Journal of Computational and Applied Mathematics 259 (2014) 464–473

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Stochastic interest rate volatility modeling with a continuous-time GARCH(1, 1) model

Selçuk Bayracı *, Gazanfer Ünal

Department of International Finance, Yeditepe University, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 15 February 2013
Received in revised form 10 September 2013

Keywords:
Lévy process
NIG process
Interest rate volatility
GARCH
COGARCH
Indirect inference method

ABSTRACT

In this work, we develop a continuous-time GARCH(1, 1) (COGARCH(1, 1)) model driven by a NIG-Lévy process in order to analyze the volatility characteristics of Turkish interest rates. To our knowledge, this is the first work considering NIG-COGARCH modeling of interest rate data that utilizes the indirect inference method for parameter estimation. The discrete-time GARCH(1, 1) model has been used as a skeleton for building the NIG-COGARCH(1, 1) model. Daily interest rates on the Turkish two-year maturity treasury bond for the period between 02/01/2006 and 31/12/2010 have been used for the analysis. The empirical results show that the NIG-COGARCH(1, 1) model successfully captures the volatility clustering and heavy-tailed behavior of the interest rate returns and yields better in-sample estimations for conditional volatility in terms of forecast error statistics than the discrete-time model.

1. Introduction

There has been a massive amount of study dealing with the stochastic modeling of interest rate movements. The first approach to specify the interest rate movements as a continuous-time Itó process was introduced by Merton [1] in 1973. Later, in his seminal work in 1977, Vasicek [2] introduced an Ornstein–Uhlenbeck type of short rate model by incorporating mean-reverting characteristics of the data. However, there is a possibility that these models could generate negative interest rates. Negativity is not a big problem with real interest rates since they can often be negative, but nominal ones are unlikely to be negative. Moreover, these models are not adequate models for the interest rates because they do not handle the conditional volatility characteristics of the data. In order to overcome these issues, Cox, Ingersoll, and Ross (CIR) [3] introduced their square-root model in the general equilibrium context. The mean-reverting and 'level effect' characteristics of the CIR model helped in eliminating the possibility of negative interest rates and represent the conditional heteroskedasticity characteristics of the data [4].

Another approach in modeling interest rate movements is the use of autoregressive conditional heteroskedasticity (ARCH) models which were introduced by Engle [5] and later developed into generalized autoregressive conditional heteroskedasticity (GARCH) by Bollerslev [6], and to exponential GARCH (EGARCH) by Nelson [7]. These models have been widely used in the literature to model interest rate movements and they indicated volatility persistence in an extremely high degree, which was not addressed in the CIR model. In financial econometrics, discrete-time GARCH processes are widely used to model the returns at regular intervals on financial assets. Specifically, a GARCH process typically represents the increments of the logarithms ($\ln P_t - \ln P_{t-1}$) of the asset price. These models capture many of the so-called stylized features of such data, tail-heaviness, volatility clustering and dependence without correlation [8].

There have been several studies dealing with the capturing of the above mentioned features of financial time series using continuous-time models. The first study that links GARCH processes with continuous-time modeling is the GARCH diffusion

* Correspondence to: Yeditepe Üniversitesi, TBF, 26 Agustos Yerlesimi, Kayısdagi Caddesi, 34755, Atasehir, Istanbul, Turkey. Tel.: +90 2163106129.
E-mail addresses: selcuk.bayraci@yeditepe.edu.tr (S. Bayracı), gunal@yeditepe.edu.tr (G. Ünal).

0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.10.017

approximation of Nelson [9]. In this ground breaking work, price and volatility processes are driven by two independent Brownian motions such as

$$dG_t = \sigma_t dW_t$$
$$d\sigma_t^2 = \theta\left(\gamma - \sigma_t^2\right)dt + \rho\sigma_t^2 dB_t, \quad t \geqslant 0. \tag{1}$$

Another approach adopts the stochastic volatility model of Barndorff-Nielsen and Shephard [10] where the stochastic volatility σ^2 follows an Ornstein–Uhlenbeck type of process. The stochastic volatility is driven by a Lévy subordinator and evolves independently of the Brownian process in the price equation. The price and volatility equations satisfy

$$dG_t = \mu dt + \sigma_t dW_t$$
$$d\sigma_t^2 = -\alpha\sigma_t^2 dt + dL_t, \quad t \geqslant 0. \tag{2}$$

These straightforward diffusion limits of the discrete-time models loose most of the properties of the discrete-time GARCH model—for example, the limit in terms of Brownian motion no longer has jumps, the discrete-time and continuous-time models are not statistically equivalent, and the mechanisms of feedback between the price and volatility processes are lost due to two sources of randomness. In order to address these problems, Klüppelberg et al. [11] adopted the idea of directly constructing a continuous-time GARCH by using a single driving Lévy motion for price and volatility processes. Their construction uses an explicit representation of the discrete-time GARCH(1, 1) process to obtain a continuous-time analogue. The price and volatility processes evolve as

$$dG_t = \sigma_{t\_} dL_t$$
$$d\sigma_t^2 = \left(\omega - \eta\sigma_{t\_}^2\right)dt + \varphi\sigma_{t\_}^2\, d\left[L, L\right]_t^{(d)}. \tag{3}$$

Although the COGARCH(1, 1) model has already been investigated and its implementation and applicability to financial time series have been demonstrated, it is still an area of active research. For this study we have developed the Klüppelberg et al. [11] COGARCH(1, 1) model by introducing a normal inverse Gaussian (NIG) process for the driving Lévy process and an indirect inference method [12,13] for the parameter estimation. The rest of the paper is organized as follows. Section 2 examines the model building methodology along with the parameter estimation algorithm, and Section 3 gives the analysis of the data used. Section 4 presents and discusses empirical results, and Section 5 concludes.

2. Methodology

We follow the methodology of Klüppelberg et al. [11] in using a discrete-time GARCH(1, 1) model to build the continuous-time analogue. Recall the discrete-time GARCH(1, 1) process given as

$$y_n = \sigma_n z_n$$
$$\sigma_n^2 = \omega + \alpha y_{n-1}^2 + \beta\sigma_{n-1}^2 \tag{4}$$

where the z_n are i.i.d. zero mean and unit variance innovations and ω, α and β are constant parameters. The idea is to replace the white noise sequence of the price equation with the increments of a Lévy process. So, the price equation of the COGARCH model reads as

$$G_t = \int_0^t \sigma_{t\_} dL_t. \tag{5}$$

And we define the volatility process as a random recurrence equation and iterate the recurrence:

$$\sigma_n^2 = \omega + \alpha y_{n-1}^2 + \beta\sigma_{n-1}^2$$
$$\sigma_n^2 = \omega + (\beta + \alpha y_{n-1}^2)\sigma_{n-1}^2$$
$$\vdots$$
$$\sigma_n^2 = \omega\sum_{i=0}^{n-1}\prod_{j=i+1}^{n-1}\left(\beta + \alpha y_j^2\right) + \sigma_0^2\prod_{j=0}^{n-1}\left(\beta + \alpha\sigma_j^2\right).$$

On replacing the sum by an integral, the volatility process becomes

$$\sigma_n^2 = \left(\omega\int_0^n \exp\left(-\sum_{j=0}^{|s|}\log\left(\beta + \alpha y_j^2\right)ds\right) + \sigma_0^2\right)\exp\left(\sum_{j=0}^{n-1}\log\left(\beta + \alpha y_j^2\right)\right).$$

On replacing y_j by jumps of a Lévy process L and taking $\omega = \omega$, $\eta = -\log\beta$, $\varphi = \frac{\alpha}{\beta}$, we get the volatility process

$$\sigma_t^2 = \left(\omega\int_0^t e^{X_s} ds + \sigma_0^2\right)e^{-X_t} \tag{6}$$

with the auxiliary process
$X_t = t\eta - \sum_{0<t\leqslant t}\log\left(1 + \varphi\left(\Delta L_s\right)^2\right)$, $t \geqslant 0$, where $(L_t)_{t\geqslant 0}$ is a Lévy process.

Expressions (5) and (6), the price and volatility equations of the COGARCH(1, 1) model, can be expressed in terms of stochastic differential equations as

$$dG_t = \sigma_{t-} dL_t \tag{7}$$

$$d\sigma_t^2 = \left(\omega - \eta\sigma_{t-}^2\right) dt + \varphi\sigma_{t-}^2 d\,[L, L]_t^{(d)} \tag{8}$$

where $d\,[L, L]_t^{(d)} = \sum_{0<s\leqslant t} (\Delta L_s)^2$ is the quadratic variation of the Lévy process. All details concerning the derivation of the COGARCH(1, 1) model can be found in the original paper by Klüppelberg et al. [11].

2.1. The estimation procedure

Haug et al. [14] proposed a method of moments procedure for estimating the model parameters from the log-returns by matching the empirical autocorrelation function and moments to their theoretical counterparts. However, their methodology is only valid if the driving Lévy process has finite variance, e.g., is a Compound Poisson process with normally distributed jumps or a Variance-Gamma process. The study by Müller [15] shows that it is also possible to use Bayesian methods to estimate COGARCH parameters. This method is also limited to the case where the driving Lévy process is Compound Poisson. Maller et al. [16] developed a pseudo-maximum likelihood (PML) method based on the GARCH approximation to COGARCH [17].

Here, we propose a simulation based indirect inference approach in order to estimate the COGARCH(1, 1) model. The basic idea of the indirect inference method (IIM) proposed by Gouriéroux et al. [12] and later developed by Gallant and Tauchen [18], and Jiang [19], is that when a model leads to a complicated structural or reduced form and therefore to intractable likelihood functions, estimation of the original model can be indirectly achieved by estimating an auxiliary model which is constructed as an approximation of the original one.

For this paper, we follow the methodology proposed by Monfardini [20], where he applied an indirect inference method to estimate the parameters of a discrete-time stochastic volatility model. Let us call the parameters of the COGARCH(1, 1) model θ, with $\theta = (\omega, \eta, \varphi)$. Then, the estimation of θ based on indirect inference can be summarized as follows.

1. *Choice of the auxiliary model*

The first step in the indirect inference estimation method is to choose an auxiliary model; the natural choice of auxiliary model for the COGARCH(1, 1) model is the discretized version of the model where variance process can be written as a GARCH-type recursion:

$$\sigma_t^2 = \omega + (1-\eta)\sigma_{t-1}^2 + \varphi y_t^2 \tag{9}$$

where y_t is the observed log-returns.

2. *Estimation of the auxiliary model*

Let us define the parameter set of the auxiliary model as $\beta = (\omega, \eta, \varphi)$, and the likelihood function as $Q_T(y_T, \beta)$. Then, the parameters of the auxiliary model can be estimated as follows:

$$\hat{\beta}_T = \text{argmax}\,\{Q_T(y_T, \beta)\} \tag{10}$$

where

$$Q_T(y_T, \beta) = -\frac{1}{2}\sum_{t=1}^{T}\left(\frac{y_t^2}{\sigma_t^2}\right) - \frac{1}{2}\sum_{i=1}^{T}\log(\sigma_t^2) - \frac{T}{2}\log(2\pi).$$

Notice that the auxiliary parameter set β is defined implicitly in terms of the structural parameter vector θ, which is called the binding function [12]:

$$b(\theta) = E[\beta(y(\theta))]. \tag{11}$$

3. *Path simulation of the original process and estimation of the auxiliary model based on a simulated sampling path*

After we simulate the original model, we estimate the parameters of the auxiliary model from the observations of the simulated sampling path via the PML method. Let us denote the simulated vector of y_t's by $y_{TH}(\theta) = \left\{y^h(\theta), h = 1 \ldots TH\right\}$. After replacing the original observation with the simulated ones in (10), the binding function estimator is given by

$$\tilde{\beta}_{TH}(\theta) = \text{argmax}\,\{Q_T\,[y_{TH}(\theta), \beta]\}\,. \tag{12}$$

4. *Indirect estimation of the model parameters*

An indirect estimator of $\hat{\beta}$, denoted by $\tilde{\theta}_{TH}$, is defined by choosing values of θ from which $\hat{\beta}_T$ and $\tilde{\beta}_{TH}(\theta)$ are as close as possible, i.e.,

$$\tilde{\theta}_{TH} = \text{argmin}\left\{[\hat{\beta}_T - \tilde{\beta}_{TH}(\theta)]'\Omega[\hat{\beta}_{TH} - \tilde{\beta}_{TH}(\theta)]\right\} \tag{13}$$

where Ω is a symmetric nonnegative matrix, defining the metric [13].

Gouriéroux et al. [12] provide the expression for the optimal choice of the matrix Ω, which minimizes the asymptotic variance–covariance matrix of the indirect estimator $\tilde{\theta}_{TH}$. Let us define the following matrices:

$$I_0 = \lim_{T\to\infty} V_0 \left\{ \sqrt{T} \frac{\partial Q_T}{\partial \beta} \left[y^h(\theta_0), \beta_0 \right] \right\}$$

$$J_0 = \underset{T\to\infty}{\text{plim}} - \frac{\partial^2 Q_T}{\partial \beta \partial \beta'} \left[y^h(\theta_0), \beta_0 \right]$$

where V_0 represents the variance with respect to the true distribution of the y's process. By using the above matrices, the asymptotic variance–covariance matrix of $\tilde{\theta}_{TH}$ can be written as

$$W\left(H, \Omega\right) = \left(1 + \frac{1}{H}\right) d(\theta_0, \Omega) \frac{\partial b'}{\partial \theta} \Omega J_0^{-1} I_0 J_0^{-1} \Omega \frac{\partial b}{\partial \theta'(\theta_0) d(\theta_0, \Omega)}$$

with

$$d(\theta_0, \Omega) = \left[\frac{\partial b'}{\partial \theta}(\theta_0) \Omega \frac{\partial b}{\partial \theta'}(\theta_0) \right]^{-1}$$

then, the optimal choice of Ω is given by:

$$\Omega^* = J_0 I_0^{-1} J_0. \tag{14}$$

On replacing $\hat{\Omega}_T$ with the consistent estimator of Ω_T^* in (13), we can compute the optimal indirect estimator $\hat{\theta}_{TH}$. The asymptotic normal variance–covariance matrix of $\hat{\theta}_{TH}$ is given by

$$W_H^* = W\left(H, \Omega^*\right) = \left(1 + \frac{1}{H}\right) \left[\frac{\partial b'}{\partial \theta}(\theta_0) J_0 I_0^{-1} J_0 \frac{\partial b}{\partial \theta'}(\theta_0) \right]^{-1}.$$

An alternative method for implementing the indirect inference procedure by calibrating the parameter set $\theta(\omega, \eta, \varphi)$ through the score function was proposed by Gallant and Tauchen [18]. By choosing the value of θ which minimizes the score function of the auxiliary model, we can estimate the parameter of interest. The score function formula is given by

$$\hat{\theta}_{TH} = \text{argmin} \left\{ \frac{\partial Q_T}{\partial \beta'} \left[y_{TH}(\theta), \hat{\beta}_T \right] \hat{\Sigma}_T \frac{\partial Q_T}{\partial \beta} \left[y_{TH}(\theta), \hat{\beta}_T \right] \right\} \tag{15}$$

where $\hat{\Sigma}_T$ converges to a positive definite matrix Σ. Gouriéroux et al. [13] prove that $\hat{\theta}_{TH}$ is asymptotically equivalent to $\hat{\theta}_{TH}(J_0 \Sigma J_0)$; then the minimum variance estimator is obtained when $\Sigma^* = J_0^{-1} \Omega^* J_0^{-1} = I_0^{-1}$.

3. Data

The data that we used for this analysis are the daily (business days) rates on the Turkish treasury bonds. We employ log-returns of the Turkish Republic central bank's two-year maturity bond rates from January 03, 2006, to December 31, 2010 (1286 observations). Fig. 1 exhibits the plots of the interest rates, log-returns, squared log-returns, the histogram of the log-returns, and the autocorrelation functions (acf) of log-returns and squared log-returns.

In order to model the volatility, financial time series should be stationary; for the stationarity condition, time series should not contain a unit root. For this purpose, augmented Dickey–Fuller, Phillips–Perron, and Zivot–Andrews unit-root tests are applied to test the presence of a unit root in the log-return series. Table 1 presents the test statistics along with critical values at 5% significance levels. According to all three test statistics, interest rate series do not have a unit root at the log-return level.

Table 2 provides the descriptive statistics of the log-returns. It is observed from the skewness and kurtosis that interest rate log-returns are not normally distributed. And the autocorrelation plot of the squared log-returns and the ARCH-LM test statistics indicate the presence of ARCH effects in the log-return series. Also, we applied Engle's ARCH-LM [5] test to investigate whether ARCH effects are present in the log-returns. The procedure for the ARCH-LM test can be summarized as follows. Consider a time series:

$$y_t = \mu + \epsilon_t$$

where μ is the conditional mean of the process y_t, and ϵ_t is the innovation process. To test for the ARCH effects, regress the squared innovations ϵ_t^2 on a constant and m lagged values:

$$\epsilon_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \epsilon_{t-i}^2.$$

The null hypothesis states that, in the absence of ARCH effects, the estimated α_i parameters are equal to 0:

$$H_0 : \alpha_0 = \alpha_1 = \cdots = \alpha_m = 0.$$

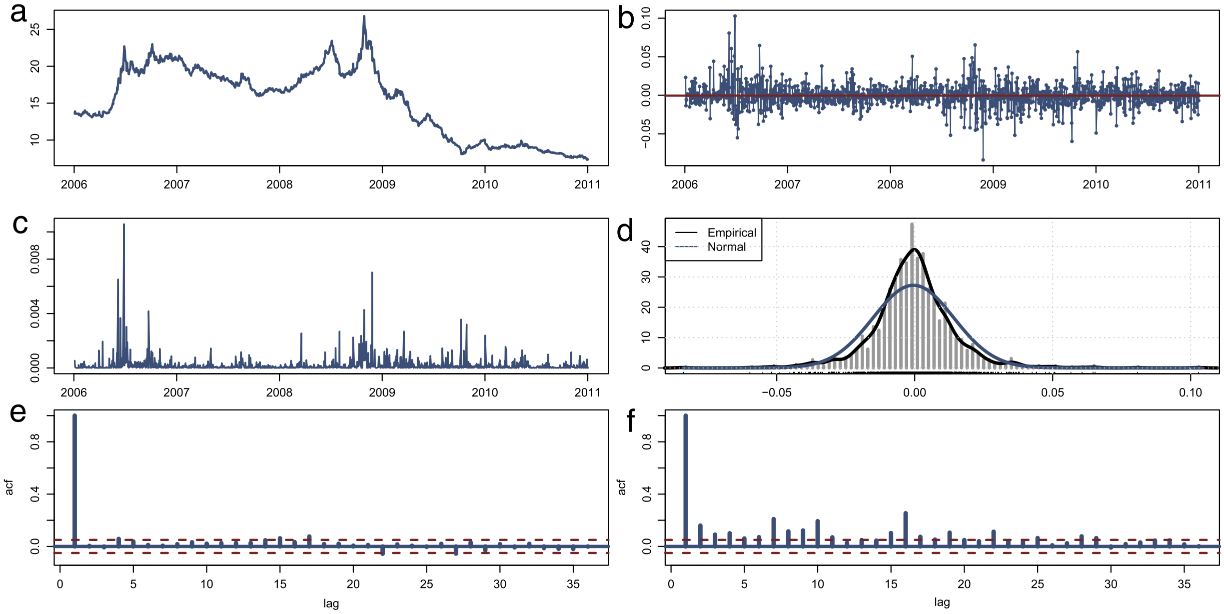

Fig. 1. (a) Interest rates, (b) log-returns, (c) squared log-returns, (d) the histogram of log-returns, (e) the acf of the log-returns, (f) the acf of the squared log-returns.

According to the alternative hypothesis, for the existence of ARCH effects, at least one of the estimated α_i parameters is statistically significant. The rejection of the null hypothesis states that there is a significant correlation of squared innovations. The ARCH-LM test statistic from Table 2 suggests the rejection of the null hypothesis. Therefore, log-return series show significant ARCH characteristics. From inspecting the empirical densities of log-returns from panel (d) of Fig. 1, we can identify the distributional properties. Compared to a normal distribution, there is mass near the origin, less mass in the flanks, and more mass in the tails, meaning that tiny movements occur with higher frequency, small and middle sized movements with lower frequency, and big changes are much more frequent than predicted by the normal law. Therefore, GARCH(p, q) models are appropriate tools to use in modeling the log-returns of Turkish interest rates, as they show leptokurtic and volatility clustering behavior.

4. Empirical results

Table 3 provides the results from the discrete-time GARCH(1, 1) model estimation for the interest rate log-returns. It is evident from the estimation results that the GARCH(1, 1) model is successful at capturing the volatility clustering behavior, as the coefficients for ARCH and GARCH terms have statistically significant t-values. The sum of the coefficients is less than 1, which means that the volatility process is covariance stationary. The Ljung–Box serial correlation test for the squared residuals indicates non-persistence of the serial correlation, and the ARCH-LM test confirms no remaining ARCH effects for the residuals. Thus, the discrete-time GARCH(1, 1) model is a good candidate model for conditional variance modeling. Although the GARCH(1, 1) model successfully eliminates the ARCH effects from the data, the normality assumption for the residuals is violated. The deviation from normality is evident for the standardized residuals, as the Kolmogorov–Smirnov and Anderson–Darling statistics from Table 4 suggest that a NIG distribution provides a better fit than the normal one. Also, density plots from panel (a) of Fig. 2 and quantile plots from panel (c) and panel (d) of Fig. 2 show that a NIG distribution is more successful for capturing the leptokurtic characteristics of the standardized residuals than the normal distribution. Hence, the distributional characteristics of the standardized residuals cannot be captured by the normal law, but can be captured by the hyperbolic law which was introduced by Eberlein [21] for financial applications. Later, Barndorff-Nielsen [22] introduced the normal inverse Gaussian (NIG) distribution to model the financial returns. In order to achieve a better fit to real-life data, we prefer the NIG process as the driving Lévy motion for the COGARCH(1, 1) model.

We fit a COGARCH(1, 1) model driven by a NIG-Lévy process by using the indirect inference algorithm, to the log-returns of Turkish interest rates. We simulated 1000 samples of $n = 1286$ observations and chose the parameters from the simulation minimizing the score function of the auxiliary model. Estimated parameters of the model are presented in Table 5. In Fig. 3 we plotted an example of simulated sample paths for the NIG-COGARCH(1, 1) process G, the log-return process dGt, the volatility process σ^2 and the driving NIG-Lévy process. The same random seed has been used in simulating all four sample paths.

In order to check the performance of the IIM for parameter estimation, by using estimated parameters from Table 5 we simulated 1000 sample paths with $n = 1286$ observations of the NIG-COGARCH(1, 1) log-return process dGt (the sample

Table 1
Unit-root tests.

	ADF test	P–P test	Z–A test
Interest rates (log-returns)	−19.5211 (−2.57)	−35.8993 (−2.86)	−8.9886 (−4.80)

Notes: Regression equations for unit-root tests contain a drift term. The table reports the statistics of the augmented Dickey–Fuller, Phillips–Perron and Zivot–Andrews tests. Critical values at the 5% level are given in brackets.

Table 2
Descriptive statistics.

	Interest rates (log-returns)	
Mean	−0.0005	
Standard deviation	0.0146	
Skewness	0.3917	
Kurtosis	5.1440	
Minimum	−0.0838	
Maximum	0.1028	
ARCH test		
	Test statistics	p-value
LM (35)	185.08	(2.2e−16)

Table 3
GARCH(1, 1) model estimation.

	Estimate	Standard error	t-value	Pr(> |t|)
ω	7.67e−6	3.60e−6	3.0132	0.0025
α	0.1091	0.0223	4.8797	0.0000
β	0.8584	0.0286	29.938	0.0000
Log-likelihood	3721.78			
	Test statistics	p-value		
LB^2 (35)	35.63	0.4790		
ARCH-LM (35)	35.71	0.4756		

Table 4
GARCH residual distributional tests.

	Kolmogorov–Smirnov		Anderson–Darling	
	Statistic	p-value	Statistic	p-value
Normal	0.066	0.007	4.11	0.007
NIG	0.027	0.697	0.390	0.569

Table 5
COGARCH(1, 1) model estimation.

$\hat{\omega}$	$\hat{\eta}$	$\hat{\varphi}$
7.77e−06	0.107	0.073
	Test statistics	p-value
LB^2 (35)	41.01	0.2546
ARCH-LM (35)	41.97	0.2229

size is chosen as 1286 in order to match the length of the empirical data). We estimated parameters of the simulated sample series via IIM and PML methods in order to allow comparison between parameter estimation methods. The results of our simulation study are given in Table 6. The estimated parameters and the empirical mean of all simulated parameters, with their standard deviations in brackets, are reported. We also estimated the root mean square error (RMSE), mean absolute error (MAE) and average bias for the parameter values. The reported error statistics RMSE, MAE and bias suggest that IIM method provides better estimates for parameters $\hat{\omega}$ and $\hat{\eta}$. For estimating parameter $\hat{\varphi}$, according to RMSE and average bias statistics, the PML method outperforms IIM while IIM is superior to the PML method in terms of MAE statistic.

To investigate the model fit, we performed a Ljung–Box test for each sample of the squared residuals. We computed the test statistic based on 35 lags, and had to reject the null hypothesis of no correlation 77 times out of 1000 simulations at

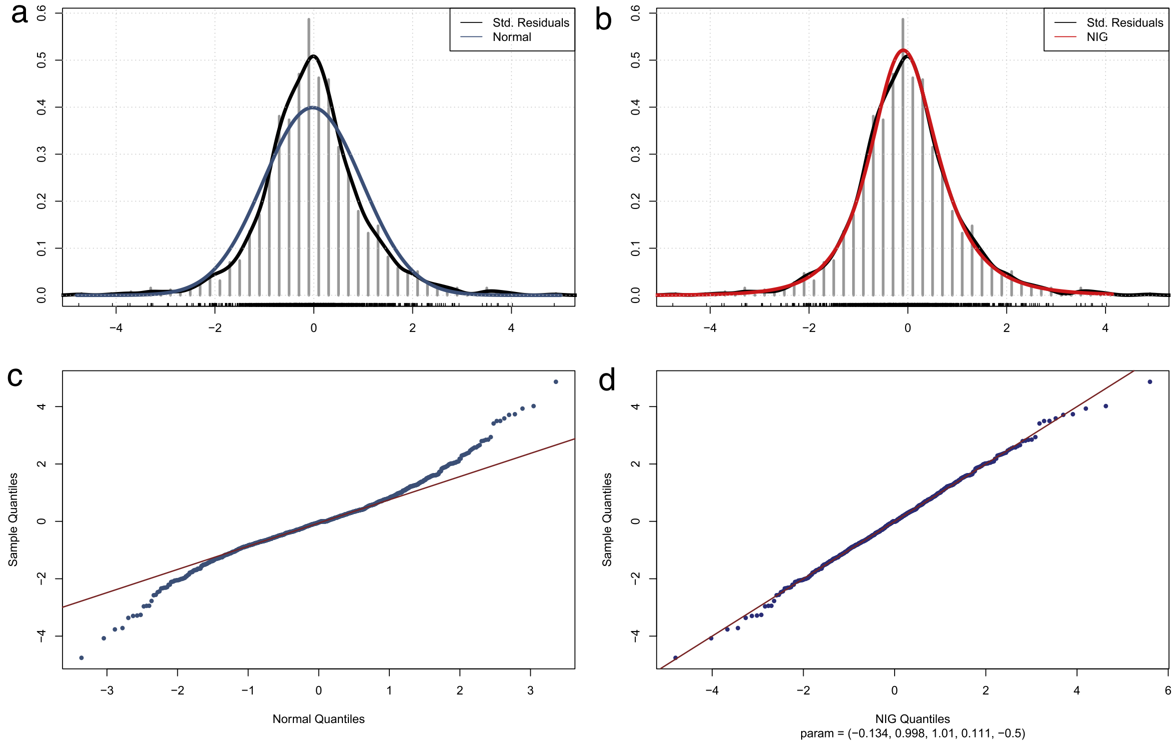

Fig. 2. (a) Density of residuals (normal fitted), (b) density of residuals (NIG fitted), (c) normal *QQ*-plot, (d) NIG *QQ*-plot.

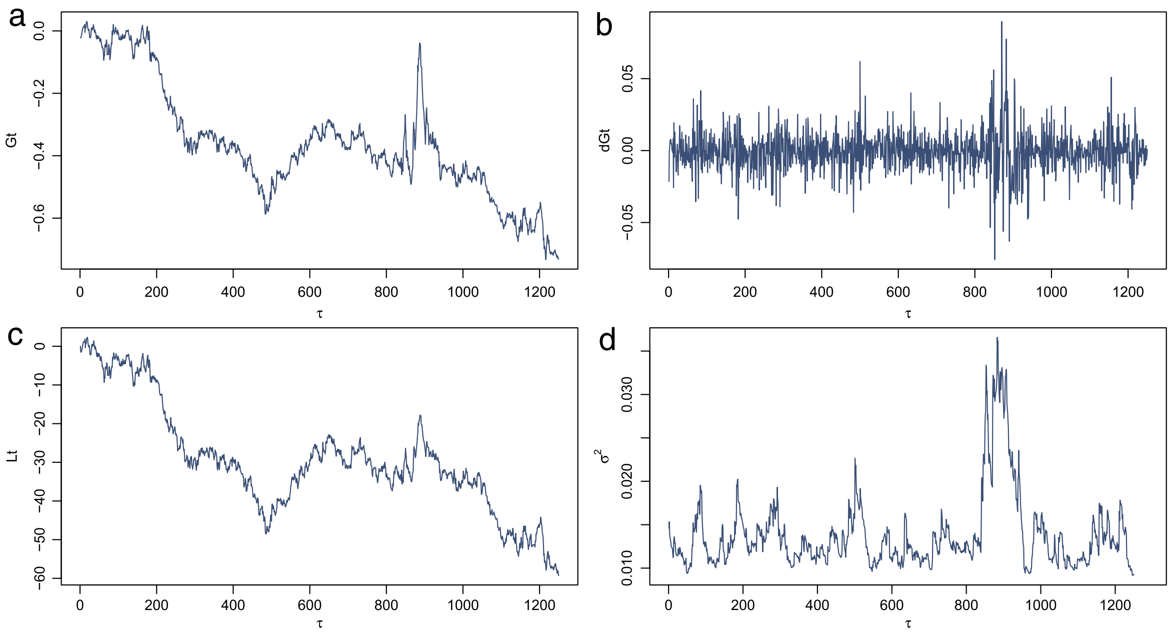

Fig. 3. Simulated (a) COGARCH(1, 1) process, (b) log-return process, (c) NIG-Lévy process, (d) volatility process.

the 0.05 level. Also, we performed an ARCH-LM test on residuals based on 35 lags, and the null hypothesis was rejected 71 times out of 1000 simulations at the 0.05 level.

4.1. Volatility estimation and comparison

It would be beneficial to estimate the volatility of the COGARCH(1, 1) model for the interest rate returns and compare its estimation error with that of the discrete-time GARCH(1, 1) model from the econometric point of view. For this purpose,

Table 6
Performance comparison of IIM and PML estimation methods.

	IIM			PML		
	$\hat{\omega}$	$\hat{\eta}$	$\hat{\varphi}$	$\hat{\omega}$	$\hat{\eta}$	$\hat{\varphi}$
Mean	9.16e−06	0.1232	0.085	9.302e−06	0.1239	0.084
	(4.74e−06)	(0.045)	(0.028)	(5.44e−06)	(0.051)	(0.033)
RMSE	4.37e−05	0.5108	0.3501	4.82e−05	0.5344	0.3475
MAE	3.24e−06	0.0324	0.0230	3.75e−06	0.0372	0.0262
Bias	0.0013	16.16	11.07	0.0015	16.89	10.98

The true parameter values are $\omega = 7.77\text{e}{-06}$, $\eta = 0.1070$, $\varphi = 0.073$.

Table 7
Volatility estimation error statistics.

	RMSE	MAE	MHSE
GARCH(1, 1)	0.138	0.0084	0.54
COGARCH(1, 1)	0.146	0.0080	0.41
t-statistic	−3.9202	−11.514	−4.5753
p-value	9.31e−06	2.2e−16	5.21e−06

we employ a recursive estimator of the volatility process for given parameters ω, η, and φ. The procedure that we follow here is that proposed by Haug et al. [14]. The estimator is then applied to the empirical data. The COGARCH(1, 1) volatility process in (6) can be arranged as

$$\sigma_t^2 = \sigma_{t-1}^2 + \omega - \eta \int_{n+1,n} \sigma_s^2 ds + \varphi \sum_{n-1<s\leq n} \sigma_s^2 (\Delta Ls)^2. \tag{16}$$

Since σ_s is latent and ΔLs is usually not observable, we have to approximate the integral and the sum on the right hand side. For the integral, we use a simple Euler approximation:

$$\int_{n+1,n} \sigma_s^2 ds \approx \sigma_{t-1}^2, \quad n \in N.$$

As we observe the log-price process G only at integer times, we approximate as follows:

$$\sum_{n-1<s\leq n} \sigma_s^2 (\Delta Ls)^2 \approx (G_t - G_{t-1})^2 = (G_t^{(1)})^2, \quad n \in N.$$

Therefore, the recursive estimator of the volatility can be calculated as

$$\hat{\sigma}_t^2 = \hat{\omega} + (1-\hat{\eta})\hat{\sigma}_{t-1}^2 + \hat{\varphi}(G_t^{(1)})^2, \quad n \in N, \tag{17}$$

where $(G_t^{(1)})^2$ is the squared log-returns of the interest rates.

Fig. 4 presents the recursive estimation of the COGARCH(1, 1) volatility process along with the conditional volatility of the GARCH(1, 1) model. For comparison purposes, we have applied root mean squared error (RMSE), mean absolute error (MAE), and mean heteroskedastic squared error (MHSE) statistics for the estimation performance of the models. The error statistics are given by the following formulas:

$$RMSE = \sqrt{\frac{\sum_{t=1}^{n} (\sigma_t - \hat{\sigma}_t)^2}{n}}$$

$$MAE = \frac{1}{n} \sum_{t=1}^{n} |\hat{\sigma}_t - \sigma_t|$$

$$MHSE = \frac{1}{n} \sum_{t=1}^{n} \left(\frac{\sigma_t}{\hat{\sigma}_t} - 1 \right)^2$$

where $\hat{\sigma}_t$ represents the predicted (GARCH or COGARCH) volatility, and σ_t represents the absolute log-returns as a proxy for the true volatility.

Table 7 shows the error statistics of the models. While MAE statistics suggest that the two models have almost the same prediction power, according to RMSE and MHSE statistics, COGARCH(1, 1) gives better in-sample predictions for the conditional volatility, as it has a lower error statistic than the discrete-time GARCH(1, 1) model. One sample t-test has been employed to test the significance of the differences of the error statistics. The t-statistics and p-values indicate that we should reject the null hypothesis which assumes that the true mean of the differences is equal to zero. Therefore, the error

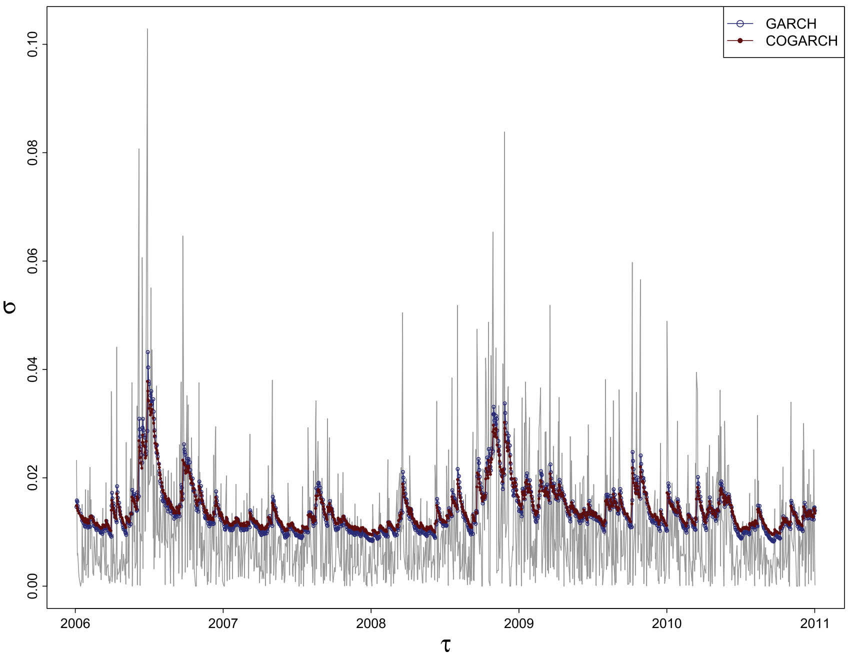

Fig. 4. Estimated conditional volatilities versus absolute log-returns.

Table 8
COGARCH residual distributional tests.

	Kolmogorov–Smirnov		Anderson–Darling	
	Statistic	p-value	Statistic	p-value
Normal	0.059	0.019	2.390	0.055
NIG	0.031	0.530	0.593	0.488

statistics (MAE, RMSE and MHSE) of the COGARCH(1, 1) model are significantly different than those of GARCH(1, 1) models. The RMSE is a very popular error measurement tool among practioners but when volatility clustering occurs it is not sufficient for accurate model comparison because it measures the error in terms of average deviations. We prefer MHSE over RMSE and MAE statistics because MHSE measures the error as an average relative error and takes high and low volatility periods into account.

4.2. Diagnostic tests

The robustness of the model estimation method is investigated via a residual analysis. The COGARCH(1, 1) residuals are given by $\varepsilon_t = \frac{dG_t}{\hat{\sigma}_t}$, where log-returns dG_t are divided by the estimated volatility $\hat{\sigma}_t$.

The correlation of squared log-returns is checked by Ljung–Box serial correlation testing. We also performed an ARCH-LM test on the residuals to investigate the ARCH effects. The test statistics used 35 lags of the corresponding empirical autocorrelation functions. The null hypothesis was not rejected up to 35 lags for both tests. The p-value for the Ljung–Box test is 0.2546, and for the ARCH-LM test it is 0.2229 (see Table 5). Thus, both test results confirm that the COGARCH(1, 1) model is successful at removing the ARCH effects from the data. These results are also confirmed by Fig. 5 where the empirical autocorrelation function of the squared residuals is plotted in panel (b), showing no significant correlations of the squared residuals.

Distributional properties of the COGARCH(1, 1) residuals are given in panel (a) and panel (b) of Fig. 4. Density and quantile plots of the residuals show that the residuals are NIG distributed. Also, Kolmogorov–Smirnov and Anderson–Darling statistics from Table 8 confirm that the NIG distribution provides a better fit than the normal distribution.

5. Conclusion

In this paper, we have applied discrete-time and continuous-time GARCH(1, 1) models to analyze the interest rate dynamics in the Turkish market. The COGARCH(1, 1) models provided excellent results in modeling the interest rate series, as they capture the characteristics of the volatility process and yielded better conditional volatility estimates than the discrete-time counterparts. The main advantage of the COGARCH(1, 1) model over other two-factor stochastic volatility models is

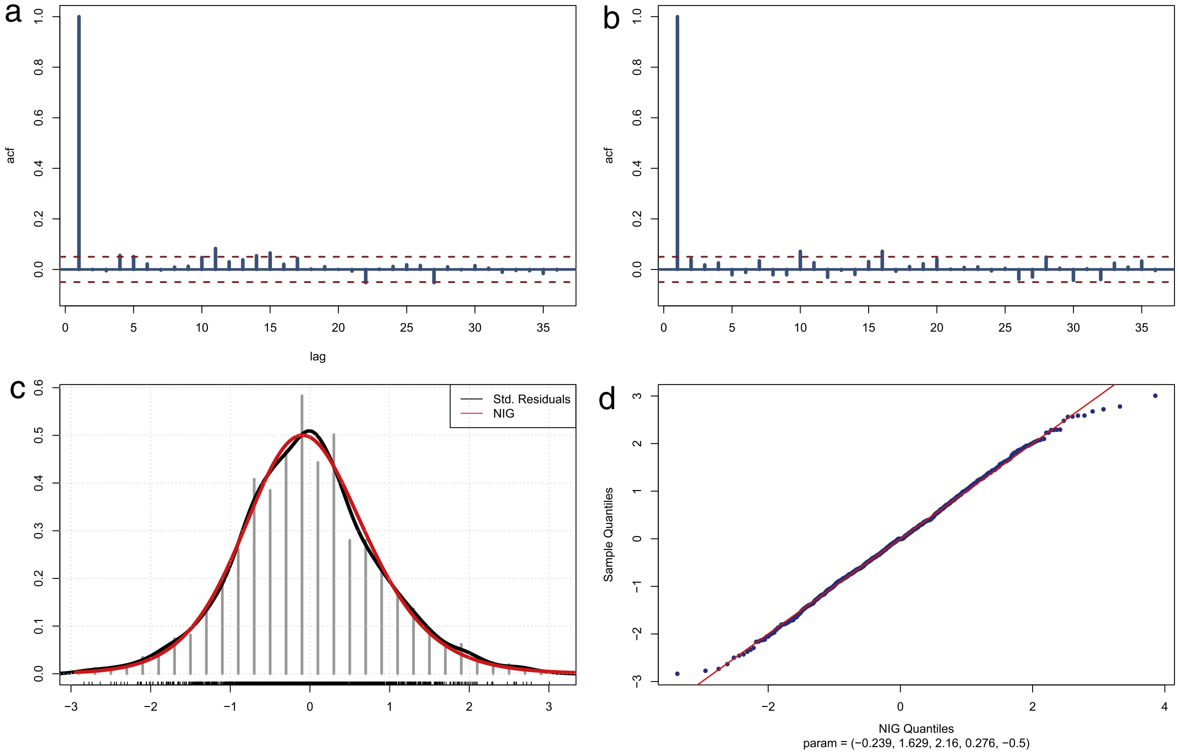

Fig. 5. (a) acf of residuals, (b) acf of squared residuals, (c) density of residuals, (d) NIG *QQ*-plot of residuals.

that it is driven by a single background Lévy process; therefore, the mechanisms of feedback between the price and volatility processes are not lost. The COGARCH(1, 1) model could be a good tool to use in extending one-factor continuous-time interest rate models such as the Vasicek and CIR models by incorporating the Lévy driven stochastic volatility process to the model. Also long-memory characteristics of the financial returns and volatility can be addressed by the COGARCH(1, 1) model by extending it with a fractional NIG process as a driving source of randomness. Since the aim of this study is to demonstrate that the COGARCH(1, 1) model is a continuous-time analogue of the discrete-time GARCH(1, 1) model, we only compared COGARCH(1, 1) with the GARCH(1, 1) model. A further study comparing the COGARCH(1, 1) model with other approaches like CIR and other GARCH families would be more beneficial in terms of testing the strength of the COGARCH(1, 1) model in volatility modeling.

References

[1] R.C. Merton, Theory of rational option pricing, Bell Journal of Economics and Management Science 4 (1973) 141–183.
[2] O. Vasicek, An equilibrium characterization of the term structure, Journal of Financial Economics 5 (1977) 177–188.
[3] J.C. Cox, J.E. Ingersoll, S.A. Ross, A theory of the term structure of interest rates, Econometrica 53 (1985) 385–407.
[4] T.G. Andersen, J. Lund, Estimating continuous-time stochastic volatility models of the short term interest rate, Journal of Econometrics 77 (1997) 343–378.
[5] R.F. Engle, Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation, Econometrica 50 (1982) 987–1007.
[6] T. Bollerslev, Generalized autoregressive conditional heteroskedasticity, Journal of Econometrics 31 (1986) 307–327.
[7] D.B. Nelson, Conditional heteroskedasticity in asset returns: a new approach, Econometrica 59 (1991) 347–370.
[8] P. Brockwell, E. Chadraa, A. Lindner, Continuous time GARCH processes, The Annals of Applied Probability 16 (2006) 790–826.
[9] D.B. Nelson, ARCH models as diffusion approximations, Journal of Econometrics 45 (1990) 7–38.
[10] O.E. Barndorff-Nielsen, N. Shephard, Non-Gaussian Ornstein–Uhlenbeck based models and some of their uses in financial economics, Journal of the Royal Statistical Society 63 (2001) 167–241.
[11] C. Klüppelberg, A. Lindner, R. Maller, A continuous time GARCH process driven by a Lévy process: stationarity and second order behaviour, Journal of Applied Probability 41 (2004) 601–622.
[12] C. Gouriéroux, A. Monfort, E. Renault, Indirect inference, Journal of Applied Econometrics 8 (1993) 85–118.
[13] C. Gouriéroux, A. Monfort, Simulation Based Econometric Methods, Oxford University Press, New York, 1996.
[14] S. Haug, C. Klüppelberg, A. Lindner, M. Zapp, Method of moments estimation in the COGARCH(1, 1) model, Econometrics Journal 10 (2007) 320–341.
[15] G. Müller, MCMC estimation of the COGARCH(1, 1) model, Journal of Financial Econometrics 8 (4) (2010) 481–510.
[16] R.A. Maller, G. Müller, A. Szimayer, GARCH modelling in continuous time for irregularly spaced time series data, Bernoulli 14 (2008) 519–542.
[17] G. Müller, R.B. Durand, R. Maller, C. Klüppelberg, Analysis of stock market volatility by continuous-time GARCH models, in: G.N. Gregoriou (Ed.), Stock Market Volatility, Chapman and Hall-CRC/Taylor and Francis, London, 2010, pp. 31–50.
[18] A.R. Gallant, G. Tauchen, Which moments to match, Econometric Theory 12 (1995) 657–681.
[19] G.J. Jiang, Jump diffusion model of exchange rate dynamics—estimation via indirect inference, Research Report, University of Groningen, 1998.
[20] C. Monfardini, Estimating stochastic volatility models through indirect inference, Econometrics Journal 1 (1998).
[21] E. Eberlein, U. Keller, Hyperbolic distributions in finance, Bernoulli 1 (1995) 281–299.
[22] O.E. Barndorff-Nielsen, Processes of normal inverse Gaussian type, Finance and Stochastics 2 (1998) 41–68.