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Multidimensional fixed point theorems under (ψ, φ)-contractive conditions in partially ordered complete metric spaces

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Abstract

In this paper, we study the existence and uniqueness of coincidence points for nonlinear mappings of any number of arguments under a weak (ψ, φ)-contractive condition. Our results generalize, extend and unify several classical and very recent related results in literature (see [1, 3, 4, 5, 6, 7, 13, 17, 18, 23]).

Keywords: Metric space, Coincidence point, Fixed point, Contraction, Mixed monotone property, Partially ordered set

1. Introduction

The concept of *coupled fixed point* for nonlinear operators was introduced and studied by Opoitsev (see [20, 21, 22]) and then, in 1987, by Guo and Lakshmikantham (see [10]) in connection with coupled quasisolutions of an initial value problem for ordinary differential equations. In a recent paper, Gnana-Bhaskar and Lakshmikantham [4] introduced the concept of the *mixed monotone property* for contractive mappings of the form $F : X \times X \to X$, where X is a partially ordered metric space, and then established some coupled fixed point theorems. Since then, many results on coupled fixed point theory in different contexts have been published (see [9, 11, 14, 15, 17, 24, 26]. Later, Berinde and Borcut [3] introduced the concept of the *tripled fixed point* and proved tripled fixed point theorems using the mixed monotone mappings (see also [2, 6, 25]).

Very recently, Roldán et al. [23] proposed the notion of *coincidence point* between mappings in any number of variables and showed some existence and uniqueness theorems that extend the mentioned previous results for this kind of nonlinear mappings, not necessarily permuted or ordered, in the framework of partially ordered complete metric spaces, using a weaker contraction condition, that also generalizes other work by Berzig and Samet [5].

In this paper, our main aim is to study a weaker contractive condition for nonlinear mappings of any number of arguments. This condition can be particularized in a variety of forms that lets us to extend the above mentioned results and other recent ones in this field (see [1, 3, 4, 5, 6, 7, 13, 17, 18, 23]).

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2. Preliminaries

Preliminaries and notation about coincidence points can also be found in [23]. Let n be a positive integer. Henceforth, X denotes a nonempty set and X^n denotes the product space $X \times X \times \cdots \times X$. Throughout this manuscript, m and k denote non-negative integers and $i, j, s \in \{1, 2, \ldots, n\}$. Unless otherwise stated, "for all m" means "for all $m \geq 0$" and "for all i" means "for all $i \in \{1, 2, \ldots, n\}$".

A *metric* d on X is a mapping $d : X \times X \to \mathbb{R}$ satisfying: for all $x, y, z \in X$,

$$\text{(a)} \quad d(x, y) = 0 \text{ if and only if } x = y; \qquad \text{(b)} \quad d(x, y) \leq d(z, x) + d(z, y).$$

From these properties, we can easily deduce that $d(x, y) \geq 0$ and $d(y, x) = d(x, y)$ for all $x, y \in X$. The last requirement is called the *triangle inequality*. If d is a metric on X, we say that (X, d) is a *metric space*.

Definition 1. ([8]) A triple $(X, d, \leq)$ is called an *ordered metric space* if (X, d) is a metric space and $(X, \leq)$ is a partially ordered set.

In the sequel, let $F : X^n \to X$ and $g : X \to X$ be two mappings.

Definition 2. ([4]) An ordered metric space $(X, d, \leq)$ is said to have the *sequential g-monotone property* if it verifies the following properties:

(a) If $\{x_m\}$ is a non-decreasing sequence and $\lim_{m\to\infty} x_m = x$, then $gx_m \leq gx$ for all m.

(b) If $\{y_m\}$ is a non-increasing sequence and $\lim_{m\to\infty} y_m = y$, then $gy_m \geq gy$ for all m.

If g is the identity mapping, then X is said to have the sequential monotone property.

Henceforth, we fix a partition $\{A, B\}$ of $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \varnothing$. Then we denote:

$$\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\}$$

and

$$\Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\}.$$

If $(X, \leq)$ is a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, then we use the following notation:

$$x \leq_i y \iff \begin{cases} x \leq y & \text{if } i \in A, \\ x \geq y & \text{if } i \in B. \end{cases}$$

Definition 3. ([23]) Let $(X, \leq)$ be a partially ordered space. We say that F has the *mixed g-monotone property* with respect to the partition $\{A, B\}$ if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, i.e., for all $x_1, x_2, \ldots, x_n, y, z \in X$ and i,

$$gy \leq gz \implies F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \leq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

Henceforth, let $\sigma_1, \sigma_2, \ldots, \sigma_n, \tau : \Lambda_n \to \Lambda_n$ be $n+1$ mappings from Λ_n into itself and let Φ be the $(n+1)$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n, \tau)$.

Definition 4. ([23]) A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a Φ-*coincidence point* of the mappings F and g if, for all i,

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_{\tau(i)}. \tag{1}$$

If g is the identity mapping on X, then $(x_1, x_2, \ldots, x_n) \in X^n$ is called a Φ-*fixed point* of the mapping F.

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Remark 5. If F and g are commuting and $(x_1, x_2, \ldots, x_n) \in X^n$ is a Φ-coincidence point of F and g, then $(gx_1, gx_2, \ldots, gx_n)$ also is a Φ-coincidence point of F and g.

The commutative condition between F and g is very restrictive. When $n = 2$, in [19], the authors introduced a weaker condition, which we will extend to any number of variables.

Definition 6. ([19, 12]) Let $(X, d, \leq)$ be an ordered metric space. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be *O-compatible* if

$$\lim_{m\to\infty} d(gF(x_m, y_m), F(gx_m, gy_m)) = 0, \quad \lim_{m\to\infty} d(gF(y_m, x_m), F(gy_m, gx_m)) = 0,$$

where $\{x_m\}$ and $\{y_m\}$ are the sequences in X such that $\{gx_m\}$, $\{gy_m\}$ are monotone and

$$\lim_{m\to\infty} F(x_m, y_m) = \lim_{m\to\infty} gx_m = x, \quad \lim_{m\to\infty} F(y_m, x_m) = \lim_{m\to\infty} gy_m = y$$

for all $x, y \in X$.

3. Preliminary results

In this section, we present some statements about real sequences and coincidence points which simplify the proofs of our main results.

Lemma 7. *Let $\{a_m^1\}_{m\in\mathbb{N}}, \ldots, \{a_m^n\}_{m\in\mathbb{N}}$ be n real lower bounded sequences such that*

$$\{\max(a_m^1, \ldots, a_m^n)\}_{m\in\mathbb{N}} \to \delta.$$

Then there exists $i_0 \in \{1, 2, \ldots, n\}$ and a subsequence $\{a_{m(k)}^{i_0}\}_{k\in\mathbb{N}}$ such that $\{a_{m(k)}^{i_0}\}_{k\in\mathbb{N}} \to \delta$.

Proof. Let $b_m = \max(a_m^1, a_m^2, \ldots, a_m^n)$ for all m. Since $\{b_m\}$ is convergent, it is bounded. Since $a_m^i \leq b_m$ for all m and i, every $\{a_m^i\}$ is bounded. Since $\{a_m^1\}_{m\in\mathbb{N}}$ is a real bounded sequence, it has a convergent subsequence $\{a_{\sigma_1(m)}^1\}_{m\in\mathbb{N}} \to a_1$.

Consider the subsequences $\{a_{\sigma_1(m)}^2\}_{m\in\mathbb{N}}, \{a_{\sigma_1(m)}^3\}_{m\in\mathbb{N}}, \ldots, \{a_{\sigma_1(m)}^n\}_{m\in\mathbb{N}}$ which are $n-1$ real bounded sequences and the sequence $\{b_{\sigma_1(m)}\}_{m\in\mathbb{N}}$ which also converges to δ. Since $\{a_{\sigma_1(m)}^2\}_{m\in\mathbb{N}}$ is a real bounded sequence, it has a convergent subsequence $\{a_{\sigma_2\sigma_1(m)}^2\}_{m\in\mathbb{N}} \to a_2$. Then the sequences $\{a_{\sigma_2\sigma_1(m)}^3\}_{m\in\mathbb{N}}$, $\{a_{\sigma_2\sigma_1(m)}^4\}_{m\in\mathbb{N}}$, …, $\{a_{\sigma_2\sigma_1(m)}^n\}_{m\in\mathbb{N}}$ also are $n-2$ real bounded sequences, $\{a_{\sigma_2\sigma_1(m)}^1\}_{m\in\mathbb{N}} \to a_1$ and $\{b_{\sigma_2\sigma_1(m)}\}_{m\in\mathbb{N}} \to \delta$. Repeating this process n times, we can find the n subsequences $\{a_{\sigma(m)}^1\}_{m\in\mathbb{N}}$, $\{a_{\sigma(m)}^2\}_{m\in\mathbb{N}}$, …, $\{a_{\sigma(m)}^n\}_{m\in\mathbb{N}}$, where $\sigma = \sigma_n \ldots \sigma_1$, such that $\{a_{\sigma(m)}^i\}_{m\in\mathbb{N}} \to a_i$ for all i and $\{b_{\sigma(m)}\}_{m\in\mathbb{N}} \to \delta$. But

$$\{b_{\sigma(m)}\}_{m\in\mathbb{N}} = \{\max(a_{\sigma(m)}^n, \ldots, a_{\sigma(m)}^n)\}_{m\in\mathbb{N}} \to \max(a_1, \ldots, a_n)$$

and so $\delta = \max(a_1, \ldots, a_n)$ and there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $a_{i_0} = \delta$. Therefore, there exists $i_0 \in \{1, 2, \ldots, n\}$ and a subsequence $\{a_{\sigma(m)}^{i_0}\}_{m\in\mathbb{N}}$ such that $\{a_{\sigma(m)}^{i_0}\}_{m\in\mathbb{N}} \to a_{i_0} = \delta$. This completes the proof. ∎

Lemma 8. *Let $\{a_m\}_{m\in\mathbb{N}}$ be a sequence of non-negative real numbers which has not any subsequence converging to zero. Then, for all $\varepsilon > 0$, there exist $\delta \in]0, \varepsilon[$ and $m_0 \in \mathbb{N}$ such that $a_m \geq \delta$ for all $m \geq m_0$.*

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Proof. Suppose that the conclusion is not true. Then there exists $\varepsilon_0 > 0$ such that, for all $\delta \in]0, \varepsilon_0[$, there exists $m_0 \in \mathbb{N}$ verifying $a_{m_0} < \delta$. Let $k_0 \in \mathbb{N}$ be such that $1/k_0 < \varepsilon_0$. For all $k \in \mathbb{N}$, take $\delta_k = 1/(k+k_0) \in]0, \varepsilon_0[$. Then there exists $m(k) \in \mathbb{N}$ verifying $0 \leq a_{m(k)} < \delta_k = 1/(k+k_0)$. Taking limit when $k \to \infty$, we deduce that $\lim_{k\to\infty} a_{m(k)} = 0$. Then $\{a_m\}$ has a subsequence converging to zero (maybe, reordering $\{a_{m(k)}\}$), but this is a contradiction. This completes the proof. ■

Lemma 9. *If $\{x_m\}_{m\in\mathbb{N}}$ is a sequence on a metric space (X, d) that is not a Cauchy sequence, then there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$, $\{x_{n(k)}\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$,*

$$k < m(k) < n(k), \quad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon_0, \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon_0.$$

Proof. We know that $\{x_m\}$ is a Cauchy sequence if and only if, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0$.

If this condition does not hold, then there exists $\varepsilon_0 > 0$ such that, for all $n_0 \in \mathbb{N}$, there exists $m, n \geq n_0$ such that $d(x_m, x_n) \geq \varepsilon_0$.

Let $n_0 = 2$. Then there exists $m_1, n_1 \in \mathbb{N}$ such that $m_1, n_1 \geq n_0$ and $d(x_{m_1}, x_{n_1}) \geq \varepsilon_0$. Let $m(1) = \min(m_1, n_1) \geq n_0 = 2 > 1$ and consider the numbers

$$d(x_{m(1)}, x_{m(1)+1}), d(x_{m(1)}, x_{m(1)+2}), \ldots, d(x_{m(1)}, x_{\max(m_1,n_1)}).$$

Since $d(x_{m(1)}, x_{\max(m_1,n_1)}) = d(x_{m_1}, x_{n_1}) \geq \varepsilon_0$, between the previous numbers, there exists a first non-negative integer $n(1) \in \{m(1)+1, m(1)+2, \ldots, \max(m_1, n_1)\}$ such that $d(x_{m(1)}, x_{n(1)}) \geq \varepsilon_0$ but $d(x_{m(1)}, x_j) < \varepsilon_0$ for all $j \in \{m(1), m(1)+1, \ldots, n(1)-1\}$. In particular, $d(x_{m(1)}, x_{n(1)-1}) < \varepsilon_0$.

Now, let $n_0 = n(1) + 1$. Then there exists $m_2, n_2 \in \mathbb{N}$ such that $m_2, n_2 \geq n(1)+1$ and $d(x_{m_2}, x_{n_2}) \geq \varepsilon_0$. Let $m(2) = \min(m_2, n_2) \geq n_0 = n(1) + 1 > n(1)$ and consider the numbers

$$d(x_{m(2)}, x_{m(2)+1}),\ d(x_{m(2)}, x_{m(2)+2}), \ldots, d(x_{m(2)}, x_{\max(m_2,n_2)}).$$

Since $d(x_{m(2)}, x_{\max(m_2,n_2)}) = d(x_{m_2}, x_{n2}) \geq \varepsilon_0$, between the previous numbers there exists a first non-negative integer $n(2) \in \{m(2)+1, m(2)+2, \ldots, \max(m_2, n_2)\}$ such that $d(x_{m(2)}, x_{n(2)}) \geq \varepsilon_0$, but $d(x_{m(2)}, x_j) < \varepsilon_0$ for all $j \in \{m(2), m(2)+1, \ldots, n(2)-1\}$. In particular, $d(x_{m(2)}, x_{n(2)-1}) < \varepsilon_0$.

Repeating this process, we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ such that, for all $k \in \mathbb{N}$:

$$k < m(k) < n(k) < m(k+1), \quad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon_0, \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon_0.$$

This completes the proof. ■

Definition 10. Let Ψ be the family of all continuous, non-decreasing mappings $\psi : [0, \infty[\to [0, \infty[$ such that $\psi(t) = 0$ if and only if $t = 0$.

These mappings are known as the *altering distance functions* (see [16]). Note that every $\psi \in \Psi$ commutes with max, i.e., $\psi(\max(s_1, s_2, \ldots, s_N)) = \max(\psi(s_1), \psi(s_2), \ldots, \psi(s_N))$ for all $s_1, s_2, \ldots, s_N \in [0, \infty)$.

Lemma 11. *If $\psi \in \Psi$ and $\lim_{m\to\infty} \psi(a_m) = 0$, then $\lim_{m\to\infty} a_m = 0$.*

Proof. Since there exists $\psi(a_m)$, then $a_m \in \operatorname{dom}\psi = [0, \infty[$. If the conclusion is not true, there exists $\varepsilon_0 > 0$ such that, for all $n_0 \in \mathbb{N}$, there exists $n \geq n_0$ verifying $a_n \geq \varepsilon_0$. This means that $\{a_m\}$ has a subsequence

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$\{a_{m(k)}\}_k$ such that $a_{m(k)} \geq \varepsilon_0$. Since ψ is non-decreasing, $\psi(\varepsilon_0) \leq \psi(a_{m(k)})$ for all $k \in \mathbb{N}$. Therefore, $\{\psi(a_m)\}_m$ has a subsequence $\{\psi(a_{m(k)})\}_k$ lower bounded by $\psi(\varepsilon_0) > 0$, but this is impossible since $\lim_{m\to\infty} \psi(a_m) = 0$. This completes the proof. ■

With regards to coincidence points, it is possible to consider the following simplification. If τ is a permutation of Λ_n and we reorder equations (1), then we deduce that every coincidence point may be seen as a coincidence point associated to the identity mapping on Λ_n.

Lemma 12. *Let τ be a permutation of Λ_n and let*

$$\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n, \tau), \quad \Phi' = (\sigma_{\tau^{-1}(1)}, \sigma_{\tau^{-1}(2)}, \ldots, \sigma_{\tau^{-1}(n)}, I_{\Lambda_n}).$$

Then a point $(x_1, x_2, \ldots, x_n) \in X^n$ is a Φ-coincidence point of the mappings F and g if and only if $(x_1, x_2, \ldots, x_n)$ is a Φ'-coincidence point of the mappings F and g.

Therefore, in the sequel, without loss of generality, we only consider Υ-coincidence points where $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, i.e., which verify $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i$ for all i.

4. Main results

In order to avoid the commutative condition between F and g, we introduce the *O-compatibility* in any number of variables.

Definition 13. *Let $(X, d, \leq)$ be an ordered metric space. The mappings $F : X^n \to X$ and $g : X \to X$ are said to be* O-compatible *if, for all sequences $\{x_m^1\}_{m\geq 0}$, $\{x_m^2\}_{m\geq 0}$, ..., $\{x_m^n\}_{m\geq 0}$ in X such that $\{gx_m^1\}_{m\geq 0}$, $\{gx_m^2\}_{m\geq 0}, \ldots, \{gx_m^n\}_{m\geq 0}$ are monotone and the following limit exists: for all i,*

$$\lim_{m\to\infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) = \lim_{m\to\infty} gx_m^i \in X,$$

we have

$$\lim_{n\to\infty} d(gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)})) = 0$$

for all i.

Obviously, if F and g are commuting, then they are O-compatible, but the converse does not hold. In the following result, we show sufficient conditions to ensure the existence of Υ-coincidence points, where $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Theorem 14. *Let $(X, d, \leq)$ be a complete ordered metric space. Let $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be a n-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X, $F(X^n) \subseteq g(X)$ and g is continuous and* O-compatible *with F. Assume that there exist $\psi, \varphi \in \Psi$ such that*

$$\psi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \leq \psi\left(\max_{1\leq i\leq n} d(gx_i, gy_i)\right) - \varphi\left(\max_{1\leq i\leq n} d(gx_i, gy_i)\right) \tag{2}$$

for which $gx_i \leq_i gy_i$ for all i. Suppose that either F is continuous or X has the sequential g-monotone property. If there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying:

$$gx_0^i \leq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$$

for all i, then F and g have at least one Υ-coincidence point.

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Proof. The proof is divided in six steps. We follow the methodology of Theorem 9 in [23].

Step 1. There exist the n sequences $\{x_m^1\}_{m\geq 0}$, $\{x_m^2\}_{m\geq 0}$, $\ldots$, $\{x_m^n\}_{m\geq 0}$ such that

$$gx_{m+1}^i = F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})$$

for all m and i.

Step 2. $gx_m^i \leq_i gx_{m+1}^i$ for all m and i.

Step 3. We claim that $\{d(gx_m^i, gx_{m+1}^i)\}_{m\geq 0} \to 0$ for all i (i.e., $\{\max_{1\leq j\leq n} d(gx_m^j, gx_{m+1}^j)\}_{m\geq 0} \to 0$).

Indeed, define $\delta_m = \max_{1\leq j\leq n} d(gx_m^j, gx_{m+1}^j)$ for all m. Since $gx_m^i \leq_i gx_{m+1}^i$ for all m and i, the condition (2) implies that, for all $m \geq 1$ and i,

$$\begin{aligned}\psi(d(gx_m^i, gx_{m+1}^i)) &= \psi(d(F(x_{m-1}^{\sigma_i(1)}, x_{m-1}^{\sigma_i(2)}, \ldots, x_{m-1}^{\sigma_i(n)}), F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})))\\ &\leq \psi\left(\max_{1\leq j\leq n} d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)})\right) - \varphi\left(\max_{1\leq j\leq n} d(gx_{m-1}^{\sigma_i(j)}, gx_m^{\sigma_i(j)})\right) \qquad (3)\\ &\leq \psi\left(\max_{1\leq j\leq n} d(gx_{m-1}^j, gx_m^j)\right) = \psi(\delta_{m-1}).\end{aligned}$$

Therefore, for all $m \geq 1$,

$$\psi(\delta_m) = \psi\left(\max_{1\leq i\leq n} d(gx_m^i, gx_{m+1}^i)\right) = \max_{1\leq i\leq n} \psi\left(d(gx_m^i, gx_{m+1}^i)\right) \leq \psi(\delta_{m-1}).$$

This means that the sequence $\{\psi(\delta_m)\}_{m\geq 1}$ is non-increasing and lower bounded. Therefore, it is convergent, i.e., there exists $\Delta \geq 0$ such that $\{\psi(\delta_m)\}_{m\geq 1} \to \Delta$.

Now, we show that $\Delta = 0$. Suppose that $\Delta > 0$. Since we have

$$\left\{\max_{1\leq i\leq n} \psi\left(d(gx_m^i, gx_{m+1}^i)\right)\right\}_m = \left\{\psi\left(\max_{1\leq i\leq n} d(gx_m^i, gx_{m+1}^i)\right)\right\}_m = \{\psi(\delta_m)\}_m \to \Delta,$$

Lemma 7 assures that there exist $i_0 \in \{1, 2, \ldots, n\}$ and a subsequence $\{\psi(d(gx_{m(k)}^{i_0}, gx_{m(k)+1}^{i_0}))\}_k$ such that $\{\psi(d(gx_{m(k)}^{i_0}, gx_{m(k)+1}^{i_0}))\}_k \to \Delta$. Repeating (3), for all $k \geq 1$,

$$\psi\left(d(gx_{m(k)}^{i_0}, gx_{m(k)+1}^{i_0})\right) \leq \psi\left(\max_{1\leq j\leq n} d(gx_{m(k)-1}^{\sigma_{i_0}(j)}, gx_{m(k)}^{\sigma_{i_0}(j)})\right) - \varphi\left(\max_{1\leq j\leq n} d(gx_{m(k)-1}^{\sigma_{i_0}(j)}, gx_{m(k)}^{\sigma_{i_0}(j)})\right). \qquad (4)$$

Consider the sequence

$$\left\{\max_{1\leq j\leq n} d(gx_{m(k)-1}^{\sigma_{i_0}(j)}, gx_{m(k)}^{\sigma_{i_0}(j)})\right\}_{k\geq 1}. \qquad (5)$$

If this sequence has a subsequence converging to zero, taking into account that ψ and φ are continuous, then we can take the limit in (4) using this subsequence and we would deduce

$$0 < \Delta \leq \psi(0) - \varphi(0) = 0,$$

which is impossible. Suppose that the sequence in (5) has no any subsequence converging to zero. Using $\varepsilon = 1$, Lemma 8 assures that there exists $\delta' \in]0, 1[$ and $k_0 \in \mathbb{N}$ such that

$$\max_{1\leq j\leq n} d(gx_{m(k)-1}^{\sigma_{i_0}(j)}, gx_{m(k)}^{\sigma_{i_0}(j)}) \geq \delta'$$

for all $k \geq k_0$. Therefore, we have

$$-\varphi\left(\max_{1\leq j\leq n} d(gx_{m(k)-1}^{\sigma_{i_0}(j)}, gx_{m(k)}^{\sigma_{i_0}(j)})\right) \leq -\varphi(\delta')$$

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for all $k \geq k_0$. Then (4) implies that

$$\begin{aligned}\psi\left(d(gx^{i_0}_{m(k)}, gx^{i_0}_{m(k)+1})\right) &\leq \psi\left(\max_{1\leq j\leq n} d(gx^{\sigma_{i_0}(j)}_{m(k)-1}, gx^{\sigma_{i_0}(j)}_{m(k)})\right) - \varphi\left(\max_{1\leq j\leq n} d(gx^{\sigma_{i_0}(j)}_{m(k)-1}, gx^{\sigma_{i_0}(j)}_{m(k)})\right)\\ &\leq \psi\left(\max_{1\leq j\leq n} d(gx^{\sigma_{i_0}(j)}_{m(k)-1}, gx^{\sigma_{i_0}(j)}_{m(k)})\right) - \varphi(\delta')\\ &\leq \psi\left(\max_{1\leq j\leq n} d(gx^{j}_{m(k)-1}, gx^{j}_{m(k)})\right) - \varphi(\delta')\\ &= \psi\left(\delta_{m(k)-1}\right) - \varphi(\delta').\end{aligned}$$

Taking the limit in k, we deduce that $\Delta \leq \Delta - \varphi(\delta') < \Delta$, which is impossible. Then we have just proved that $\Delta = 0$. Therefore, $\{\psi(\delta_m)\}_{m\geq 1} \to \Delta = 0$ and Lemma 11 assures that $\{\delta_m\}_{m\geq 1} \to 0$, which means that

$$\lim_{m\to\infty} \delta_m = \lim_{m\to\infty}\left(\max_{1\leq j\leq n} d(gx^j_m, gx^j_{m+1})\right) = 0.$$

Since $0 \leq d(gx^j_m, gx^j_{m+1}) \leq \delta_m$ for all m and j, $\{d(gx^j_m, gx^j_{m+1})\} \to 0$ for all j.

Step 4. Every sequence $\{gx^i_m\}_{m\geq 0}$ is a Cauchy sequence. Suppose that $\{gx^{i_1}_m\}_{m\geq 0}, \ldots, \{gx^{i_s}_m\}_{m\geq 0}$ are not Cauchy sequences and $\{gx^{i_{s+1}}_m\}_{m\geq 0}, \ldots, \{gx^{i_n}_m\}_{m\geq 0}$ are Cauchy sequences if $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. By Lemma 9, for all $r \in \{1, 2, \ldots, s\}$, there exists $\varepsilon_r > 0$, the subsequences $\{gx^{i_r}_{m'_r(k)}\}_{k\in\mathbb{N}}$ and $\{gx^{i_r}_{n'_r(k)}\}_{k\in\mathbb{N}}$ such that

$$k < m'_r(k) < n'_r(k), \quad d(gx^{i_r}_{m'_r(k)}, gx^{i_r}_{n'_r(k)}) \geq \varepsilon_r, \quad d(gx^{i_r}_{m'_r(k)}, gx^{i_r}_{n'_r(k)-1}) < \varepsilon_r$$

for all $k \in \mathbb{N}$. Let $\varepsilon_0 = \min(\varepsilon_1, \ldots, \varepsilon_s) > 0$. Since $\{gx^{i_{s+1}}_m\}_{m\geq 0}, \ldots, \{gx^{i_n}_m\}_{m\geq 0}$ are Cauchy sequences, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $d(gx^j_m, gx^j_n) < \varepsilon_0/2$ for all $j \in \{i_{s+1}, \ldots, i_n\}$.

Let $k_0 \in \mathbb{N}$ such that $n_0 < \min(m'_1(k_0), m'_2(k_0), \ldots, m'_s(k_0))$ and define

$$m(1) = \min(m'_1(k_0), m'_2(k_0), \ldots, m'_s(k_0)).$$

Since $m(1) = m'_r(k_0)$ for some $r \in \{1, 2, \ldots, s\}$, there exists $n'_r(k_0)$ such that $d(gx^{i_r}_{m'_r(k_0)}, gx^{i_r}_{n'_r(k_0)}) \geq \varepsilon_r \geq \varepsilon_0$. Thus we can consider the numbers $m(1)+1, m(1)+2, \ldots$ until finding the first positive integer $n(1) > m(1)$ verifying

$$d(gx^{i_{r(1)}}_{m(1)}, gx^{i_{r(1)}}_{n(1)}) = \max_{1\leq r\leq s} d(gx^{i_r}_{m(1)}, gx^{i_r}_{n(1)}) \geq \varepsilon_0, \quad d(gx^{i_j}_{m(1)}, gx^{i_j}_{n(1)-1}) < \varepsilon_0$$

for all $j \in \{1, 2, \ldots, s\}$, where $r(1) \in \{1, 2, \ldots, s\}$.

Now, let $k_1 \in \mathbb{N}$ such that $n(1) < \min(m'_1(k_1), m'_2(k_1), \ldots, m'_s(k_1))$ and define

$$m(2) = \min(m'_1(k_1), m'_2(k_1), \ldots, m'_s(k_1)).$$

Since $m(2) \in \{m'_1(k_1), m'_2(k_1), \ldots, m'_s(k_1)\}$, we can consider the numbers $m(2)+1, m(2)+2, \ldots$ until finding the first positive integer $n(2) > m(2)$ verifying

$$d(gx^{i_{r(2)}}_{m(2)}, gx^{i_{r(2)}}_{n(2)}) = \max_{1\leq r\leq s} d(gx^{i_r}_{m(2)}, gx^{i_r}_{n(2)}) \geq \varepsilon_0, \quad d(gx^{i_j}_{m(2)}, gx^{i_j}_{n(2)-1}) < \varepsilon_0$$

for all $j \in \{1, 2, \ldots, s\}$, where $r(2) \in \{1, 2, \ldots, s\}$. Repeating this process, we can find the sequences such that, for all $k \geq 1$,

$$n_0 < m(k) < n(k) < m(k+1), \quad d(gx^{i_{r(k)}}_{m(k)}, gx^{i_{r(k)}}_{n(k)}) = \max_{1\leq r\leq s} d(gx^{i_r}_{m(k)}, gx^{i_r}_{n(k)}) \geq \varepsilon_0,$$

where $r(k) \in \{1, 2, \ldots, s\}$, and

$$d(gx^{i_j}_{m(k)}, gx^{i_j}_{n(k)-1}) < \varepsilon_0$$

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for all $j \in \{1, 2, \ldots, s\}$. Since $n_0 < m(k) < n(k)$, we know that

$$d(gx^j_{m(k)}, gx^j_{n(k)}), d(gx^j_{m(k)}, gx^j_{n(k)-1}), d(gx^j_{m(k)-1}, gx^j_{n(k)-1}) < \varepsilon_0/2$$

for all $j \in \{i_{s+1}, \ldots, i_n\}$. Therefore, it follows that, for all k,

$$\begin{aligned} &\max_{1\le j\le n} d(gx^j_{m(k)}, gx^j_{n(k)}) = \max_{1\le r\le s} d(gx^{i_r}_{m(k)}, gx^{i_r}_{n(k)}) = d(gx^{i_{r(k)}}_{m(k)}, gx^{i_{r(k)}}_{n(k)}) \ge \varepsilon_0, \\ &\max_{1\le j\le n} d(gx^j_{m(k)}, gx^j_{n(k)-1}) < \varepsilon_0. \end{aligned} \tag{6}$$

Applying the contractive condition (2), it follows, for all $k \ge 1$,

$$\begin{aligned} &0 < \psi(\varepsilon_0) \\ &\le \psi\left(d(gx^{i_{r(k)}}_{m(k)}, gx^{i_{r(k)}}_{n(k)})\right) \\ &\le \psi\left(\max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1})\right) - \varphi\left(\max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1})\right). \end{aligned} \tag{7}$$

If the sequence:

$$\left\{\max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1})\right\}_{k\ge 1}$$

has a subsequence that converges to zero, we can take the limit when $k \to \infty$ in (4.7) using this subsequence, so that we would have $0 < \psi(\varepsilon_0) \le \psi(0) - \varphi(0) = 0$, which is impossible. Therefore, this sequence has no any subsequence converging to zero. In this case, taking $\varepsilon_0 > 0$ in Lemma 8, there exist $\delta \in]0, \varepsilon_0[$ and $k_0 \in \mathbb{N}$ such that

$$\max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1}) \ge \delta$$

for all $k \ge k_0$. Therefore, since φ is non-decreasing, for all $k \ge k_0$,

$$\delta \le \max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1}) \quad \Longrightarrow \quad -\varphi\left(\max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1})\right) \le -\varphi(\delta).$$

Thus, by (7), we have

$$\begin{aligned} &0 < \psi(\varepsilon_0) \\ &\le \psi\left(d(gx^{i_{r(k)}}_{m(k)}, gx^{i_{r(k)}}_{n(k)})\right) \\ &\le \psi\left(\max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1})\right) - \varphi\left(\max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1})\right) \\ &\le \psi\left(\max_{1\le j\le n} d(gx^{\sigma_{i_{r(k)}}(j)}_{m(k)-1}, gx^{\sigma_{i_{r(k)}}(j)}_{n(k)-1})\right) - \varphi(\delta) \\ &\le \psi\left(\max_{1\le j\le n} d(gx^j_{m(k)-1}, gx^j_{n(k)-1})\right) - \varphi(\delta) \\ &\le \psi\left(\max_{1\le j\le n} d(gx^j_{m(k)-1}, gx^j_{m(k)}) + \max_{1\le j\le n} d(gx^j_{m(k)}, gx^j_{n(k)-1})\right) - \varphi(\delta) \\ &\le \psi\left(\max_{1\le j\le n} d(gx^j_{m(k)-1}, gx^j_{m(k)}) + \varepsilon_0\right) - \varphi(\delta). \end{aligned}$$

Since $\{\max_{1\le j\le n} d(gx^j_{m(k)-1}, gx^j_{m(k)})\}_k \to 0$ by Step 3 and ψ is continuous, we can take the limit when $k \to \infty$ and deduce

$$0 < \psi(\varepsilon_0) \le \psi(0) + \psi(\varepsilon_0) - \varphi(\delta) = \psi(\varepsilon_0) - \varphi(\delta) < \psi(\varepsilon_0),$$

which is impossible. Thus every sequence $\{gx^i_m\}_{m\ge 0}$ is a Cauchy sequence. Since (X, d) is complete, there exist $x_1, x_2, \ldots, x_n \in X$ such that $x_i = \lim_{m\to\infty} gx^i_m$ for all i. Since g is continuous,

$$\lim_{m\to\infty} gF(x^{\sigma_i(1)}_m, x^{\sigma_i(2)}_m, \ldots, x^{\sigma_n(n)}_m) = \lim_{m\to\infty} ggx^i_{m+1} = gx_i \tag{8}$$

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for all i. Hence we know that every sequence $\{gx_m^i\}_m$ is monotone (by Step 2) and there exist the limits

$$\lim_{m\to\infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) = \lim_{m\to\infty} gx_m^i = x_i \in X$$

for all i. Since F and g are O-compatible, we have

$$\begin{aligned}&\lim_{m\to\infty} d(ggx_{m+1}^i, F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)}))\\ &\quad = \lim_{m\to\infty} d(gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)})) = 0\end{aligned} \tag{9}$$

for all i.

Step 5. Suppose that F is continuous. In this case, since $\{gx_m^{\sigma_i(j)}\}_m \to x_{\sigma_i(j)}$ for all i, j and F is continuous, $\{F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)})\}_m \to F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)})$ for all i and, taking into account (8) and (9), we conclude that $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i$ for all i, that is, $(x_1, x_2, \ldots, x_n)$ is a Υ-coincidence point of F and g.

Step 6. Suppose that X has the sequential g-monotone property. In this case, by Step 2, we know that $gx_m^i \preceq_i gx_{m+1}^i$ for all m and i. This means that the sequence $\{gx_m^i\}_{m\geq 0}$ is monotone. Since $x_i = \lim_{m\to\infty} gx_m^i$, we deduce that $ggx_m^i \preceq_i gx_i$ for all m and i. This condition implies that, for all m and j, either $ggx_m^{\sigma_j(i)} \preceq_i gx_{\sigma_j(i)}$ for all i or $gx_{\sigma_j(i)} \preceq_i ggx_m^{\sigma_j(i)}$ for all i (the first case occurs when $j \in A$ and the second one when $j \in B$). Then, by (2), for all j, we have

$$\begin{aligned}&\psi\left(d\left(ggx_{m+1}^j, F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(n)})\right)\right)\\ &= \psi\left(d\left(F(gx_m^{\sigma_j(1)}, gx_m^{\sigma_j(2)}, \ldots, gx_m^{\sigma_j(n)}, F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(n)})\right)\right)\\ &\leq \psi\left(\max_{1\leq i\leq n} d\left(ggx_m^{\sigma_j(i)}, gx_{\sigma_j(i)}\right)\right) - \varphi\left(\max_{1\leq i\leq n} d\left(ggx_m^{\sigma_j(i)}, gx_{\sigma_j(i)}\right)\right)\\ &\leq \psi\left(\max_{1\leq i\leq n} d\left(ggx_m^i, gx_i\right)\right).\end{aligned} \tag{10}$$

Taking into account that $gx_i = \lim_{m\to\infty} ggx_{m+1}^i$ for all i, we can deduce that

$$\lim_{m\to\infty} d\left(ggx_{m+1}^i, gx_i\right) = 0,$$

and so $\lim_{m\to\infty} \max_{1\leq i\leq n} d\left(ggx_m^i, gx_i\right) = 0$. Taking the limit in (4.10) when $m \to \infty$, we deduce that

$$\lim_{m\to\infty} \psi\left(d\left(ggx_{m+1}^j, F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(n)})\right)\right) = 0$$

since ψ is continuous. From $\psi \in \Psi$, Lemma 11 guaranties that

$$\lim_{m\to\infty} d\left(ggx_{m+1}^j, F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(n)})\right) = 0$$

and so $F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(n)}) = \lim_{m\to\infty} ggx_{m+1}^j = gx_j$ for all j. Therefore, $(x_1, x_2, \ldots, x_n)$ is a Υ-coincidence point of F and g. This completes the proof. ■

Remark 15. In the previous Theorem, if the image $\operatorname{Im} d$ of the metric d is not the whole set $[0, \infty[$, then ψ and φ can only be defined on $\operatorname{Im} d$ and we can consider a wider range of mappings since it is only necessary to impose that they are continuous and non-decreasing on $\operatorname{Im} d$.

Corollary 16. *Under hypothesis of Theorem 14, if $(y_1, y_2, \ldots, y_n) \in X^n$ is a Υ-coincidence point of F and g, then $(gy_1, gy_2, \ldots, gy_n)$ also is a Υ-coincidence point of F and g.*

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Proof. If we suppose that $gy_i = F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \ldots, y_{\sigma_i(n)})$ for all i, we can choose $x_m^i = y_i$ for all m and all i as in the previous proof. We have just proved that $\{gx_m^i\} \to x_i$ for all i and $(x_1, x_2, \ldots, x_n)$ is a Υ-coincidence point of F and g. In this case, $\{gx_m^i\} = \{gy_i\} \to gy_i$ for all i and so $(gy_1, gy_2, \ldots, gy_n)$ also is a Υ-coincidence point of F and g. This completes the proof. ■

The following example is based on Examples 1.9 and 2.2 in [12].

Example 17. Let $X = \{0\} \cup [1/2, 2]$ provided with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is complete and d generates the discrete topology on X. Consider on X the following partial order:

$$x, y \in X, \qquad x \leq y \Leftrightarrow x = y \text{ or } (x, y) \in \{(0, 0), (0, 1), (1, 1)\}.$$

Consider $F : X^n \to X$ and $g : X \to X$ defined by

$$F(x_1, x_2, ..., x_n) = \begin{cases} 0 & \text{if } x_1, x_2, ..., x_n \in \{0\} \cup [1/2, 1] \\ 1 & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1, \\ 2 - x & \text{if } 1 \leq x \leq \frac{3}{2}, \\ \frac{1}{2} & \text{if } \frac{3}{2} \leq x \leq 2. \end{cases}$$

Then it is not difficult to prove the following statements:

(1) F and g are continuous mappings (since d generates the discrete topology on X).

(2) F has the mixed $(g, \preccurlyeq)$-monotone property on X.

(3) If $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ verify $gx_i \preccurlyeq_i gy_i$ for all i, then

$$F(x_1, x_2, \ldots, x_n) = F(y_1, y_2, \ldots, y_n).$$

In particular, (2) holds (whatever ψ and φ; for instance, $\psi(t) = t$ and $\varphi(t) = (1 - k)t$).

For simplicity, henceforth, suppose that n is even and A (resp., B) denotes the set of all odd (resp., even) numbers in $\{1, 2, \ldots, n\}$.

(4) For a mapping $\sigma : \Lambda_n \to \Lambda_n$, we use the notation $\sigma \equiv (\sigma(1), \sigma(2), \ldots, \sigma(n))$ and consider

$$\sigma_i \equiv (i, i + 1, \ldots, n - 1, n, 1, 2, \ldots, i - 1)$$

for all i. Then $\sigma_i \in \Omega_{A,B}$ if i is odd and $\sigma_i \in \Omega'_{A,B}$ if i is even. Let $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_n)$.

(5) F and g are O-compatible but they do not commute.

Indeed, let $\{x_m^1\}_{m\geq 0}$, $\{x_m^2\}_{m\geq 0}$, ..., $\{x_m^n\}_{m\geq 0}$ in X such that $\{gx_m^1\}_{m\geq 0}$, $\{gx_m^2\}_{m\geq 0}, \ldots, \{gx_m^n\}_{m\geq 0}$ are monotone and

$$\lim_{m\to\infty} F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) = \lim_{m\to\infty} gx_m^i = x^i \in X,$$

Since $F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}) = F(x_m^{\sigma_j(1)}, x_m^{\sigma_j(2)}, \ldots, x_m^{\sigma_j(n)}) \in \{0, 1\}$ for all i, j and for all n, $x^i = x^j \in \{0, 1\}$, for all i, j. The case, $x^i = 1$, for all i is impossible. Thus $x^i = 0$, for all i. That implies that $gx_m^i = 0$, for all i and all $m \geq n_0$, for some n_0. That is, $x_m^i = 0$, for all i and all $m \geq n_0$. Thus, for all i and all $m \geq n_0$,

$$d(gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)}), F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)})) = 0.$$

(6) Take $x_0^i = 0$ if i is odd and $x_0^i = 1$ if i is even. Then $gx_0^i \preccurlyeq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all i.

(7) $(X, d, \preccurlyeq)$ has the sequential g-monotone property.

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In conclusion, taking into account that all the conditions of Theorem 14 are satisfied, F and g have, at least, one Υ-coincidence point. However, Theorem 9 in [23] cannot be applied since F and g are not commuting as well as other results in dimensions 2 or 3.

5. Consequences

Note that Theorem 9 in [23] is an easy consequence of Theorem 14 if we take $\psi(t) = t$ and $\varphi(t) = (1-k)t$ for all $t \in [0, \infty[$.

In the next result, let Γ the family of all non-decreasing on each argument, continuous mappings $\phi : [0, \infty[^n \to [0, \infty[$ verifying $\phi(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_1 = x_2 = \ldots = x_n = 0$. Examples of such mappings are as follows:

(1) $\phi(x_1, x_2, \ldots, x_n) = k \ \max_{1 \leq i \leq n} x_i$, where $k > 0$;
(2) $\phi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i x_i$;
(3) $\phi(x_1, x_2, \ldots, x_n) = \sqrt[m]{\alpha_1 x_1^2 + \alpha_2 x_2^2 + \ldots + \alpha_n x_n^2}$;
(4) $\phi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \alpha_i x_i^{n_i}$, where $\alpha_i > 0$ and $n_i \in \mathbb{N}$ for all i.

Lemma 18. *Let $\phi \in \Gamma$ and define a function $\varphi : [0, \infty[\to [0, \infty[$*

$$\varphi(t) = \min(\phi(te_1), \phi(te_2), \ldots, \phi(te_n))$$

for all $t \geq 0$, where $\{e_1, e_2, \ldots, e_n\}$ is the usual basis of $\mathbb{R}^n$. Then $\varphi \in \Psi$ and $\varphi(\max_{1 \leq i \leq n} x_i) \leq \phi(x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in [0, \infty[$.

Proof. The first part is clear. Let $x_1, x_2, \ldots, x_n \in [0, \infty[$ and let $x_{i_0} = \max_{1 \leq i \leq n} x_i$. Since ϕ is non-decreasing on each argument, we have

$$\phi(x_{i_0} e_{i_0}) = \phi(0, 0, \ldots, 0, x_{i_0}, 0, \ldots, 0) \leq \phi(x_1, x_2, \ldots, x_{i_0 - 1}, x_{i_0}, x_{i_0}, \ldots, x_n) = \phi(x_1, x_2, \ldots, x_n).$$

Therefore, we have

$$\begin{aligned} \varphi(\max_{1 \leq i \leq n} x_i) &= \varphi(x_{i_0}) \\ &= \min(\phi(x_{i_0} e_1), \phi(x_{i_0} e_2), \ldots, \phi(x_{i_0} e_n)) \\ &\leq \phi(x_{i_0} e_{i_0}) \\ &\leq \phi(x_1, x_2, \ldots, x_n). \end{aligned}$$

This completes the proof. ■

Corollary 19. *The conclusion of Theorem 14 also holds if we replace the contractive condition* (2) *by one of the following conditions:*

(A) ([5, pag. 1322],[23, pag. 538]) *There exist $\psi \in \Psi$ and $\phi \in \Gamma$ such that*

$$\begin{aligned} &\psi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \\ &\leq \psi\left(\max_{1 \leq i \leq n} d(gx_i, gy_i)\right) - \phi\left(d(gx_1, gy_1), \ldots, d(gx_n, gy_n)\right), \end{aligned} \tag{5.1}$$

for which $gx_i \preceq_i gy_i$ for all i.

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(B) ([1, pag. 4451]) *There exist* $\psi, \varphi \in \Psi$ *and* $\beta_1, \beta_2, \ldots, \beta_n \in [0,1]$ *such that* $\beta_1 + \beta_2 + \ldots + \beta_n \leq 1$ *and*

$$\psi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \leq \psi\left(\sum_{i=1}^{n} \beta_i d(gx_i, gy_i)\right) - \varphi\left(\max_{1\leq i\leq n} d(gx_i, gy_i)\right), \tag{5.2}$$

for which $gx_i \leq_i gy_i$ *for all* i.

(C) *There exist* $\psi, \varphi \in \Psi$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ *such that*

$$\psi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \leq \psi\left(\max_{1\leq i\leq n} d(gx_i, gy_i)\right) - \varphi\left(\sum_{i=1}^{n} \alpha_i d(gx_i, gy_i)\right),$$

for which $gx_i \leq_i gy_i$ *for all* i.

(D) ([4, pag. 1380],[3, pag.4891],[6, pag. 5931]) *There exist* $\psi \in \Psi$, $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ *and* $\beta_1, \beta_2, \ldots, \beta_n \geq 0$ *such that* $\beta_1 + \beta_2 + \ldots + \beta_n \leq 1$ *and*

$$\psi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \leq \psi\left(\sum_{i=1}^{n} \beta_i d(gx_i, gy_i)\right) - \sum_{i=1}^{n} \alpha_i d(gx_i, gy_i),$$

for which $gx_i \leq_i gy_i$ *for all* i.

(E) ([17, pag 4343]) *There exist* $\psi \in \Psi$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ *such that*

$$\psi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \leq \psi\left(\frac{1}{n}\sum_{i=1}^{n} d(gx_i, gy_i)\right) - \sum_{i=1}^{n} \alpha_i d(gx_i, gy_i),$$

for which $gx_i \leq_i gy_i$ *for all* i.

(F) ([13, pag. 1841],[18, pag. 984]) *There exist* $\psi, \varphi \in \Psi$ *such that* ψ *is subadditive* ($\psi(s+t) \leq \psi(s) + \psi(t)$ *for all* $t, s \in [0, \infty)$) *and*

$$\psi(d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n))) \leq \frac{1}{n}\psi\left(\sum_{i=1}^{n} d(gx_i, gy_i)\right) - \varphi\left(\max_{1\leq i\leq n} d(gx_i, gy_i)\right),$$

for which $gx_i \leq_i gy_i$ *for all* i.

Of course, it is also interesting to particularize all the previous conditions to the following cases: $\psi(t) = \lambda t$ (where $\lambda > 0$), $\varphi(t) = \mu t$ (where $\mu > 0$) or $gx = x$ for all $x \in X$.

(A) By Lemma 5.1, there exists $\varphi \in \Psi$ such that $-\phi(x_1, x_2, \ldots, x_n) \leq -\varphi(\max_{1\leq i\leq n} x_i)$ for all $x_1, x_2, \ldots, x_n \in [0, \infty[$ and so (5.1) implies (2).

(B) It is obvious that

$$\sum_{i=1}^{n} \beta_i d(gx_i, gy_i) \leq \left(\sum_{i=1}^{n} \beta_i\right) \max_{1\leq j\leq n} d(gx_j, gy_j) \leq \max_{1\leq j\leq n} d(gx_j, gy_j)$$

and so (5.2) implies (2).

(C) We only take $\phi(x_1, x_2, \ldots, x_n) = \varphi\left(\sum_{i=1}^{n} \alpha_i x_i\right)$ in (A).

(D) It is a mixture of (B) and (C).

(E) It is a particular case of (D), where $\beta_i = 1/n$ for all i.

(F) If ψ is subadditive, then $\frac{1}{n}\psi(t) \leq \psi\left(\frac{t}{n}\right)$ for all $t \geq 0$ and so we may choose $\beta_i = 1/n$ for all i in (B).

6. The uniqueness of Υ-coincidence points

Consider on the product space X^n the following partial order: for $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$,

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \iff x_i \leq_i y_i, \text{ for all } i.$$

We say that $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are *comparable* if

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \text{ or } (x_1, x_2, \ldots, x_n) \geq (y_1, y_2, \ldots, y_n).$$

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Theorem 20. *Under hypothesis of Theorem 14, assume that, for all Υ-coincidence points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in X^n$ of F and g, there exists $(u_1, u_2, \ldots, u_n) \in X^n$ such that $(gu_1, gu_2, \ldots, gu_n)$ is comparable to $(gx_1, gx_2, \ldots, gx_n)$ and $(gy_1, gy_2, \ldots, gy_n)$, respectively. Then F and g have a unique Υ-coincidence point $(z_1, z_2, \ldots, z_n) \in X^n$ such that $gz_i = z_i$ for all i.*

Proof. From Theorem 14, the set of Υ-coincidence points of F and g is non-empty. The proof is divided in two steps.

Step 1. We claim that, if $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are two Υ-coincidence points of F and g, then we have

$$gx_i = gy_i \tag{6.1}$$

for all i. Let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ be two Υ-coincidence points of F and g and let $(u_1, u_2, \ldots, u_n) \in X^n$ be a point such that $(gu_1, gu_2, \ldots, gu_n)$ is comparable, at the same time, to $(gx_1, gx_2, \ldots, gx_n)$ and to $(gy_1, gy_2, \ldots, gy_n)$. Using $(u_1, u_2, \ldots, u_n)$ define the following sequences. Let $u_0^i = u_i$ for all i. Reasoning as in Theorem 14, we can determine the sequences $\{u_m^1\}_{m\geq 0}$, $\{u_m^2\}_{m\geq 0}$, $\ldots$, $\{u_m^n\}_{m\geq 0}$ such that $gu_{m+1}^i = F(u_m^{\sigma_i(1)}, u_m^{\sigma_i(2)}, \ldots, u_m^{\sigma_i(n)})$ for all m and i.

Now, prove that $gx_i = \lim_{m\to 0} gu_m^i = gy_i$ for all i and so (6.1) is true.

First, we concern with $(gu_1, gu_2, \ldots, gu_n)$, $(gx_1, gx_2, \ldots, gx_n)$ and the same argument will be true for $(gu_1, gu_2, \ldots, gu_n)$ and $(gy_1, gy_2, \ldots, gy_n)$. Since $(gu_1, gu_2, \ldots, gu_n)$ and $(gx_1, gx_2, \ldots, gx_n)$ are comparable, we can suppose that $(gu_1, gu_2, \ldots, gu_n) \leq (gx_1, gx_2, \ldots, gx_n)$ (the other case is similar), i.e., $gu_0^i = gu_i \leq_i gx_i$ for all i. Using that F has the mixed g-monotone property, as in Theorem 14, it is possible to prove that $gu_m^i \leq_i gx_i$ for all $m \geq 1$ and i. This condition implies that, for all j and $m \geq 1$, either $gu_m^{\sigma_j(i)} \leq_i gx_{\sigma_j(i)}$ for all i or $gx_{\sigma_j(i)} \leq_i gu_m^{\sigma_j(i)}$ for all i.

Define $\beta_m = \max_{1\leq i\leq n} d(gu_m^i, gx_i)$ for all m. As in Theorem 14, it is not difficult to prove that $\{\beta_m\}_{m\geq 1} \to 0$, which means that

$$\lim_{m\to\infty} \beta_m = \lim_{m\to\infty} \left(\max_{1\leq i\leq n} d(gu_m^i, gx_i) \right) = 0.$$

Since $0 \leq d(gu_m^i, gx_i) \leq \beta_m$ for all m and i, we deduce that $\{d(gu_m^i, gx_i)\}_{m\geq 1} \to 0$ for all i, i.e.,

$$\lim_{m\to\infty} gu_m^i = gx_i \tag{6.2}$$

for all i.

If we had supposed that $(gx_1, gx_2, \ldots, gx_n) \leq (gu_1, gu_2, \ldots, gu_n)$, we can obtain the same property (6.2). Since $(gu_1, gu_2, \ldots, gu_n)$ also is comparable to $(gy_1, gy_2, \ldots, gy_n)$, similarly, we can prove that $gy_i = \lim_{m\to 0} gu_m^i = gx_i$ for all i.

Let $(x_1, x_2, \ldots, x_n) \in X^n$ be a Υ-coincidence point of F and g and define $z_i = gx_i$ for all i. Since $(z_1, z_2, \ldots, z_n) = (gx_1, gx_2, \ldots, gx_n)$, Corollary 16 assures that $(z_1, z_2, \ldots, z_n)$ also is a Υ-coincidence point of F and g.

Step 2. We claim that $(z_1, z_2, \ldots, z_n)$ is the unique Φ-coincidence point of F and g such that $gz_i = z_i$ for all i. It is similar to Step 2 in Theorem 11 in [23]. This completes the proof. ∎

It is natural to say that g is *injective* on the set of all Υ-coincidence points of F and g when $gx_i = gy_i$ for all i implies $x_i = y_i$ for all i when $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are two Υ-coincidence points of F and g. For example, this is true if g is injective on X.

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Corollary 21. *In addition to hypotheses of Theorem 20, suppose that g is injective on the set of all Υ-coincidence points of F and g. Then F and g have a unique Υ-coincidence point.*

Proof. If $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are two coincidence points of F and g, then we have proved in (6.1) that $gx_i = gy_i$ for all i. Since g is injective on these points, then $x_i = y_i$ for all i. ■

Corollary 22. *In addition to hypotheses of Theorem 20, suppose that $(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)})$ is comparable to $(z_{\sigma_j(1)}, z_{\sigma_j(2)}, \ldots, z_{\sigma_j(n)})$ for all i, j. Then $z_1 = z_2 = \ldots = z_n$. In particular, there exists a unique $z \in X$ such that $F(z, z, \ldots, z) = z$, which verifies $gz = z$.*

Proof. Let $M = \max_{1\leq i,j\leq n} d(z_i, z_j)$, let $j_0, s_0 \in \{1, 2, \ldots, n\}$ such that $M = d(z_{j_0}, z_{s_0})$ and let

$$\Lambda_{j_0 s_0} = \max_{1\leq i\leq n} d(z_{\sigma_{j_0}(i)}, z_{\sigma_{s_0}(i)}) \leq M.$$

Fix $j, s \in \{1, 2, \ldots, n\}$. Since $(z_{\sigma_j(1)}, z_{\sigma_j(2)}, \ldots, z_{\sigma_j(n)})$ is comparable to $(z_{\sigma_s(1)}, z_{\sigma_s(2)}, \ldots, z_{\sigma_s(n)})$, it follows that either $z_{\sigma_j(i)} \leq_i z_{\sigma_s(i)}$ for all i or $z_{\sigma_s(i)} \leq_i z_{\sigma_j(i)}$ for all i. Since $gz_i = z_i$ for all i, we know that either $gz_{\sigma_j(i)} \leq_i gz_{\sigma_s(i)}$ for all i or $gz_{\sigma_s(i)} \leq_i gz_{\sigma_j(i)}$ for all i. In any case, applying (2), we have

$$\begin{aligned}
\psi(M) &= \psi(d(z_{j_0}, z_{s_0})) \\
&= \psi(d(gz_{j_0}, gz_{s_0})) \\
&= \psi(d(F(z_{\sigma_{j_0}(1)}, z_{\sigma_{j_0}(2)}, \ldots, z_{\sigma_{j_0}(n)}), F(z_{\sigma_{s_0}(1)}, z_{\sigma_{s_0}(2)}, \ldots, z_{\sigma_{s_0}(n)}))) \\
&\leq \psi\left(\max_{1\leq i\leq n} d(gz_{\sigma_{j_0}(i)}, gz_{\sigma_{s_0}(i)})\right) - \varphi\left(\max_{1\leq i\leq n} d(gz_{\sigma_{j_0}(i)}, gz_{\sigma_{s_0}(i)})\right) \\
&= \psi\left(\max_{1\leq i\leq n} d(z_{\sigma_{j_0}(i)}, z_{\sigma_{s_0}(i)})\right) - \varphi\left(\max_{1\leq i\leq n} d(z_{\sigma_{j_0}(i)}, z_{\sigma_{s_0}(i)})\right) \\
&= \psi\left(\Lambda_{j_0 s_0}\right) - \varphi\left(\Lambda_{j_0 s_0}\right).
\end{aligned}$$

If $\Lambda_{j_0 s_0} > 0$, then $\psi(M) \leq \psi\left(\Lambda_{j_0 s_0}\right) - \varphi\left(\Lambda_{j_0 s_0}\right) \leq \psi\left(M\right) - \varphi\left(\Lambda_{j_0 s_0}\right) < \psi(M)$, which is impossible. Therefore, $\Lambda_{j_0 s_0} = 0$ and $\psi(M) \leq \psi\left(\Lambda_{j_0 s_0}\right) - \varphi\left(\Lambda_{j_0 s_0}\right) = \psi\left(0\right) - \varphi\left(0\right) = 0$. Then $M = 0$ and $z_1 = z_2 = \ldots = z_n$. This completes the proof. ■

Example 23. Let $X = \mathbb{R}$ provided with the usual partial order $\leq$ and the usual metric $d(x, y) = |x - y|$. Let $n \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_n \in \mathbb{R}\setminus\{0\}$ be real numbers such that there exist $i_0, j_0 \in \{1, 2, \ldots, n\}$ verifying $a_{i_0} < 0 < a_{j_0}$. Let $N > |a_1| + |a_2| + \ldots + |a_n|$ and consider $F(x_1, x_2, \ldots, x_n) = (a_1 x_1 + a_2 x_2 + \ldots + a_n x_n)/N$ and $gx = x$, for all $x, x_1, x_2, \ldots, x_n \in X$. Then F is monotone, non-decreasing in those arguments for which $a_i > 0$ and monotone non-increasing in those arguments for which $a_i < 0$.

Furthermore, taking $k = (|a_1| + |a_2| + \ldots + |a_n|)/N \in (0, 1)$, F verifies the condition (2). Actually, all conditions of Theorems 14 and 20 (and Corollaries 21 and 22) are satisfied. Indeed, it is clear that $(0, 0, \ldots, 0)$ is the unique fixed point of F.

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