lambda} \left\| r_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 + \left\| \Delta_t \chi_h^{n+\frac{1}{2}} \right\|_{0,\Omega}^2 \leq \left\| \Delta_t^{\bullet+\frac{1}{2}} \chi_h \right\|_{L^\infty(L^2)}^2 + (\Delta t)^2 \left( \sum_{j=1}^N \left\| (u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_j) \right\|_{0,\Omega} \right)^2. \quad (6.30)$$

Taking the supremum on  $n$  on the left-hand side of (6.30) we get

$$\begin{aligned} & \frac{1}{2\mu} \left\| \varepsilon_h^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)}^2 + \frac{1}{\lambda} \left\| r_h^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)}^2 + \left\| \Delta_t^{\bullet+\frac{1}{2}} \chi_h \right\|_{L^\infty(L^2)}^2 \\ & \leq \left\| \Delta_t^{\bullet+\frac{1}{2}} \chi_h \right\|_{L^\infty(L^2)}^2 + (\Delta t)^2 \left( \sum_{j=1}^N \left\| (u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_j) \right\|_{0,\Omega} \right)^2, \end{aligned}$$

where  $\left\| \varepsilon_h^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)} := \sup_{0 \leq j \leq N-1} \left\| \varepsilon_h^{j+\frac{1}{2}} \right\|_{0,\Omega}$  and  $\left\| r_h^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)} := \sup_{0 \leq j \leq N-1} \left\| r_h^{j+\frac{1}{2}} \right\|_{0,\Omega}$ .

This last inequality and our assumptions on  $\mu$  and  $\lambda$  yield (6.21).  $\square$

Now, in order to bound the right-hand side of (6.21), we write

$$(u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 \widehat{u}_h(t_j) = (u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 u(t_j) + \Delta_t^2 u(t_j) - \Delta_t^2 \widehat{u}_h(t_j), \quad (6.31)$$

and we start with the estimation of  $\left\| \Delta_t^2 u(t_j) - \Delta_t^2 \widehat{u}_h(t_j) \right\|_{0,\Omega}$ .

**Lemma 6.7.** Under the hypotheses of Lemma 6.6, we have

$$\begin{aligned} \Delta t \sum_{j=1}^N \left\| \Delta_t^2 u(t_j) - \Delta_t^2 \widehat{u}_h(t_j) \right\|_{0,\Omega} & \lesssim h \left[ |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |p_{tt}|_{L^\infty(H_\phi^{0,1})} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} \right. \\ & \quad \left. + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})} \right]. \end{aligned} \quad (6.32)$$

**Proof.** Although the proof of this result is similar to the one of estimation (6.37) in [1], we include it for the sake of completeness. If we denote by  $R_h$  the elliptic projection operator, we can write

$$\Delta_t^2 u(t_j) - \Delta_t^2 \widehat{u}_h(t_j) = (I - R_h) \Delta_t^2 u(t_j) = (I - R_h) \frac{u(t_{j+1}) - 2u(t_j) + u(t_{j-1}))}{(\Delta t)^2}.$$

By the Taylor expansion with integral remainder up to the second order term, we have

$$u(t_{j-1}) = u(t_j) - \Delta t u_t(t_j) + \frac{(\Delta t)^2}{2} u_{tt}(t_j) + \frac{1}{2} \int_{t_j}^{t_{j-1}} (t_{j-1} - s)^2 u_{ttt}(s) ds,$$

and

$$u(t_{j+1}) = u(t_j) + \Delta t u_t(t_j) + \frac{(\Delta t)^2}{2} u_{tt}(t_j) + \frac{1}{2} \int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 u_{ttt}(s) ds.$$

Summing these two equations and dividing by  $(\Delta t)^2$ , we obtain

$$\begin{aligned} \Delta_t^2 u(t_j) & = u_{tt}(t_j) + \frac{1}{2(\Delta t)^2} \left( \int_{t_j}^{t_{j-1}} (t_{j-1} - s)^2 u_{ttt}(s) ds + \int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 u_{ttt}(s) ds \right) \\ & = u_{tt}(t_j) + \frac{1}{2(\Delta t)^2} \left( - \int_{-\Delta t}^0 (\Delta t + t)^2 u_{ttt}(t + t_j) dt + \int_0^{\Delta t} (\Delta t - t)^2 u_{ttt}(t + t_j) dt \right). \end{aligned} \quad (6.33)$$

Applying the operator  $I - R_h$  to both sides, we obtain

$$\begin{aligned} (I - R_h) \Delta_t^2 u(t_j) & = (I - R_h) u_{tt}(t_j) + \frac{1}{2(\Delta t)^2} \left( - \int_{-\Delta t}^0 (\Delta t + t)^2 (I - R_h) u_{ttt}(t + t_j) dt \right. \\ & \quad \left. + \int_0^{\Delta t} (\Delta t - t)^2 (I - R_h) u_{ttt}(t + t_j) dt \right). \end{aligned}$$

Taking norms, we obtain

$$\begin{aligned} \|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} &\leq \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{1}{2} \frac{\sqrt{\Delta t}}{\sqrt{5}} \left[ \left( \int_{-\Delta t}^0 \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2 dt \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left( \int_0^{\Delta t} \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2 dt \right)^{\frac{1}{2}} \right] \\ &\leq \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{2}}{2} \frac{\sqrt{\Delta t}}{\sqrt{5}} \left( \int_{-\Delta t}^{\Delta t} \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2 dt \right)^{\frac{1}{2}} \\ &= \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{\Delta t}}{\sqrt{10}} \left( \int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I - R_h)u_{ttt}(s)\|_{0,\Omega}^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

And hence, we have found

$$\|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} \leq \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{\Delta t}}{\sqrt{10}} \left( \int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I - R_h)u_{ttt}(s)\|_{0,\Omega}^2 ds \right)^{\frac{1}{2}}.$$

From the continuous in time error estimate (5.8), we obtain

$$\|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} \lesssim h \left[ |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |p_{tt}|_{L^\infty(H_\phi^{0,1})} \right],$$

and by using (5.10), we get

$$\begin{aligned} \Delta t \int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I - R_h)u_{ttt}(s)\|_{0,\Omega}^2 ds &\lesssim \Delta t h^2 \left[ \int_{-\Delta t+t_j}^{\Delta t+t_j} (|u_{ttt}(s)|_{1,1;\phi,\Omega}^2 + |u_{ttt}(s)|_{1,\Omega}^2 + |p_{ttt}(s)|_{0,1;\phi,\Omega}^2) ds \right] \\ &\lesssim (\Delta t)^2 h^2 \left[ |u_{ttt}|_{L^\infty(H_\phi^{1,1})}^2 + |u_{ttt}|_{L^\infty(H^1)}^2 + |p_{ttt}|_{L^\infty(H_\phi^{0,1})}^2 \right]. \end{aligned}$$

As  $\Delta t < 1$ , we can write

$$\sqrt{\Delta t} \left( \int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I - R_h)u_{ttt}(s)\|_{0,\Omega}^2 ds \right)^{\frac{1}{2}} \lesssim h \left[ |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})} \right].$$

Summing these last inequalities from  $j = 1$  to  $j = N$ , we get

$$\begin{aligned} \Delta t \sum_{j=1}^N \|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} &\lesssim h \left[ |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |p_{tt}|_{L^\infty(H_\phi^{0,1})} \right. \\ &\quad \left. + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})} \right] \times \sum_{j=1}^N \Delta t \\ &\lesssim h \left[ |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |p_{tt}|_{L^\infty(H_\phi^{0,1})} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})} \right], \end{aligned}$$

which is the desired result.  $\square$

**Lemma 6.8.** Under the hypotheses of Lemma 6.6, we have

$$\begin{aligned} \left\| \varepsilon_h^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)} + \left\| r_h^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)} &\lesssim h \left[ |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} \right] \\ &\quad + (\Delta t)^2 \|u_{ttt}\|_{L^\infty(L^2)}. \end{aligned} \quad (6.34)$$

**Proof.** By (6.21), (6.31) and (6.32), it is clear that to obtain (6.34), we have to bound  $\|(u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 u(t_j)\|_{0,\Omega}$ . To this end, we may write

$$(u_{tt})^{j;\frac{1}{4}} - \Delta_t^2 u(t_j) = (u_{tt})^{j;\frac{1}{4}} - u_{tt}(t_j) + u_{tt}(t_j) - \Delta_t^2 u(t_j). \quad (6.35)$$

By the Taylor expansion, we have

$$u_{tt}(t_{j-1}) = u_{tt}(t_j) - \Delta t u_{ttt}(t_j) + \int_{t_j}^{t_{j-1}} (t_{j-1} - s) u_{tttt}(s) ds,$$

and

$$u_{tt}(t_{j+1}) = u_{tt}(t_j) + \Delta t u_{ttt}(t_j) + \int_{t_j}^{t_{j+1}} (t_{j+1} - s) u_{tttt}(s) ds.$$

Together, we get

$$u_{tt}(t_{j+1}) + u_{tt}(t_{j-1}) = 2u_{tt}(t_j) + \int_{t_j}^{t_{j+1}} (t_{j+1} - s) u_{tttt}(s) ds + \int_{t_j}^{t_{j-1}} (t_{j-1} - s) u_{tttt}(s) ds.$$

Thus

$$\begin{aligned} (u_{tt})^{j; \frac{1}{4}} - u_{tt}(t_j) &= \frac{1}{4} \left( \int_{t_j}^{t_{j+1}} (t_{j+1} - s) u_{tttt}(s) ds + \int_{t_j}^{t_{j-1}} (t_{j-1} - s) u_{tttt}(s) ds \right) \\ &= \frac{1}{4} \left( \int_0^{\Delta t} (\Delta t - t) u_{tttt}(t_j + t) dt + \int_{-\Delta t}^0 (\Delta t + t) u_{tttt}(t_j + t) dt \right), \end{aligned}$$

which implies:

$$\begin{aligned} \|(u_{tt})^{j; \frac{1}{4}} - u_{tt}(t_j)\|_{0,\Omega} &\leq \frac{1}{4} \frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{3}} \left[ \left( \int_{-\Delta t}^0 \|u_{tttt}(t + t_j)\|_{0,\Omega}^2 dt \right)^{\frac{1}{2}} + \left( \int_0^{\Delta t} \|u_{tttt}(t + t_j)\|_{0,\Omega}^2 dt \right)^{\frac{1}{2}} \right] \\ &\lesssim (\Delta t)^{\frac{3}{2}} \left( \int_{-\Delta t}^{\Delta t} \|u_{tttt}(t + t_j)\|_{0,\Omega}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\|(u_{tt})^{j; \frac{1}{4}} - u_{tt}(t_j)\|_{0,\Omega} \lesssim (\Delta t)^{\frac{3}{2}} \left( \int_{-\Delta t+t_j}^{\Delta t+t_j} \|u_{tttt}(s)\|_{0,\Omega}^2 ds \right)^{\frac{1}{2}}. \quad (6.36)$$

So that

$$\begin{aligned} \Delta t \sum_{j=1}^N \|(u_{tt})^{j; \frac{1}{4}} - u_{tt}(t_j)\|_{0,\Omega} &\lesssim (\Delta t)^2 \|u_{tttt}\|_{L^\infty(L^2)} \sum_{j=1}^N \Delta t \\ &\lesssim (\Delta t)^2 \|u_{tttt}\|_{L^\infty(L^2)}. \end{aligned} \quad (6.37)$$

Using once again the Taylor expansion with integral remainder, but up to the third order term this time, we get:

$$u(t_{j-1}) = u(t_j) - \Delta t u_t(t_j) + \frac{(\Delta t)^2}{2} u_{tt}(t_j) - \frac{(\Delta t)^3}{6} u_{ttt}(t_j) + \frac{1}{6} \int_{t_j}^{t_{j-1}} (t_{j-1} - s)^3 u_{tttt}(s) ds,$$

and

$$u(t_{j+1}) = u(t_j) + \Delta t u_t(t_j) + \frac{(\Delta t)^2}{2} u_{tt}(t_j) + \frac{(\Delta t)^3}{6} u_{ttt}(t_j) + \frac{1}{6} \int_{t_j}^{t_{j+1}} (t_{j+1} - s)^3 u_{tttt}(s) ds.$$

Thus

$$\begin{aligned} |\Delta_t^2 u(t_j) - u_{tt}(t_j)| &= \left| \frac{1}{6(\Delta t)^2} \left( \int_{-\Delta t}^0 (\Delta t + t)^3 u_{tttt}(t + t_j) dt + \int_0^{\Delta t} (\Delta t - t)^3 u_{tttt}(t + t_j) dt \right) \right| \\ &\leq \frac{1}{6} \frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{7}} \left[ \left( \int_{-\Delta t}^0 (u_{tttt}(t + t_j))^2 dt \right)^{\frac{1}{2}} + \left( \int_0^{\Delta t} (u_{tttt}(t + t_j))^2 dt \right)^{\frac{1}{2}} \right] \\ &\lesssim \Delta t^{\frac{3}{2}} \left( \int_{-\Delta t}^{\Delta t} (u_{tttt}(t + t_j))^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\|\Delta_t^2 u(t_j) - u_{tt}(t_j)\|_{0,\Omega} \lesssim (\Delta t)^{\frac{3}{2}} \left( \int_{-\Delta t+t_j}^{\Delta t+t_j} \|u_{tttt}(s)\|_{0,\Omega}^2 ds \right)^{\frac{1}{2}}. \quad (6.38)$$

So that

$$\begin{aligned} \Delta t \sum_{j=1}^N \|\Delta_t^2 u(t_j) - u_{tt}(t_j)\|_{0,\Omega} &\lesssim (\Delta t)^2 \|u_{tttt}\|_{L^\infty(L^2)} \sum_{j=1}^N \Delta t \\ &\lesssim (\Delta t)^2 \|u_{tttt}\|_{L^\infty(L^2)}. \end{aligned} \quad (6.39)$$

Finally using (6.21), (6.31), (6.32), (6.35), (6.37) and (6.39) and the triangle inequality, we obtain (6.34).  $\square$

We are now in a position to establish optimal error estimates between the solution of the fully discrete implicit-in-time mixed finite element problem and the solution of the continuous problem. To this end, we recall the following result:

**Lemma 6.9** ([8]). *Under hypotheses  $R_1$ – $R_2$ , the following error estimate holds for every  $q \in H_\phi^{0,1}(\Omega)$ ,*

$$\|q - P_h^1 q\|_{0,\Omega} \lesssim h|q|_{0,1;\phi,\Omega}, \quad (6.40)$$

where  $P_h^1$  denotes the  $L^2$ -orthogonal projection on  $\{\theta_h \in L^2(\Omega); \theta_h|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\}$ .

**Theorem 6.10.** *Let  $(\mathcal{T}_h)_{h>0}$  be a regular family of triangulations on  $\Omega$ . We suppose that  $(\mathcal{T}_h)_{h>0}$  satisfies the refinement conditions  $R_1$ – $R_3$ . Under the hypotheses of Proposition 4.4, the following error estimates hold*

$$\begin{aligned} &\sup_{0 \leq n \leq N-1} \left( \left\| \sigma \left( t_{n+\frac{1}{2}} \right) - \sigma_h^{n+\frac{1}{2}} \right\|_{0,\Omega} + \left\| p \left( t_{n+\frac{1}{2}} \right) - p_h^{n+\frac{1}{2}} \right\|_{0,\Omega} \right) \\ &\lesssim h \left[ |u|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} \right] \\ &\quad + (\Delta t)^2 \left[ \|u_{tttt}\|_{L^\infty(L^2)} + |u_{tt}|_{L^\infty(H^1)} + \|u_{tt}\|_{L^\infty(L^2)} \right], \end{aligned} \quad (6.41)$$

$$\begin{aligned} &\sup_{0 \leq n \leq N-1} \left( \left\| P_h^0 u \left( t_{n+\frac{1}{2}} \right) - u_h^{n+\frac{1}{2}} \right\|_{0,\Omega} + \left\| \omega \left( t_{n+\frac{1}{2}} \right) - \omega_h^{n+\frac{1}{2}} \right\|_{0,\Omega} \right) \\ &\lesssim h \left[ |u|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} \right] \\ &\quad + (\Delta t)^2 \left[ \|u_{tttt}\|_{L^\infty(L^2)} + |u_{tt}|_{L^\infty(H^1)} + \|u_{tt}\|_{L^\infty(L^2)} \right], \end{aligned} \quad (6.42)$$

$$\begin{aligned} &\sup_{0 \leq n \leq N-1} \left\| u \left( t_{n+\frac{1}{2}} \right) - u_h^{n+\frac{1}{2}} \right\|_{0,\Omega} \\ &\lesssim h \left[ |u|_{L^\infty(H_\phi^{1,1})} + |u|_{L^\infty(H^1)} + |u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H_\phi^{0,1})} + |u_{ttt}|_{L^\infty(H^1)} \right] \\ &\quad + (\Delta t)^2 \left[ \|u_{tttt}\|_{L^\infty(L^2)} + |u_{tt}|_{L^\infty(H^1)} + \|u_{tt}\|_{L^\infty(L^2)} \right]. \end{aligned} \quad (6.43)$$

**Proof.** By (5.5), we get:

$$\left\| \widehat{\sigma}_h^{\bullet+\frac{1}{2}} - \sigma^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)} + \left\| \widehat{p}_h^{\bullet+\frac{1}{2}} - p^{\bullet+\frac{1}{2}} \right\|_{L^\infty(L^2)} \lesssim h \left[ |u|_{L^\infty(H_\phi^{1,1})} + |p|_{L^\infty(H_\phi^{0,1})} \right]. \quad (6.44)$$

By the Taylor formula with integral remainder, we obtain

$$\left\| \sigma^{\bullet+\frac{1}{2}} - \sigma \left( t_{\bullet+\frac{1}{2}} \right) \right\|_{L^\infty(L^2)} + \left\| p^{\bullet+\frac{1}{2}} - p \left( t_{\bullet+\frac{1}{2}} \right) \right\|_{L^\infty(L^2)} \lesssim (\Delta t)^2 \left[ \|u_{tt}\|_{L^\infty(H^1)} + \|p_{tt}\|_{L^\infty(L^2)} \right]. \quad (6.45)$$

By (6.34), (6.44), (6.45), and the triangle inequality, we obtain (6.41).

Now, let us prove inequality (6.42). Subtracting from the first equation of system (6.3) with  $n + 1$  replaced by  $n + \frac{1}{2}$ , the first equation of system (3.3) at  $t = t_{n+\frac{1}{2}}$  and applying the uniform inf-sup inequality (6.4), we obtain

$$\begin{aligned} & \left\| u_h^{n+\frac{1}{2}} - P_h^0 u \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} + \left\| \omega_h^{n+\frac{1}{2}} - P_h^1 \omega \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} \\ & \lesssim \left[ \left\| \sigma_h^{n+\frac{1}{2}} - \sigma \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} + \left\| p_h^{n+\frac{1}{2}} - p \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} + \left\| P_h^1 \omega \left( t_{n+\frac{1}{2}} \right) - \omega \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} \right], \end{aligned}$$

which implies

$$\begin{aligned} & \left\| u_h^{n+\frac{1}{2}} - P_h^0 u \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} + \left\| \omega_h^{n+\frac{1}{2}} - \omega \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} \\ & \lesssim \left[ \left\| \sigma_h^{n+\frac{1}{2}} - \sigma \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} + \left\| p_h^{n+\frac{1}{2}} - p \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} + \left\| P_h^1 \omega \left( t_{n+\frac{1}{2}} \right) - \omega \left( t_{n+\frac{1}{2}} \right) \right\|_{0,\Omega} \right]. \end{aligned}$$

Then using the  $P_h^1$  projection error inequality (6.40), taking the supremum on  $n$  and applying (6.41), we obtain (6.42).

From the triangle inequality, the  $P_h^0$  projection error inequality (1.47) p. 27 of [16] (or (45) p. 624 of [17]), and (6.42), (6.43) now follows.  $\square$

## 7. Implementation and numerical results

In this section we will confirm our theoretical analysis by numerical tests. We begin by introducing the so called “Hybrid formulations” [6,18,19] for solving the implicit scheme (6.2)–(6.3). The numerical results are presented on an L-shaped domain. Given  $f : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ , a surface force density  $g : [0, T] \times \Gamma_N \rightarrow \mathbb{R}^2$  and the initial conditions  $u_0$  and  $u_1$ , the displacement field  $u$  satisfies the following equations:

$$\begin{cases} u_{tt} - \operatorname{div} \sigma_s(u) = f & \text{in } [0, T] \times \Omega, \\ u = 0 & \text{on } [0, T] \times \Gamma_D, \\ \sigma_s(u) \cdot n = g & \text{on } [0, T] \times \Gamma_N, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \\ u_t(0, \cdot) = u_1 & \text{in } \Omega. \end{cases} \quad (7.1)$$

### 7.1. Implicit-in-time scheme

#### 7.1.1. Hybrid formulation

Firstly, we introduce the enlarged space  $\tilde{\Sigma}_h$  (with respect to  $\Sigma_{h,0}$ ) by suppressing the requirement for its elements to have continuous normal component at the interfaces of the triangulation  $\mathcal{T}_h$ :

$$\tilde{\Sigma}_h := \{(\tau_h, q_h) \in [L^2(\Omega)]^{2 \times 2} \times L^2(\Omega); \forall T \in \mathcal{T}_h : q_{h|T} \in \mathbb{P}_1(T) \text{ and } \tau_{h|T} \in [\mathbb{P}_1(T)]^{2 \times 2} \oplus [\mathbb{R} \operatorname{Curl} b_T]^2\}.$$

The hybrid formulation corresponding to the mixed formulation (6.2)–(6.3) is as follows:

Find  $(\tilde{\sigma}_h^{n+1}, \tilde{p}_h^{n+1}, \lambda_h^{n+1}) \in \tilde{\Sigma}_h \times \Lambda_h$  and  $(\tilde{u}_h^{n+1}, \tilde{\omega}_h^{n+1}) \in V_h \times W_h$  such that

$$\tilde{u}_h^0 = \hat{u}_h(0), \quad \tilde{u}_h^1 = \hat{u}_h(\Delta t) \simeq \hat{u}_h(0) + \Delta t \hat{u}_{h,t}(0) + \frac{\Delta t^2}{2} \hat{u}_{h,tt}(0),$$

and

$$\begin{cases} \frac{1}{2\mu} (\tilde{\sigma}_h^{n+1}, \tau_h) + \frac{1}{\lambda} (\tilde{p}_h^{n+1}, q_h) + \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div} (\tau_h - q_h \delta) \cdot \tilde{u}_h^{n+1} dx + (\operatorname{as} (\tau_h), \tilde{\omega}_h^{n+1}) \\ \quad - \sum_{T \in \mathcal{T}_h} \int_{\partial T} \lambda_h^{n+1} (\tau_h - q_h \delta) \cdot n_T ds = 0, \quad \forall (\tau_h, q_h) \in \tilde{\Sigma}_h, \forall n \geq -1, \\ (\operatorname{as} (\tilde{\sigma}_h^{n+1}), \theta_h) = 0, \quad \forall \theta_h \in W_h, \forall n \geq -1, \\ (\Delta_t^2 \tilde{u}_h^n, v_h) - \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div} \left( \tilde{\sigma}_h^{n;\frac{1}{4}} - \tilde{p}_h^{n;\frac{1}{4}} \delta \right) \cdot v_h dx - (f^{n;\frac{1}{4}}, v_h) = 0, \quad \forall v_h \in V_h, \forall n \geq 1, \\ \sum_{T \in \mathcal{T}_h} \int_{\partial T} \mu_h (\tilde{\sigma}_h^{n+1} - \tilde{p}_h^{n+1} \delta) \cdot n_T ds = \sum_{T \in \mathcal{T}_h} \int_{\partial T \cap \Gamma_N} \mu_h \cdot g^{n+1} ds, \quad \forall \mu_h \in \Lambda_h, \forall n \geq -1, \end{cases} \quad (7.2)$$

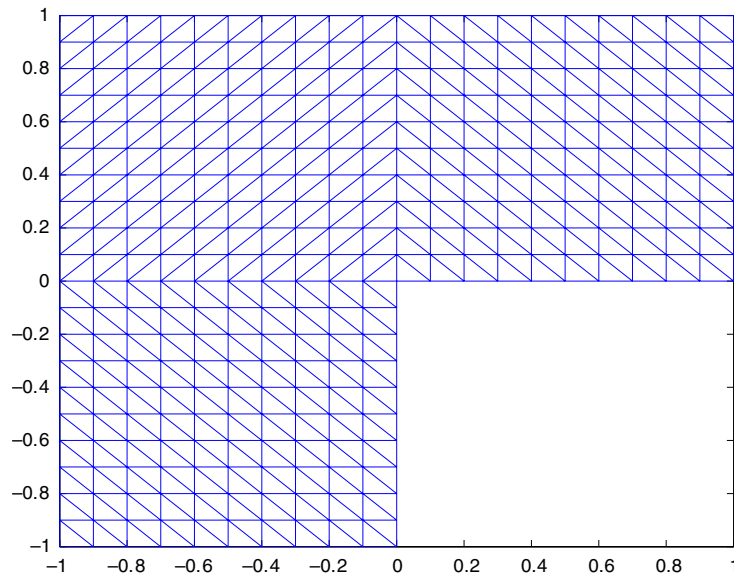


Fig. 1. Uniform meshes.

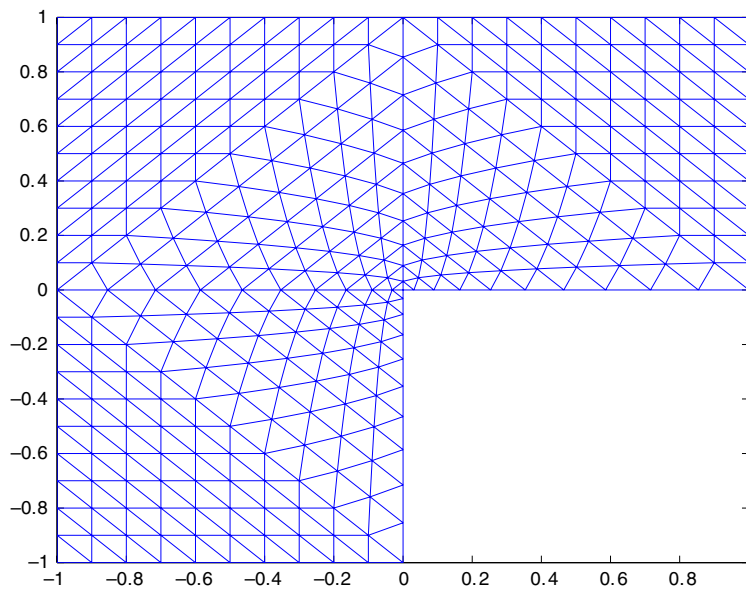


Fig. 2. Refined meshes.

where

$$\Lambda_h := \{\mu_h \in [L^2(\mathcal{E}_h)]^2; \mu_{h|e} \in [\mathbb{P}_1(e)]^2, \forall e \in \mathcal{E}_h \text{ and } \mu_{h|e} = 0, \forall e \subset \Gamma_D\}.$$

$\mathcal{E}_h$  denotes the set of all edges in  $\mathcal{T}_h$ . It is easy to prove that the hybrid formulation (7.2) is equivalent to the implicit-in-time discrete mixed formulation (6.3) i.e.  $\tilde{\sigma}_h^{n+1} = \sigma_h^{n+1}$ ,  $\tilde{p}_h^{n+1} = p_h^{n+1}$ ,  $\tilde{u}_h^{n+1} = u_h^{n+1}$  and  $\tilde{\omega}_h^{n+1} = \omega_h^{n+1}$ . The corresponding algebraic system takes the following form

$$\begin{cases} A\sigma_h^{n+1} + C^T u_h^{n+1} - E^T \lambda_h^{n+1} + H^T \omega_h^{n+1} = 0, \\ Pp_h^{n+1} - B^T u_h^{n+1} + G^T \lambda_h^{n+1} = 0, \\ H\sigma_h^{n+1} = 0, \\ Mu_h^{n+1} - C\sigma_h^{n+1} + Bp_h^{n+1} = F^{n+1}, \\ E\sigma_h^{n+1} - Gp_h^{n+1} = T^{n+1}, \end{cases} \quad (7.3)$$

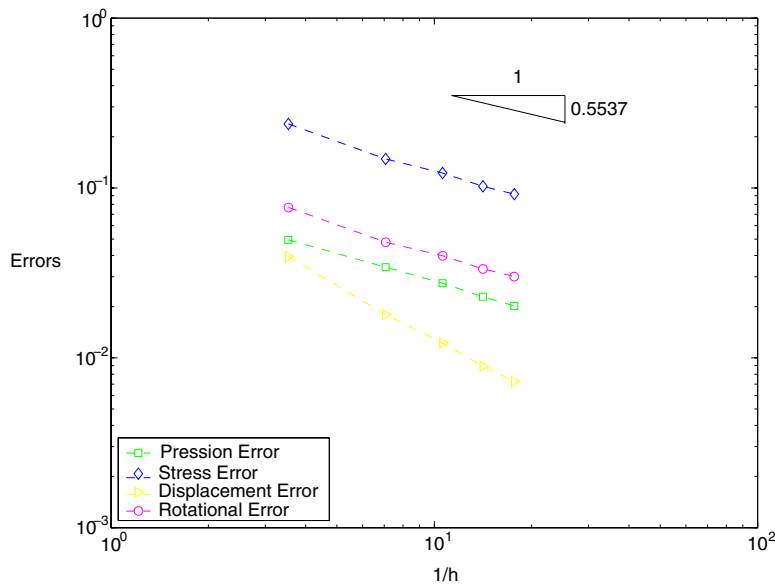


Fig. 3. Errors as a function of  $1/h$  for uniform meshes with time step  $\Delta t = h$ .

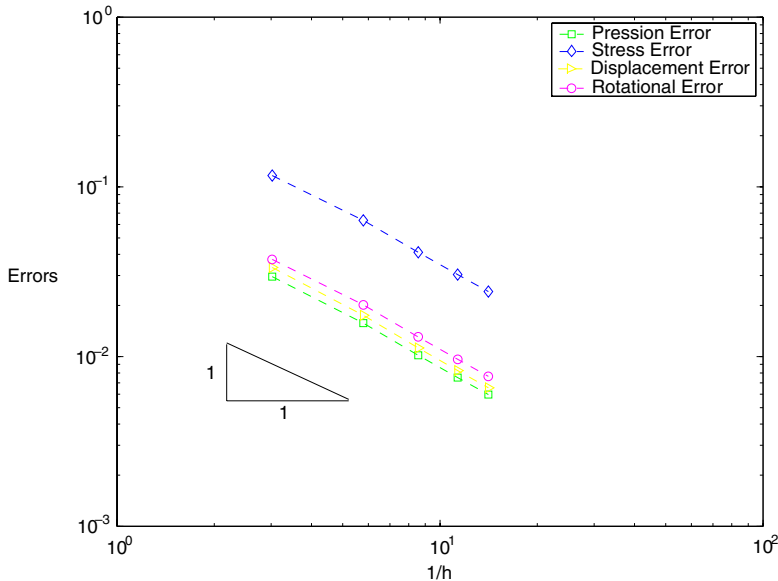


Fig. 4. Errors as a function of  $1/h$  for refined meshes with time step  $\Delta t = h$ .

where  $A, C, E, H, P, B, G, M$  are the corresponding matrices of the bilinear forms of the different terms appearing in system (7.2), and  $F^{n+1}$  is the second member vector at the  $n + 1$  time step obtained by putting, on the right-hand side of the third equation of system (7.2) multiplied by 4, the massic force term and all terms former to the time step  $n + 1$ . Thus we get the explicit form of  $F^{n+1}$ :

$$F_{|T}^{n+1} = 4P_{h|T}^0 f^{n+1/4} + 2 \operatorname{div}(\sigma_{h|T}^n - p_{h|T}^n \delta) + \operatorname{div}(\sigma_{h|T}^{n-1} - p_{h|T}^{n-1} \delta) + \frac{4}{(\Delta t)^2} (2u_{h|T}^n - u_{h|T}^{n-1}).$$

Finally  $T^{n+1}$  corresponds to the traction on the Neumann boundary  $\Gamma_N$  at the  $n + 1$  time step. The quantities  $\sigma_h^{n+1}, p_h^{n+1}, \omega_h^{n+1}$  and  $\lambda_h^{n+1}$  for  $n = -1$  and  $n = 0$ , needed to start the implicit hybrid scheme (7.3), are obtained by resolving the following system at  $n = -1$  and  $n = 0$ :

$$\begin{cases} A\sigma_h^{n+1} - E^T \lambda_h^{n+1} + H^T \omega_h^{n+1} = F_1^{n+1}, \\ Pp_h^{n+1} + G^T \lambda_h^{n+1} = F_2^{n+1}, \\ H\sigma_h^{n+1} = 0, \\ E\sigma_h^{n+1} - Gp_h^{n+1} = T^{n+1}, \end{cases} \quad (7.4)$$

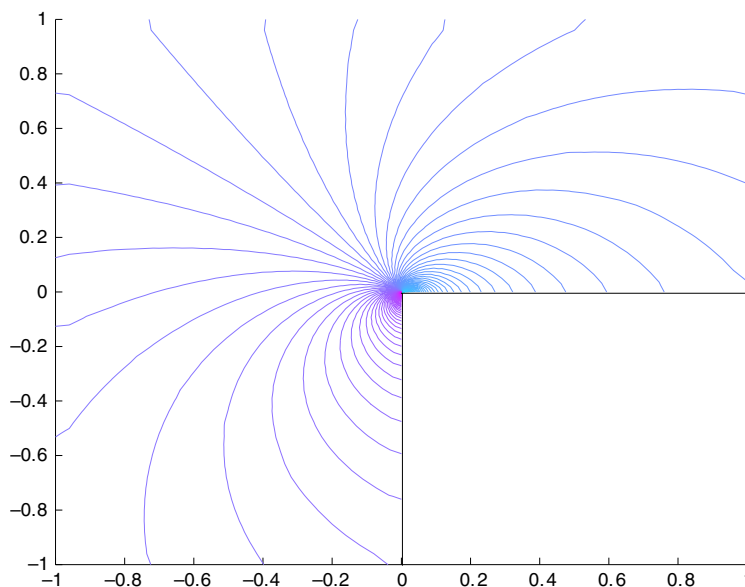


Fig. 5. Contourlines of the pressure.

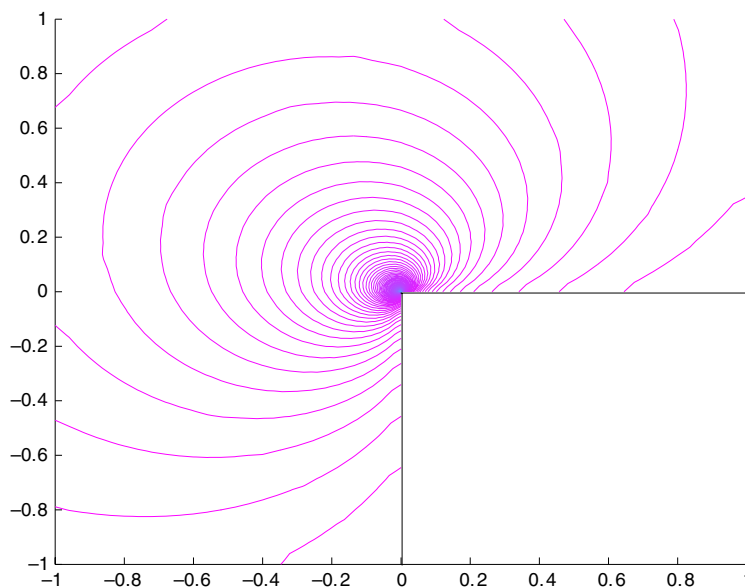


Fig. 6. Contourlines of the rotational.

where  $F_1^{n+1}, F_2^{n+1}$  are second members vector at the  $n+1$  time step obtained by replacing the variable  $u_h^{n+1}$  by its initial value in the first and second equations of system (7.3) and putting these terms on the right-hand side. In system (7.3), we start by eliminating  $\sigma_h^{n+1}$  and  $p_h^{n+1}$  and after we eliminate firstly  $\omega_h^{n+1}$  and secondly  $u_h^{n+1}$ . These eliminations are made element by element. After this procedure, we find the following system:

$$R\lambda_h^{n+1} = \mathcal{F}^{n+1}, \quad (7.5)$$

where

$$\begin{aligned} R = & GP^{-1}G^T + EA^{-1}E^T - EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}E^T - (EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}C^T \\ & - EA^{-1}C^T - GP^{-1}B^T)(M + CA^{-1}C^T + BP^{-1}B^T - CA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}C^T)^{-1} \\ & \times (CA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}E^T - CA^{-1}E^T - BP^{-1}G^T), \end{aligned}$$

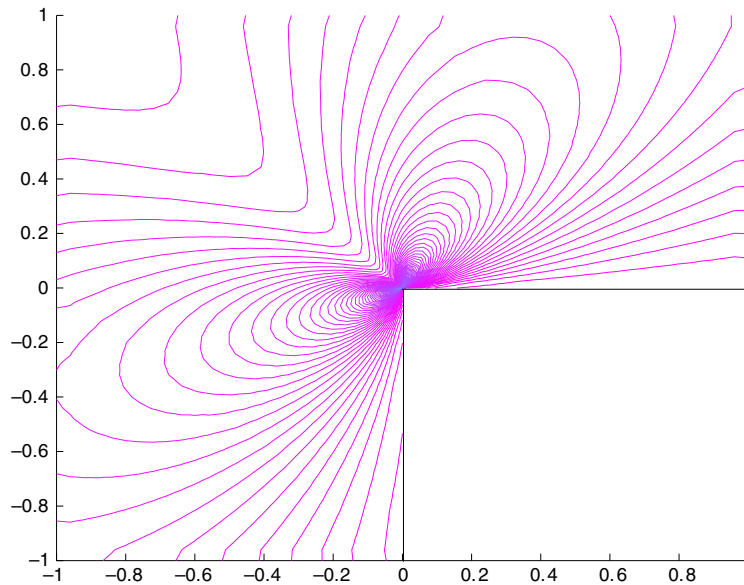


Fig. 7. Contourlines of the strain in the x direction.

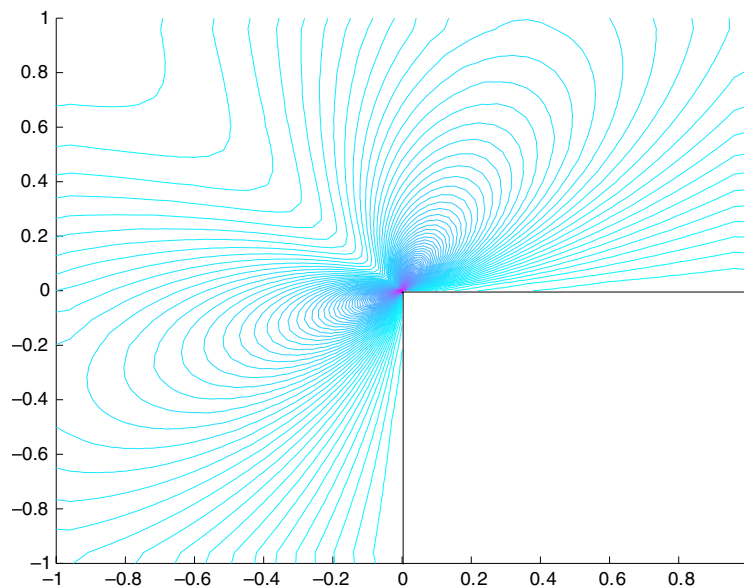


Fig. 8. Contourlines of the strain in the y direction.

and

$$\mathcal{F}^{n+1} = T^{n+1} - (EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}C^T - EA^{-1}C^T - GP^{-1}B^T)(M + CA^{-1}C^T + BP^{-1}B^T - CA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}C^T)^{-1}F^{n+1}.$$

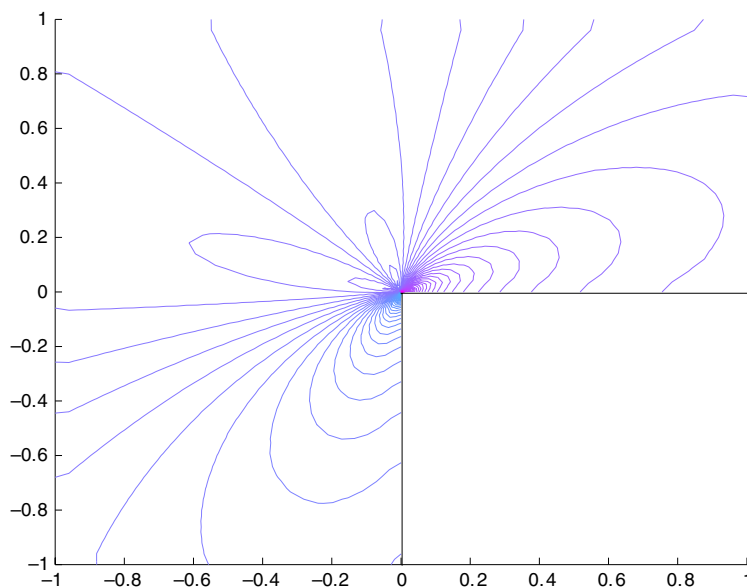
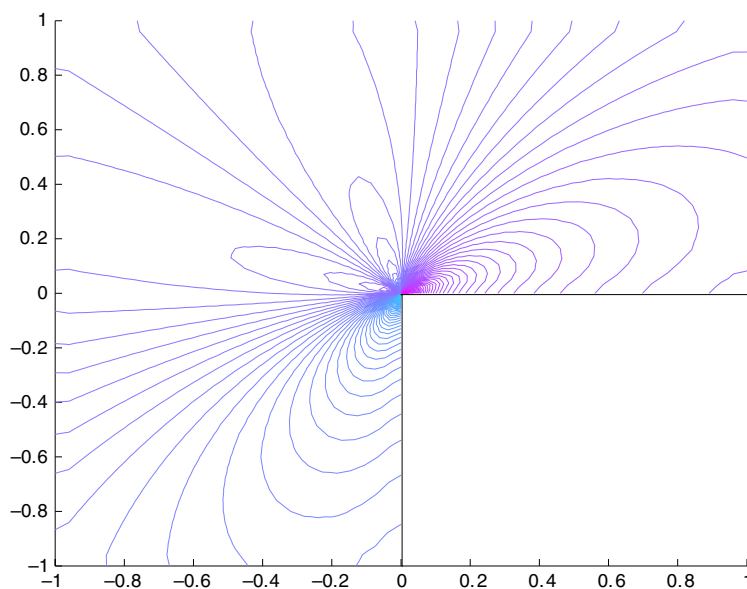
The matrix  $R$  is symmetric and positive definite.

#### 7.1.2. Numerical test with the implicit Newmark scheme

We now present some numerical results on a test problem in the L-shaped domain

$$\Omega = ]-1, 1[^2 \setminus ([0, 1[ \times ]-1, 0])$$

whose exact solution is the tensor product of the function  $t \mapsto e^{-t}$  with a singularity of the stationary Navier equation [20, Section 4.2, p. 52] arising at the reentrant corner of the L-shaped domain. The numerical tests are performed with

Fig. 9. Contourlines of  $\sigma_{1,2}$ .Fig. 10. Contourlines of  $\sigma_{2,1}$ .

$T = 1$  (second). Using polar coordinates  $(r, \theta)$ ,  $0 \leq \theta \leq \frac{3\pi}{2}$ , which are centered at the reentrant corner, we consider as analytical solution

$$u(r, \theta, t) = e^{-t} r^{\alpha_1} \vec{\phi}_{\alpha_1}(\theta)$$

where

$$\vec{\phi}_{\alpha_1}(\theta)_1 = C_1(\rho + \tau)\{\cos((\alpha_1 - 2)\theta) - \cos(\alpha_1\theta)\} + C_2((\rho + \tau)\sin((\alpha_1 - 2)\theta) + (\rho - 3\tau)\sin(\alpha_1\theta)),$$

$$\vec{\phi}_{\alpha_1}(\theta)_2 = C_1(-(\rho + \tau)\sin((\alpha_1 - 2)\theta) + (3\rho - \tau)\sin(\alpha_1\theta)) + C_2(\rho + \tau)\{\cos((\alpha_1 - 2)\theta) - \cos(\alpha_1\theta)\}.$$

The parameters are

$$C_1 = (\rho + \tau)\sin((\alpha_1 - 2)\omega) - (3\tau - \rho)\sin(\alpha_1\omega),$$

$$C_2 = (\rho + \tau)\{\cos(\alpha_1\omega) - \cos((\alpha_1 - 2)\omega)\},$$

$$\rho = \frac{\lambda + \mu}{\mu}(\alpha_1 - 1) - 2, \quad \tau = \frac{\lambda + \mu}{\mu}(\alpha_1 + 1) + 2,$$

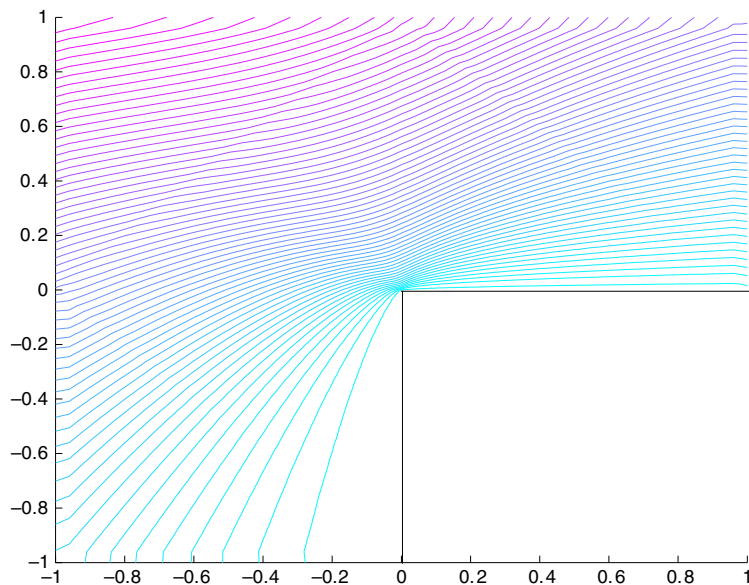


Fig. 11. Displacement in the x direction.

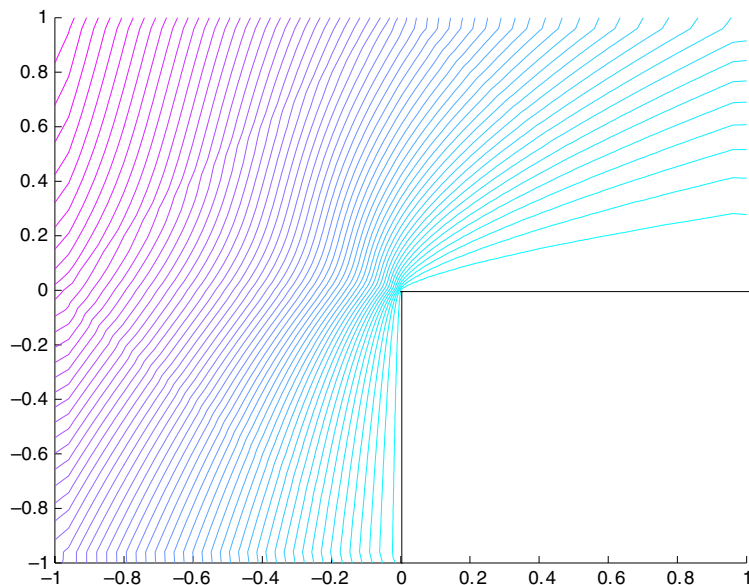


Fig. 12. Displacement in the y direction.

where  $\alpha_1 = 0.548643149483$  is the smallest strictly positive solution of the transcendental equation (4.2) for  $\omega = \frac{3\pi}{2}$ ,  $\lambda = 1000$ ,  $\mu = 20$ . In conformity with what we have said just after Definition 4.2, we have to choose  $\alpha \in ]1 - \alpha_1, \frac{1}{2}[$ . We use two kind of meshes. The first one (uniform) is obtained by dividing each of the intervals  $[0, 1]$  and  $[-1, 0]$  into  $n$  subintervals of length  $\frac{1}{n}$ , and then each square of sidelength  $\frac{1}{n}$  is divided into two triangles (see Fig. 1 where we have chosen  $n = 10$ ). The second kind of meshes (refined meshes) are obtained from the first ones by refinement near the reentrant corner  $(0, 0)$  according to Raugel's procedure [13] in order to satisfy the refinement rules  $R_1$ – $R_3$ . Namely,  $\Omega$  is divided into six big triangles; on the three which do not contain  $(0, 0)$ , a uniform mesh is used; on the other hand each big triangle admitting  $(0, 0)$  as a vertex is divided into strips according to the ratios  $(\frac{i}{n})^\beta$ ,  $1 \leq i \leq n$ , where  $\beta \geq \frac{1}{(1-\alpha)}$  along the sides which end up at  $(0, 0)$  and finally each of these strips divided uniformly (see Fig. 2 where we have chosen  $n = 10$  and  $\beta = 1.8$ ).

We represent the variations of the errors  $\|\sigma_h^N - \sigma(T)\|_{0,\Omega}$ ,  $\|p_h^N - p(T)\|_{0,\Omega}$ ,  $\|u_h^N - u(T)\|_{0,\Omega}$  and  $\|\omega_h^N - \omega(T)\|_{0,\Omega}$ , with respect to the mesh size  $h$ , in Figs. 3 and 4. The time step  $\Delta t$  is selected equal to the mesh size  $h$ . A double logarithmic scale was used such that the slope of the curves yields the order of convergence  $O(h)$  for refined meshes (see Fig. 4) according to

**Table 1**Convergence results when using uniform meshes at  $T = 1$  s with  $\Delta t = h$ .

$h = \Delta t$	Pressure errors	Strain errors	Displacement errors	Rotational errors
2.828427e–001	4.950115e–002	2.379090e–001	3.944810e–002	7.684252e–002
1.414214e–001	3.416687e–002	1.481635e–001	1.794047e–002	4.790442e–002
9.428090e–002	2.746780e–002	1.223985e–001	1.223487e–002	3.985001e–002
7.071068e–002	2.288484e–002	1.021814e–001	8.927111e–003	3.337248e–002
5.656854e–002	2.022493e–002	9.194350e–002	7.247770e–003	3.013416e–002

**Table 2**Convergence results when using refined meshes at  $T = 1$  s with  $\Delta t = h$ .

$h = \Delta t$	Pressure errors	Strain errors	Displacement errors	Rotational errors
3.307907e–001	2.957709e–002	1.164499e–001	3.349528e–002	3.736893e–002
1.727505e–001	1.574251e–002	6.344335e–002	1.751535e–002	2.017349e–002
1.167855e–001	1.017447e–002	4.111925e–002	1.124376e–002	1.304880e–002
8.819391e–002	7.523381e–003	3.039981e–002	8.271745e–003	9.635101e–003
7.084489e–002	5.969955e–003	2.412921e–002	6.540737e–003	7.641664e–003

the theoretical results, and  $O(h^{\frac{2}{3}})$  for uniform meshes (see Fig. 3) due to the singular behavior of the solutions. In Tables 1 and 2, we summarize the results on the errors for both uniform and refined meshes. Let us mention that the numerical example considered in this paper is exactly the same as in [1]. For the explicit in time scheme (see [1]), we have fixed  $\Delta t = 10^{-5}$  a very small time step because of the CFL stability condition. While the implicit-in-time scheme (7.2) requires no limitation on the time step and convergence results seem to be better than those concerning the explicit in time scheme (see [1]). This suggests that the implicit-in-time scheme (7.2) is preferable to the explicit in time scheme (Figs. 5–12).

## 8. Conclusion

We have constructed and analyzed a finite element method for approximating the elastodynamic system using the dual mixed formulation for spatial discretization and an implicit Newmark scheme in the time variable. In our analysis, we take into account the singularities of the solution due to the geometric singularities of the boundary. Optimal order  $L^\infty$ -in-time and  $L^2$ -in-space a priori error estimates are derived and a quadratic convergence rate in time for the fully discretized scheme has been established for the implicit Newmark numerical scheme. As mentioned above, the implicit-in-time scheme requires no limitation on the time step and convergence results seem to be better than those concerning the explicit in time scheme.

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