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A backward parabolic equation with a time-dependent coefficient: Regularization and error estimates

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ABSTRACT

We consider the problem of determining the temperature $u(x, t)$, for $(x, t) \in [0, \pi] \times [0, T)$ in the parabolic equation with a time-dependent coefficient. This problem is severely ill-posed, i.e., the solution (if it exists) does not depend continuously on the given data. In this paper, we use a modified method for regularizing the problem and derive an optimal stability estimation. A numerical experiment is presented for illustrating the estimate.

1. Introduction

Let T be a positive number, we consider the backward problem for the nonhomogeneous linear parabolic equation

$$u_t(x, t) - a(t)u_{xx}(x, t) = f(x, t), \quad (x, t) \in [0, \pi] \times (0, T] \tag{1}$$

$$u(0, t) = u(\pi, t) = 0, \quad t \in [0, T] \tag{2}$$

$$u(x, T) = g(x), \quad x \in [0, \pi] \tag{3}$$

where $a(t)$ is a function such that there exists $p, q > 0$

$$0 < p \le a(t) \le q. \tag{4}$$

Many physical and engineering problems in areas require the solution of the backward problem for the parabolic equation with a time-dependent coefficient. In general, the backward problem for the parabolic equation is ill-posed in the sense that the solution (if it exists) does not depend continuously on the given data. It means that a small perturbation on the data can affect the exact solution largely. Hence, it is difficult to calculate the regularized solution closing the exact solution and a regularization is necessary. In fact, the linear case has been studied in the past four decades by many scientists all over the

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world. Moreover, there were so many papers relating to the backward problem for the parabolic equation (see, e.g., [1–7]). In [8], the authors introduced a method which was called the quasi-reversibility method (QR method). They regularized the problem by using a "corrector term" in order to add it into the main equation. In a particular case, they investigated the problem

$$u_t + Ku - K^*Ku = 0,$$
$$u(x, T) = g(x).$$

We see that the above problem is useful if we can construct the adjoint operator K^*. In fact, the other approximated problem was more practical than the problem given in [9,10]

$$u_t + Ku - Ku_t = 0,$$
$$u(x, T) = g(x).$$

On the other hand, in 1983, Showalter presented the quasi-boundary value method (QBV method). By using the QBV method, they regularized the problem by adding the "corrector term" into the final condition. Applying this method, Dense and Bessila [2] used the final condition as follows

$$u(x, T) - \epsilon u_x(x, 0) = g(x).$$

As stated above, there are many works on the backward problem for the parabolic equation with a constant coefficient, the paper related to the time-dependent coefficient is very scarce. Recently, in [11], the authors consider the backward problem for the heat equation (with a constant coefficient) and obtain the error estimates between the regularized solution and the exact solution as follows

$$\|u^\epsilon(., t) - u(., t)\| \le C\epsilon^{\frac{t}{T}}, \quad \text{for } t > 0$$
$$\|u^\epsilon(., 0) - u(., 0)\| \le \sqrt[4]{8}C\sqrt[4]{T}\,(\ln(1/\epsilon))^{-\frac{1}{4}}, \quad \text{for } t = 0.$$

We can easily see that the above estimate tends to zero slowly when t is in a neighborhood of zero. That is the one disadvantage of this method (using in [11]). However, in [12], by requiring some acceptable assumptions of f and the exact solution u, the authors also improved the method (using in [12]) in order to obtain the better error estimate than [11]

$$\|u^\epsilon(., t) - u(., t)\| \le T_1(1 + \sqrt{M}) \exp\left\{\frac{3L^2TT_1^2(T - t)}{2}\right\} \epsilon^{t/T} \ln\left(\frac{T}{\epsilon}\right)^{\frac{t}{T}-1}.$$

Hence, in this paper a modified method is given for regularizing the backward problem with the time-dependent coefficient and obtain the error estimate that tends to zero more quickly than the logarithmic order.

In this paper, we also approximate (1)–(3) by using the regularization problem

$$u^\epsilon(g)(x, t) = \sum_{m=1}^{\infty}\left[\frac{\exp\{-m^2F(t)\}}{\beta + \exp\{-m^2F(T)\}}g_m - \int_t^T \frac{\exp\{m^2(F(s) - F(t) - F(T))\}}{\beta + \exp\{-m^2F(T)\}}f_m(s)ds\right]\sin(mx) \tag{5}$$

where

$$g_m = \frac{2}{\pi}\int_0^\pi g(x)\sin(mx)dx,$$
$$f_m(s) = \frac{2}{\pi}\int_0^\pi f(x, s)\sin(mx)dx,$$
$$F(t) = \int_0^t a(s)ds,$$

and $\beta = \beta(\epsilon)$ (denoting by β) are chosen later. The rest of this paper is divided into two sections. In Section 2, the regularization results and the proof of main results are presented. A numerical experiment is shown in Section 3 to illustrate the main results.

2. Regularization and error estimates

For clarity, we denote that $\|.\|$ is the norm in $L^2[0, \pi]$.

2.1. Main results

In this section, we shall give the regularized solution of (1)–(3) and estimate the error between the regularized solution and the exact solution. Hence, we need to find out the exact solution of (1)–(3). In fact, the exact solution of (1)–(3) satisfies

$$u(x,t)=\sum_{m=1}^{\infty}\left[\exp\{m^2(F(T)-F(t))\}g_m-\int_t^T \exp\{m^2(F(s)-F(t))\}f_m(s)ds\right]\sin(mx), \tag{6}$$

where

$$f_m(s)=\frac{2}{\pi}\int_0^{\pi} f(x,s)\sin(mx)dx,$$
$$g_m=\frac{2}{\pi}\int_0^{\pi} g(x)\sin(mx)dx,$$
$$F(t)=\int_0^t a(s)ds.$$

Hereafter, we need the following inequalities in order to evaluate error estimates.

Lemma 1. *Let $n\in\mathbb{R}$, $\gamma>0$, $0\le a\le b$ and $b\neq 0$ then we have*

$$\frac{e^{na}}{1+\gamma e^{nb}}\le \gamma^{-\frac{a}{b}}.$$

Lemma 2. *Let $a(t)$ be a function satisfying (4) and $t\in[0,T]$, $0<\beta<1$. Then for $m>0$, one has*

$$\frac{\exp\left\{-m^2\int_0^t a(s)ds\right\}}{\beta+\exp\left\{-m^2\int_0^T a(s)ds\right\}}\le \beta^{\frac{qt}{pT}-\frac{q}{p}}.$$

Lemma 3. *Let $a(t)$ be a function satisfying (4) and $t\in[0,T]$, $0<\beta<1$. Then for $m>0$, one has*

$$\frac{\beta\exp\left\{-m^2\int_0^t a(s)ds\right\}}{\beta+\exp\left\{-m^2\int_0^T a(s)ds\right\}}\le \beta^{\frac{pt}{qT}}.$$

The next theorem is proved for the stability of the modified method given in this paper.

Theorem 1 (*Stability of The Modified Method*)**.** *Let $u^{\epsilon}(g)$ and $u^{\epsilon}(h)$ be defined by (5) corresponding to the final values g and h in $L^2(0,\pi)$, respectively. Then we get*

$$\|u^{\epsilon}(g)(.,t)-u^{\epsilon}(h)(.,t)\|\le \beta^{\frac{qt}{pT}-\frac{q}{p}}\|g-h\|,$$

for every $t\in[0,T)$.

The following theorems give us the error estimate between the exact solution of (1)–(3) and the regularized solution (5) corresponding to the noise data g_{ϵ}.

Theorem 2. *Let $\epsilon\in(0,T)$, $g_{\epsilon}, g_{ex}\in L^2[0,\pi]$, u be the exact solution of problem (1)–(3) such that $Q=2\|u(.,0)\|^2<\infty$ and*

$$M=4\pi T\sum_{m=1}^{\infty}\int_0^T\int_0^{\pi}\left[\left|\exp\{m^2F(s)\}u_t(x,s)\right|^2+\left|\exp\{m^2F(s)\}a(s)u_{xx}(x,s)\right|^2\right]dxds<\infty.$$

If $\beta(\epsilon)=\epsilon^{\frac{p}{q}}$ and $u_{\epsilon}(g_{\epsilon})(.,t)$ is given by (5) then one has, for every $t\in[0,T)$,

$$\|u_{\epsilon}(g_{\epsilon})(.,t)-u(.,t)\|\le C_1\epsilon^{\frac{p^2t}{q^2T}},$$

where $C_1=1+\sqrt{Q+M}$.

Finally, the error estimate between the exact solution of (1)–(2) and the regularized solution is presented.

Theorem 3. *Let $\epsilon\in(0,T)$, $g_{\epsilon}, g_{ex}\in L^2[0,\pi]$ and u be the exact solution of problem (1)–(3) such that there exists a positive number $\gamma\in(0,qT)$ satisfying for all $t\in[0,T]$,*

$$\frac{\pi}{2}\sum_{m=1}^{\infty}\exp\{2\gamma m^2\}u_m^2(t)<A_2^2,$$

where $u_m(t) = \frac{2}{\pi}\int_0^\pi u(x,t)\sin(mx)dx$. *If we assume that* $\beta = \epsilon^{\frac{pT}{q(T+\gamma)}}$ *and* $u_\epsilon(g_\epsilon)(.,t)$ *is given by* (5) *then one has, for every* $t \in [0,T]$,

$$\|u(.,t) - u^\epsilon(g_\epsilon)(.,t)\| \le \epsilon^{\frac{t+\gamma}{T+\gamma}} + A_2\epsilon^{\frac{p\gamma}{q^2(T+\gamma)}}.$$

Remark.

1. In Theorem 2, it is easy to see that the error estimate between the exact solution and the regularized solution is $C_1\epsilon^{\frac{p^2t}{q^2T}}$. Hence, if t closes to the initial time $t = 0$, the convergence rate is very slow. Especially, if $t = 0$ then the regularized solution may not converge to the exact solution. To improve this point, we suggested another regularized parameter $\beta = \epsilon^{\frac{pT}{q(T+\gamma)}}$ (in Theorem 3) and obtain the better error than Theorem 2.
2. In Theorem 3, we required the condition on the expansion coefficient $u_m(t)$ and we consider that the assumption $\frac{\pi}{2}\sum_{m=1}^\infty e^{2\gamma m^2}u_m^2(t) < A_2^2$ does not depend on the function $f(x,t)$. Hence, this condition is acceptable.

2.2. Proof of the main theorem

Proof of Lemma 1. We have

$$\begin{aligned}\frac{e^{na}}{1+\gamma e^{nb}} &= \frac{e^{na}}{(1+\gamma e^{nb})^{\frac{a}{b}}(1+\gamma e^{nb})^{1-\frac{a}{b}}}\\ &\le \frac{e^{na}}{(1+\gamma e^{nb})^{\frac{a}{b}}}\\ &\le \gamma^{-\frac{a}{b}}.\end{aligned}$$

This completes the proof of Lemma 1. □

Proof of Lemma 2. From Lemma 1, we obtain

$$\frac{\exp\left\{-m^2\int_0^t a(s)ds\right\}}{\beta + \exp\left\{-m^2\int_0^T a(s)ds\right\}} \le \left(\frac{1}{\beta}\right)^{c(t)},$$

where $c(t) = \frac{\int_0^T a(s)ds - \int_0^t a(s)ds}{\int_0^T a(s)ds}$.

From (4), we get

$$F(T) = \int_0^T a(s)ds \ge \int_0^T pds = pT,$$

$$F(T) - F(t) = \int_t^T a(s)ds \le \int_t^T qds = q(T-t).$$

Then we have

$$\frac{\exp\left\{-m^2\int_0^t a(s)ds\right\}}{\beta + \exp\left\{-m^2\int_0^T a(s)ds\right\}} \le \left(\frac{1}{\beta}\right)^{\frac{q(T-t)}{pT}} = \beta^{\frac{qt}{pT}-\frac{q}{p}}.$$

This completes the proof of Lemma 2. □

Proof of Lemma 3. We have

$$\begin{aligned}\frac{\beta\exp\left\{-m^2\int_0^t a(s)ds\right\}}{\beta + \exp\left\{-m^2\int_0^T a(s)ds\right\}} &\le \beta\left(\frac{1}{\beta}\right)^{c(t)}\\ &= \beta^{1-c(t)}\\ &\le \beta^{\frac{pt}{qT}}.\end{aligned}$$

This completes the proof of Lemma 3. □

Proof of Theorem 1. From $u^\epsilon(g)$ and $u^\epsilon(h)$ as in (5) corresponding to the final values g and h, we get

$$u^\epsilon(g)(x,t) = \sum_{m=1}^{\infty} u_m^\epsilon(g)(t)\sin(mx),$$
$$u^\epsilon(h)(x,t) = \sum_{m=1}^{\infty} u_m^\epsilon(h)(t)\sin(mx),$$

where

$$u_m^\epsilon(g)(t) = \frac{\exp\{-m^2F(t)\}}{\beta+\exp\{-m^2F(T)\}}g_m - \int_t^T \frac{\exp\{m^2(F(s)-F(t)-F(T))\}}{\beta+\exp\{-m^2F(T)\}}f_m(s)ds,$$
$$u_m^\epsilon(h)(t) = \frac{\exp\{-m^2F(t)\}}{\beta+\exp\{-m^2F(T)\}}h_m - \int_t^T \frac{\exp\{m^2(F(s)-F(t)-F(T))\}}{\beta+\exp\{-m^2F(T)\}}f_m(s)ds$$

and

$$g_m = \frac{2}{\pi}\int_0^\pi g(x)\sin(mx)dx,$$
$$h_m = \frac{2}{\pi}\int_0^\pi h(x)\sin(mx)dx,$$
$$f_m(s) = \frac{2}{\pi}\int_0^\pi f(x,s)\sin(mx)dx.$$

By applying Lemma 2, we obtain the following estimate

$$\begin{aligned}\|u^\epsilon(g)(.,t)-u^\epsilon(h)(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=1}^{\infty}\left|u_m^\epsilon(g)(x,t)-u_m^\epsilon(h)(x,t)\right|^2\\ &= \frac{\pi}{2}\sum_{m=1}^{\infty}\left|\frac{\exp\{-m^2F(t)\}}{\beta+\exp\{-m^2F(T)\}}(g_m-h_m)\right|^2\\ &\le \beta^{\frac{2qt}{pT}-\frac{2q}{p}}\|g-h\|^2.\end{aligned}$$

Therefore, we get

$$\|u^\epsilon(g)(.,t)-u^\epsilon(h)(.,t)\| \le \beta^{\frac{qt}{pT}-\frac{q}{p}}\|g-h\|.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. From (5), we construct the regularized solution corresponding to the final values g_ϵ and g_{ex}

$$u^\epsilon(g_\epsilon)(x,t) = \sum_{m=1}^{\infty} u_m^\epsilon(g_\epsilon)(x,t)\sin(mx),$$
$$u^\epsilon(g_{ex})(x,t) = \sum_{m=1}^{\infty} u_m^\epsilon(g_{ex})(x,t)\sin(mx),$$

where

$$u_m^\epsilon(g_\epsilon)(t) = \frac{\exp\{-m^2F(t)\}}{\beta(\epsilon)+\exp\{-m^2F(T)\}}g_m^\epsilon - \int_t^T \frac{\exp\{m^2(F(s)-F(t)-F(T))\}}{\beta(\epsilon)+\exp\{-m^2F(T)\}}f_m(s)ds,$$
$$u_m^\epsilon(g_{ex})(t) = \frac{\exp\{-m^2F(t)\}}{\beta(\epsilon)+\exp\{-m^2F(T)\}}g_m^{ex} - \int_t^T \frac{\exp\{m^2(F(s)-F(t)-F(T))\}}{\beta(\epsilon)+\exp\{-m^2F(T)\}}f_m(s)ds.$$

From Theorem 1, we get

$$\begin{aligned}\|u^\epsilon(g_\epsilon)(.,t)-u^\epsilon(g_{ex})(.,t)\| &\le \beta^{\frac{qt}{pT}-\frac{q}{p}}\|g_\epsilon-g_{ex}\|\\ &\le \beta^{\frac{qt}{pT}-\frac{q}{p}}\epsilon.\end{aligned} \tag{7}$$

We get the exact solution of (1)–(3)

$$u(x,t) = \sum_{m=1}^{\infty} u_m(t)\sin(mx), \quad (x,t)\in[0,\pi]\times[0,T],$$

where

$$u_m(t) = \frac{\exp\{-m^2F(t)\}}{\exp\{-m^2F(T)\}}g_m - \int_t^T \frac{\exp\{m^2(F(s)-F(t)-F(T))\}}{\exp\{-m^2F(T)\}}f_m(s)ds.$$

Thus, we have

$$u_m(0) = \exp\{m^2F(T)\}g_m - \int_0^T \exp\{m^2F(s)\}f_m(s)ds.$$

Hence, we get

$$\begin{aligned} u_m(t) - u_m^{\epsilon}(g_{ex})(t) &= \frac{\exp\{-m^2F(t)\}}{\exp\{-m^2F(T)\}}g_m - \int_t^T \frac{\exp\{m^2(F(s)-F(t)-F(T))\}}{\exp\{-m^2F(T)\}}f_m(s)ds \\ &\quad - \left(\frac{\exp\{-m^2F(t)\}}{\beta+\exp\{-m^2F(T)\}}g_m - \int_t^T \frac{\exp\{m^2(F(s)-F(t)-F(T))\}}{\beta+\exp\{-m^2F(T)\}}f_m(s)ds\right) \\ &= \frac{\beta\exp\{-m^2F(t)\}}{\beta+\exp\{-m^2F(T)\}}\left(\exp\{m^2F(T)\}g_m - \int_t^T \exp\{m^2F(s)\}f_m(s)ds\right). \end{aligned} \tag{8}$$

From Lemma 3, we obtain

$$\begin{aligned} \left|u_m(t) - u_m^{\epsilon}(g_{ex})(t)\right| &\le \frac{\beta\exp\{-m^2F(t)\}}{\beta+\exp\{-m^2F(T)\}}\left|\exp\{m^2F(T)\}g_m - \int_t^T \exp\{m^2F(s)\}f_m(s)ds\right| \\ &\le \beta^{\frac{pt}{qT}}\left|u_m(0) + \int_0^t \exp\{m^2F(s)\}f_m(s)ds\right|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|u_\epsilon(g_{ex})(.,t) - u(.,t)\|^2 &= \frac{\pi}{2}\sum_{m=1}^{\infty}\left|u_m^{\epsilon}(g_{ex})(t) - u_m(t)\right|^2 \\ &\le \beta^{\frac{2pt}{qT}}\frac{\pi}{2}\sum_{m=1}^{\infty}\left|u_m(0) + \int_0^t \exp\{m^2F(s)\}f_m(s)ds\right|^2 \\ &\le \beta^{\frac{2pt}{qT}}\frac{\pi}{2}\sum_{m=1}^{\infty}\left[2\left|u_m(0)\right|^2 + 2\left|\int_0^t \exp\{m^2F(s)\}f_m(s)ds\right|^2\right]. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\|u_\epsilon(g_{ex})(.,t) - u(.,t)\|^2 \\ &\le \beta^{\frac{2pt}{qT}}\pi\sum_{m=1}^{\infty}|u_m(0)|^2 + \beta^{\frac{2pt}{qT}}\pi\sum_{m=1}^{\infty}\left(\int_0^T \left|\exp\{m^2F(s)\}f_m(s)\right|ds\right)^2 \\ &\le \beta^{\frac{2pt}{qT}}\pi\sum_{m=1}^{\infty}|u_m(0)|^2 + 2\beta^{\frac{2pt}{qT}}\sum_{m=1}^{\infty}\left(\int_0^T \left|\exp\{m^2F(s)\}\int_0^\pi f(x,s)\sin(mx)dx\right|ds\right)^2 \\ &= \beta^{\frac{2pt}{qT}}\pi\sum_{m=1}^{\infty}|u_m(0)|^2 + 2\beta^{\frac{2pt}{qT}}\sum_{m=1}^{\infty}\left(\int_0^T \left|\exp\{m^2F(s)\}\int_0^\pi (u_t(x,s) - a(s)u_{xx}(x,s))\sin(mx)dx\right|ds\right)^2 \\ &= 2\beta^{\frac{2pt}{qT}}\|u_{ex}(.,0)\|^2 + 2\beta^{\frac{2pt}{qT}}\sum_{m=1}^{\infty}\left(\int_0^T \left|\int_0^\pi \exp\{m^2F(s)\}(u_t(x,s) - a(s)u_{xx}(x,s))\sin(mx)dx\right|ds\right)^2 \\ &\le 2\beta^{\frac{2pt}{qT}}\|u_{ex}(.,0)\|^2 + 2T\beta^{\frac{2pt}{qT}}\sum_{m=1}^{\infty}\int_0^T \left|\int_0^\pi \left|\exp\{m^2F(s)\}(u_t(x,s) - a(s)u_{xx}(x,s))\right|dx\right|^2 ds. \end{aligned}$$

Thus, we get

$$\begin{aligned} \|u_\epsilon(g_{ex})(.,t) - u(.,t)\|^2 \le 2\beta^{\frac{2pt}{qT}}\|u_{ex}(.,0)\|^2 + 2\pi T\beta^{\frac{2pt}{qT}}\sum_{m=1}^{\infty}\int_0^T\int_0^\pi &\left|\exp\{m^2F(s)\}u_t(x,s)\right. \\ &\left. - \exp\{m^2F(s)\}a(s)u_{xx}(x,s)\right|^2 dxds \end{aligned}$$

$$\leq 2\beta^{\frac{2pt}{qT}} \|u_{ex}(.,0)\|^2 + 4\pi T \beta^{\frac{2pt}{qT}} \sum_{m=1}^{\infty} \int_0^T \int_0^\pi \left|\exp\{m^2F(s)\}u_t(x,s)\right|^2 + \left|\exp\{m^2F(s)\}a(s)u_{xx}(x,s)\right|^2 dxds$$

$$\leq \beta^{\frac{2pt}{qT}}(Q+M), \tag{9}$$

where

$$Q = 2\|u_{ex}(.,0)\|^2,$$

$$M = 4\pi T \sum_{m=1}^{\infty} \int_0^T \int_0^\pi \left[\left|\exp\{m^2F(s)\}u_t(x,s)\right|^2 + \left|\exp\{m^2F(s)\}a(s)u_{xx}(x,s)\right|^2\right] dxds.$$

Hence, from (7) and (9), we get

$$\|u_\epsilon(g_\epsilon)(.,t) - u(.,t)\| \leq \|u_\epsilon(g_\epsilon)(.,t) - u_\epsilon(g_{ex})(.,t)\| + \|u_\epsilon(g_{ex})(.,t) - u_{ex}(.,t)\|$$
$$\leq \beta^{\frac{qt}{pT}-\frac{q}{p}}\epsilon + \beta^{\frac{pt}{qT}}\sqrt{Q+M}.$$

Let $\beta(\epsilon) = \epsilon^{\frac{p}{q}}$, we obtain the estimate

$$\|u_\epsilon(g_\epsilon)(.,t) - u(.,t)\| \leq \left(\epsilon^{\frac{p}{q}}\right)^{\frac{qt}{pT}-\frac{q}{p}}\epsilon + \left(\epsilon^{\frac{p}{q}}\right)^{\frac{pt}{qT}}\sqrt{Q+M}$$
$$\leq \epsilon^{\frac{t}{T}} + \epsilon^{\frac{p^2t}{q^2T}}\sqrt{Q+M}$$
$$\leq C_1\epsilon^{\frac{p^2t}{q^2T}},$$

where $C_1 = 1 + \sqrt{Q+M}$.

This completes the proof of Theorem 2. □

Proof of Theorem 3. From Theorem 1, we get

$$\|u^\epsilon(g_\epsilon)(.,t) - u^\epsilon(g_{ex})(.,t)\| \leq \beta^{\frac{qt}{pT}-\frac{q}{p}}\|g_\epsilon - g_{ex}\|$$
$$\leq \beta^{\frac{qt}{pT}-\frac{q}{p}}\epsilon. \tag{10}$$

From (8), we have

$$u_m(t) - u_m^\epsilon(g_{ex})(t) = \frac{\beta}{\beta + \exp\{-m^2F(T)\}}u_m(t).$$

Thus, we obtain

$$\left|u_m(t) - u_m^\epsilon(g_{ex})(t)\right| = \frac{\beta\exp\{-\gamma m^2\}}{\beta + \exp\{-m^2F(T)\}}\exp\{\gamma m^2\}|u_m(t)|$$
$$\leq \beta^{\frac{\gamma}{qT}}\exp\{\gamma m^2\}|u_m(t)|.$$

Hence, we have

$$\|u_\epsilon(g_{ex})(.,t) - u(.,t)\|^2 = \frac{\pi}{2}\sum_{m=1}^{\infty}\left|u_m^\epsilon(g_{ex})(t) - u_m(t)\right|^2$$
$$\leq \beta^{\frac{2\gamma}{qT}}\frac{\pi}{2}\sum_{m=1}^{\infty}\exp\{2\gamma m^2\}|u_m(t)|^2$$
$$\leq \beta^{\frac{2\gamma}{qT}}A_2^2. \tag{11}$$

Thus, from (10) and (11), we get

$$\|u_\epsilon(g_\epsilon)(.,t) - u(.,t)\| \leq \|u_\epsilon(g_\epsilon)(.,t) - u_\epsilon(g_{ex})(.,t)\| + \|u_\epsilon(g_{ex})(.,t) - u(.,t)\|$$
$$\leq \beta^{\frac{qt}{pT}-\frac{q}{p}}\epsilon + \beta^{\frac{\gamma}{qT}}A_2.$$

Let $\beta = \epsilon^{\frac{pT}{q(T+\gamma)}}$, we obtain the estimate

$$\|u_\epsilon(g_\epsilon)(.,t) - u(.,t)\| \le \left(\epsilon^{\frac{pT}{q(T+\gamma)}}\right)^{\frac{qt}{pT}-\frac{q}{p}} \epsilon + \left(\epsilon^{\frac{pT}{q(T+\gamma)}}\right)^{\frac{\gamma}{qT}} A_2$$
$$= \epsilon^{\frac{t+\gamma}{T+\gamma}} + A_2 \epsilon^{\frac{p\gamma}{q^2(T+\gamma)}}.$$

This completes the proof of Theorem 3. □

3. Numerical experiment

Consider the linear nonhomogeneous parabolic equation with the time-dependent coefficient

$$u_t(x,t) - a(t)u_{xx}(x,t) = f(x,t), \quad (x,t) \in [0,\pi] \times (0,1]$$

where

$$a(t) = 2t+1, \qquad f(x,t) = -\frac{\sin(t)\sin(x)}{\exp(t^2+t)}. \tag{12}$$

The exact solution of the equation is

$$u_{ex}(x,t) = \frac{\cos(t)\sin(x)}{\exp(t^2+t)}. \tag{13}$$

Then we obtain

$$u_{ex}(x,1) = g_{ex}(x) = \frac{\cos(1)\sin(x)}{\exp(2)}. \tag{14}$$

From (3), we obtain

$$u_{ex}(x,1) = g_{ex}(x) = \frac{\cos(1)\sin(x)}{\exp(2)}. \tag{15}$$

Let $t = 0$, from (13), we have

$$u_{ex}(x,0) = \sin(x). \tag{16}$$

Consider the measured data

$$g_\varepsilon(x) = \left(1 + \varepsilon\sqrt{\frac{4\exp(4)}{\pi(\cos(2)+1)}}\right) g_{ex}(x), \tag{17}$$

then we have

$$\|g_\varepsilon - g_{ex}\|_2 = \varepsilon\sqrt{\frac{4\exp(4)}{\pi(\cos(2)+1)}}\,\|g_{ex}\|_2 = \varepsilon. \tag{18}$$

From (5) and (17), we have the regularized solution for the case $t = 0$

$$u_\varepsilon(g_\varepsilon)(x,0) = \sum_{m=1}^{\infty} u_m^\epsilon(g_\epsilon)(0)\sin(mx),$$

where

$$u_m^\epsilon(g_\epsilon)(0) = \frac{1}{\beta(\epsilon) + \exp\{-2m^2\}} g_m^\epsilon - \int_t^T \frac{\exp\{m^2(s^2+s-2)\}}{\beta(\epsilon)+\exp\{-2m^2\}} f_m(s)ds.$$

We consider $\varepsilon_1 = 10^{-1}$, $\varepsilon_2 = 10^{-5}$, $\varepsilon_3 = 10^{-50}$, $\varepsilon_4 = 10^{-60}$, $\varepsilon_5 = 10^{-100}$. We get the following table for the case $t = 0$.

ε	$\left\|u_{\varepsilon_i}(g_{\epsilon_i})(.,0) - u_{ex}(.,0)\right\|$
$\varepsilon_1 = 10^{-1}$	9.231542e−001
$\varepsilon_2 = 10^{-5}$	6.520434e−001
$\varepsilon_3 = 10^{-50}$	4.298486e−008
$\varepsilon_4 = 10^{-60}$	9.260810e−010
$\varepsilon_5 = 10^{-100}$	3.253301e−016

We have the following graphs (Figs. 1 and 2) of the exact solution $u_{ex}(.,t)$ and the regularized solution $u_{\varepsilon_i}(g_{\varepsilon_i})(.,t)$, $i = 1, 2, 3, 4, 5$.

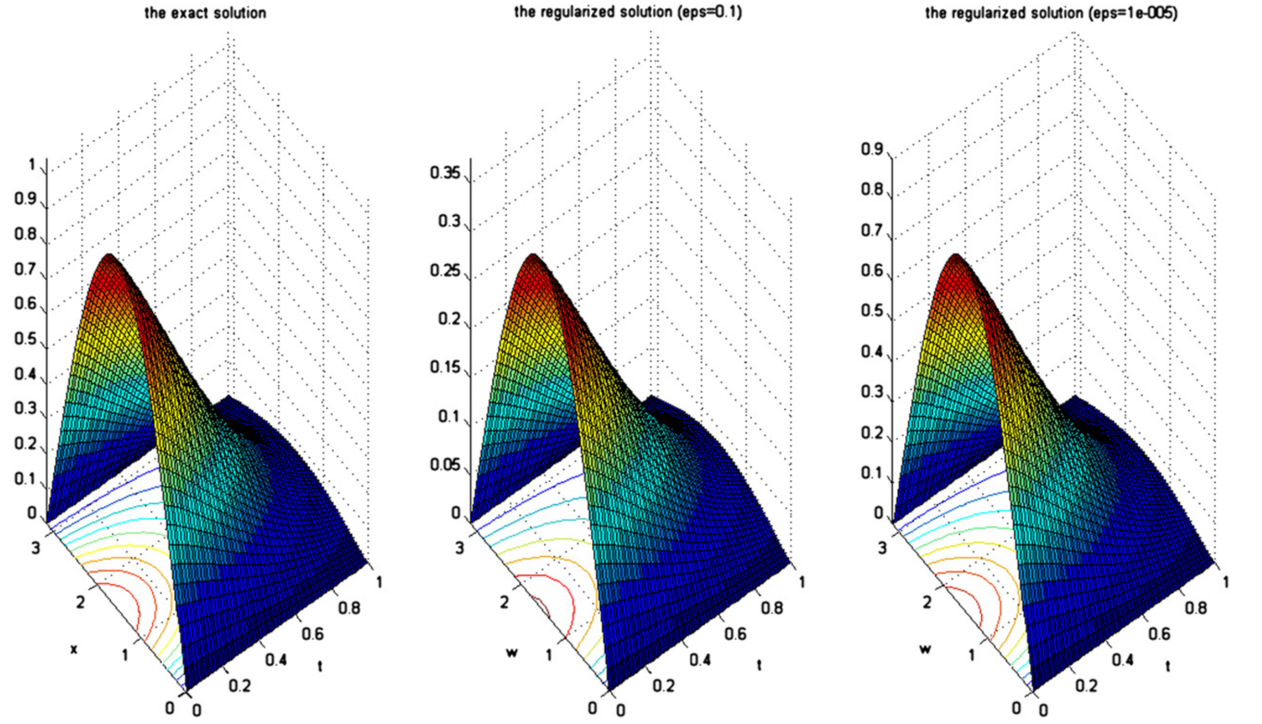

Fig. 1. The exact solution $u_{ex}(., t)$ and the regularized solution $u_{\varepsilon_i}(g_{\varepsilon_i})(., t)$, $i = 1, 2$.

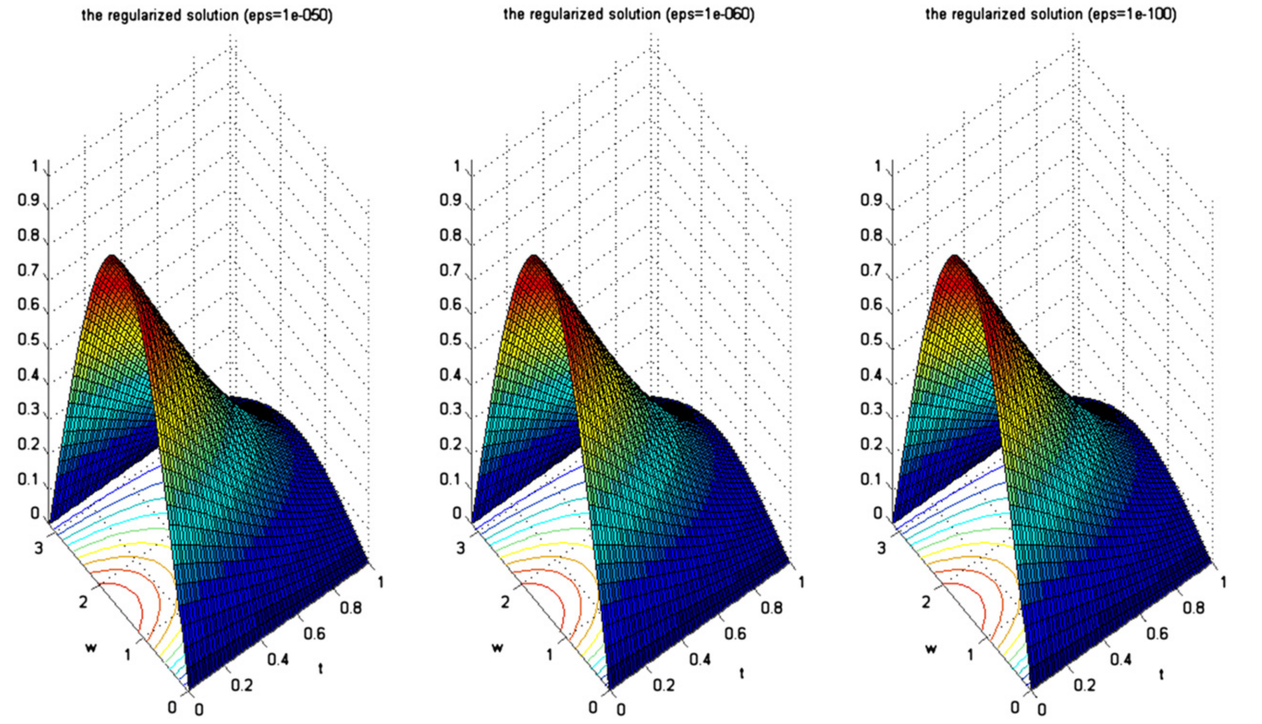

Fig. 2. The regularized solution $u_{\varepsilon_i}(g_{\varepsilon_i})(., t)$, $i = 3, 4, 5$.

Now, the figure can represent visually the exact solution and the regularized solution at initial time $t = 0$.

Notice that, in Fig. 3, the curve number 0 expressing the exact solution is indistinguishable from the curve number i expressing the regularized solution corresponding ε_i, $i = 3, \ldots, 5$.

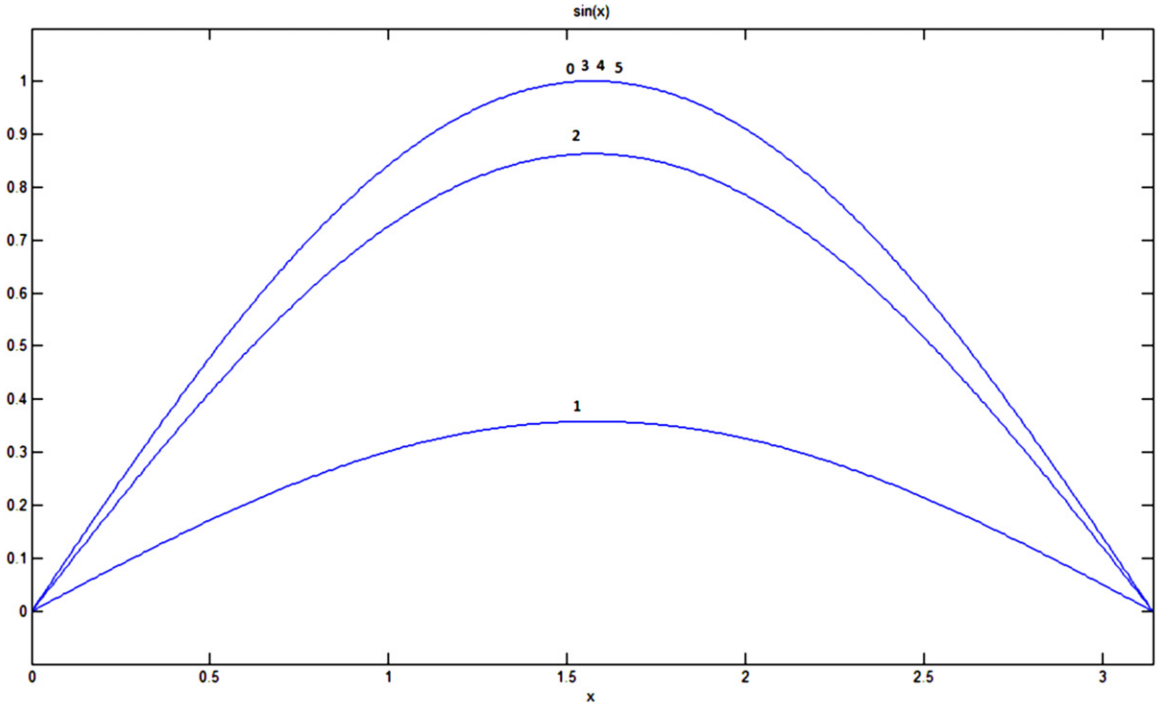

Fig. 3. The exact solution and the regularized solution at initial time $t = 0$.

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References

[1] Chu-Li Fu, Zhi Qian, Rui Shi, A modified method for a backward heat conduction problem, Appl. Math. Comput. 185 (2007) 564–573.
[2] M. Denche, K. Bessila, A modified quasi-boundary value method for ill-posed problems, J. Math. Anal. Appl. 301 (2005) 419–426.
[3] R.E. Ewing, The approximation of certain parabolic equations backward in time by Sobolev equations, SIAM J. Math. Anal. 6 (2) (1975) 283–294.
[4] Xiao-Li Feng, Zhi Qian, Chu-Li Fu, Numerical approximation of solution of nonhomogeneous backward heat conduction problem in bounded region, Math. Comput. Simul. 79 (2) (2008) 177–188.
[5] Chu-Li Fu, Xiang-Tuan Xiong, Zhi Qian, Fourier regularization for a backward heat equation, J. Math. Anal. Appl. 331 (1) (2007) 472–480.
[6] Dinh Nho Hao, Nguyen Trung Thanh, Hichem Sahli, Splitting-based conjugate gradient method for a multi-dimensional linear inverse heat conduction problem, J. Comput. Appl. Math. 232 (2) (2009) 361–377.
[7] Houde Han, Dongsheng Yin, A non-overlap domain decomposition method for the forward–backwardheat equation, J. Comput. Appl. Math. 159 (1) (2003) 35–44.
[8] G.W. Clark, S.F. Oppenheimer, Quasireversibility methods for non-well posed problems, Electron. J. Differential Equations (8) (1994) 1–9.
[9] S.M. Alekseeva, N.I. Yurchuk, The quasi-reversibility method for the problem of the control of an initial condition for the heat equation with an integral boundary condition, J. Differential Equations 34 (4) (1998) 493–500.
[10] Y. Huang, Q. Zhneg, Regularization for ill-posed Cauchy problems associated with generators of analytic semigroups, J. Differential Equations 203 (1) (2004) 38–54.
[11] D.D. Trong, P.H. Quan, T.V. Khanh, N.H. Tuan, A nonlinear case of the 1-D backward heat problem: regularization and error estimate, Z. Anal. Anwend. 26 (2) (2007) 231–245.
[12] Dang Duc Trong, Pham Hoang Quan, Nguyen Huy Tuan, A quasi-boundary value method for regularizing nonlinear ill-posed problems, Electron. J. Differential Equations 2009 (109) (2009) 1–16.