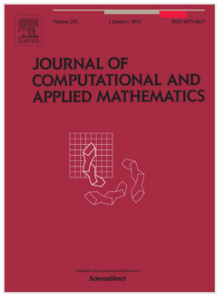

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New method for computing the upper bound of optimal value in interval quadratic program

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Abstract We consider the interval quadratic programming problems. The aim of this paper is to present a new method to compute the upper bound of the optimal values, under weaker conditions. Moreover, we discuss the relations between the new method and previous results. The features of the proposed methods are illustrated by some examples.

Keywords: Quadratic programming; Interval systems; Optimal value range; Duality gap;

Mathematics Subject Classification (2010) 15A06; 65G40

1 Introduction

In many real world applications, system parameters or model coefficients are not always known exactly and may be bounded between lower and upper bounds due to a variety of uncertainties[1–3]. Over the past decades, interval mathematical programming methods were developed to tackle such uncertainties [4–13]. Many papers studied the problem of computing the range of optimal values of interval linear programming problems, see e.g.,[7, 10, 14–17] among others. Some authors studied the problem of computing the range of optimal values of interval quadratic programs (IvQP). It is known that finding the lower bound of the optimal value in IvQP is polynomially solvable, whereas finding the upper bound of the optimal value function is a computationally hard problem when the constraints include interval linear equalities. While to determine the upper bound of the optimal value function, the existing methods have to consider the dual of the primal problem, and the condition that the duality gap is zero should be specified[18–20].

We study IvQP and our aim is to establish a new method to compute the upper bound of optimal values, which is an analogue of the results in interval linear program [7, 15, 16]. In this method, only primal program is taken into consideration. The dual problem is not required and thus the condition that the duality gap is zero is also removed.

2 Preliminaries

Following notations from [15], an interval matrix is defined as

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, $\underline{A} \leq \overline{A}$, and "$\leq$" is understood componentwise. By

$$A_c = \frac{1}{2}(\underline{A} + \overline{A}), A_\Delta = \frac{1}{2}(\overline{A} - \underline{A}),$$

we denote the center and the radius of A, respectively. Then $\mathbf{A} = [A_c - A_\Delta, A_c + A_\Delta]$. An interval vector $\mathbf{b} = [\underline{b}, \overline{b}] = \{b \in \mathbb{R}^m : \underline{b} \leq b \leq \overline{b}\}$ is understood as one-column interval matrix.

Let $\{\pm 1\}^m$ be the set of all $\{-1, 1\}$ m-dimensional vectors, i.e.

$$\{\pm 1\}^m = \{y \in \mathbb{R}^m | \mid y \mid = e\},$$

where $e = (1, \cdots, 1)^T$ is the m-dimensional vector of all $1's$ and the absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. For a given $y \in \{\pm 1\}^m$, let

$$T_y = diag(y_1, \ldots, y_m)$$

denote the corresponding diagonal matrix. For each $x \in \mathbb{R}^n$, we define its sign vector sgn x by

$$(\text{sgn } x)_i = \begin{cases} 1 & \text{if } x_i \geq 0, \\ -1 & \text{if } x_i < 0, \end{cases}$$

where $i = 1, \cdots, n$. Then we have $|x| = T_z x$, where $z = \text{sgn } x \in \{\pm 1\}^n$.

Given an interval matrix $\mathbf{A} = [A_c - A_\Delta, A_c + A_\Delta]$, for each $y \in \{\pm 1\}^m$ and $z \in \{\pm 1\}^n$, we define matrices

$$A_{yz} = A_c - T_y A_\Delta T_z.$$

Similarly, for an interval vector $\mathbf{b} = [b_c - b_\Delta, b_c + b_\Delta]$ and for each $y \in \{\pm 1\}^m$, we define vectors

$$b_y = b_c + T_y b_\Delta.$$

Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{k\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, d \in \mathbb{R}^k$ and $Q \in \mathbb{R}^{n\times n}$ be given, consider the quadratic programming problem

$$\min \frac{1}{2}x^T Qx + c^T x \text{ subject to } Ax \leq b, Bx = d, x \geq 0,$$

where Q is positive semidefinite. Briefly, we rewrite the problem as

$$\text{Min}\{\frac{1}{2}x^T Qx + c^T x | Ax \leq b, Bx = d, x \geq 0\}. \tag{1}$$

The Dorn dual problem [21, 22] of the quadratic program (1) is

$$\text{Max}\{-\frac{1}{2}u^T Qu - b^T v - d^T w | Qu + A^T v + B^T w + c \geq 0, v \geq 0\}. \tag{2}$$

Let

$$f(A,B,b,c,d,Q)=\inf\{\frac{1}{2}x^TQx+c^Tx|Ax\le b,Bx=d,x\ge 0\}$$

and

$$g(A,B,b,c,d,Q)=\sup\{-\frac{1}{2}u^TQu-b^Tv-d^Tw|Qu+A^Tv+B^Tw+c\ge 0,v\ge 0\}$$

denote the optimal value of (1) and (2), respectively.

Clearly, the following result of weak duality holds.

Theorem 2.1. *(Weak duality) We have*

$$f(A,B,b,c,d,Q)\ge g(A,B,b,c,d,Q).$$

The following result of strong duality is from [21].

Theorem 2.2. *([21]) (i) If $x=x_0$ is an optimal solution to problem (1) then an optimal solution $(u,v,w)^T=(u_0,v_0,w_0)^T$ exists to problem (2). (ii) Conversely, if an optimal solution $(u,v,w)^T=(u_0,v_0,w_0)^T$ to problem (2) exists then an optimal solution $x=x_0$ to problem (1) also exists. In either case,*

$$f(A,B,b,c,d,Q)=g(A,B,b,c,d,Q).$$

The set of all m-by-n interval matrices will be denoted by $\mathbb{IR}^{m\times n}$ and the set of all m-dimensional interval vectors by $\mathbb{IR}^m$. Given $\mathbf{A}\in\mathbb{IR}^{m\times n},\mathbf{B}\in\mathbb{IR}^{k\times n},\mathbf{b}\in\mathbb{IR}^m,\mathbf{c}\in\mathbb{IR}^n,\mathbf{d}\in\mathbb{IR}^k$ and $\mathbf{Q}\in\mathbb{IR}^{n\times n}$, the interval convex quadratic program

$$\text{Min}\{\frac{1}{2}x^T\mathbf{Q}x+\mathbf{c}^Tx|\mathbf{A}x\le\mathbf{b},\mathbf{B}x=\mathbf{d},x\ge 0\} \tag{3}$$

is the family of convex quadratic programs (1) with data satisfying

$$A\in\mathbf{A},B\in\mathbf{B},b\in\mathbf{b},c\in\mathbf{c},d\in\mathbf{d},Q\in\mathbf{Q},$$

where Q is positive semidefinite for all $Q\in\mathbf{Q}$. The lower and upper bound of the optimal values are respectively defined as

$$\underline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q})=inf\{f(A,B,b,c,d,Q)|A\in\mathbf{A},B\in\mathbf{B},b\in\mathbf{b},c\in\mathbf{c},d\in\mathbf{d},Q\in\mathbf{Q}\},$$
$$\overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q})=sup\{f(A,B,b,c,d,Q)|A\in\mathbf{A},B\in\mathbf{B},b\in\mathbf{b},c\in\mathbf{c},d\in\mathbf{d},Q\in\mathbf{Q}\}.$$

For given $\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q}$ denote $\underline{f}=\underline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q})$, $\overline{f}=\overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q})$.

Theorem 2.3. *(Proposition 6 in [20]) We have*

$$\underline{f}=\inf\frac{1}{2}x^T\underline{Q}x+\underline{c}^Tx \quad \textit{subject to} \quad \underline{A}x\le\overline{b},\underline{B}x\le\overline{d},\overline{B}x\ge\underline{d},x\ge 0.$$

The following theorem from [23] characterizes solvability of interval linear systems, together with Theorem 2.1 and Theorem 2.2, will be used to obtain our main results.

Theorem 2.4. *Let* $\mathbf{A} \in \mathbb{IR}^{k\times m},\ \mathbf{B} \in \mathbb{IR}^{k\times n},\ \mathbf{C} \in \mathbb{IR}^{l\times m},\ \mathbf{D} \in \mathbb{IR}^{l\times n},\ \mathbf{b} \in \mathbb{IR}^{k}$*, and* $\mathbf{d} \in \mathbb{IR}^{l}$*, the following system*

$$Ax + By = b,\ Cx + Dy \le d,\ x \ge 0$$

is solvable for each $A \in \mathbf{A},\ B \in \mathbf{B},\ C \in \mathbf{C},\ D \in \mathbf{D},\ b \in \mathbf{b}$ *and* $d \in \mathbf{d}$ *if and only if for each* $s \in \{\pm 1\}^k$ *the system*

$$\begin{aligned} A_{-se}x + B_{-se}y^1 - B_{se}y^2 &= b_{-s}, \\ \overline{C}x + \overline{D}y^1 - \underline{D}y^2 &\le \underline{d}, \\ x,\ y^1,\ y^2 &\ge 0 \end{aligned}$$

is solvable.

3 Computing the upper bound

It is known that the computing of lower bound $\underline{f}$ is an easy task, but the calculation for upper bound $\overline{f}$ is difficult [20]. The upper bound $\overline{f}$ is given in [20] by using the Dorn dual problem of IvQP (3) when duality gap is zero. In this section, we present a new method for computing $\overline{f}$ directly from scenarios of primal problem (3). No dual problem is required and thus the condition that duality gap is zero is removed.

Theorem 3.1. *We have*

$$\underline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q}) = inf\{\frac{1}{2}x^T\underline{Q}x + \underline{c}^Tx | \underline{A}x \le \overline{b}, \overline{B}x \ge \underline{d}, \underline{B}x \le \overline{d}, x \ge 0\}, \quad (4)$$

$$\overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q}) = \sup_{y\in\{\pm1\}^k} f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}). \quad (5)$$

Proof: The formula (4) is given by Theorem 2.3 [20]. Here, we only prove (5). Note that for each $x \ge 0$ we have

$$\frac{1}{2}x^T\mathbf{Q}x + \mathbf{c}^Tx \le \frac{1}{2}x^T\overline{Q}x + \overline{c}^Tx,$$

so for each $A \in \mathbf{A},\ B \in \mathbf{B},\ b \in \mathbf{b},\ c \in \mathbf{c},\ d \in \mathbf{d},\ Q \in \mathbf{Q}$

$$f(A, B, b, c, d, Q) \le f(A, B, b, \overline{c}, d, \overline{Q})$$

and hence

$$\overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q}) \le \overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\overline{c},\mathbf{d},\overline{Q}). \quad (6)$$

On the other hand, from $\overline{Q} \in \mathbf{Q}, \overline{c} \in \mathbf{c}$ we know that

$$\overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\overline{c},\mathbf{d},\overline{Q}) \le \overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q}). \quad (7)$$

Thus, from (6) and (7) we can obtain

$$\overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q}) = \overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\overline{c},\mathbf{d},\overline{Q}). \quad (8)$$

Furthermore, the upper bound $\overline{f}$ can be determined by the least feasible region, then we have

$$\overline{f}(\mathbf{A}, \mathbf{B}, \mathbf{b}, \overline{c}, \mathbf{d}, \overline{Q}) = \overline{f}(\overline{A}, \mathbf{B}, \underline{b}, \overline{c}, \mathbf{d}, \overline{Q}), \tag{9}$$

since we are easy to obtain that $\overline{A}x \leq \underline{b}, x \geq 0$ is the least feasible region inequality defined by $\mathbf{A}x \leq \mathbf{b}, x \geq 0$.

From (8) and (9) we are only required to prove

$$\overline{f}(\overline{A}, \mathbf{B}, \underline{b}, \overline{c}, \mathbf{d}, \overline{Q}) = \sup_{y \in \{\pm 1\}^k} f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}).$$

Let

$$\overline{\varphi} = \sup_{y \in \{\pm 1\}^k} f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}).$$

(i) Since $B_{ye} \in \mathbf{B}, d_y \in \mathbf{d}$ for each $y \in Y_m$, we are easily to obtain that

$$\overline{\varphi} \leq \sup\{f(\overline{A}, B, \underline{b}, \overline{c}, d, \overline{Q}) | B \in \mathbf{B}, d \in \mathbf{d}\} = \overline{f}(\overline{A}, \mathbf{B}, \underline{b}, \overline{c}, \mathbf{d}, \overline{Q}).$$

(ii) Now we prove $\overline{f}(\overline{A}, \mathbf{B}, \underline{b}, \overline{c}, \mathbf{d}, \overline{Q}) \leq \overline{\varphi}$ by showing that

$$f(\overline{A}, B, \underline{b}, \overline{c}, d, \overline{Q}) \leq \overline{\varphi} \tag{10}$$

holds for each $B \in \mathbf{B}, d \in \mathbf{d}$.

Note that for each $B \in \mathbf{B}, d \in \mathbf{d}$, obviously, the value of $f(\overline{A}, B, \underline{b}, \overline{c}, d, \overline{Q})$ can be divided into three cases, they are $-\infty$, ∞ and finite value. We discuss them separately.

If $f(\overline{A}, B, \underline{b}, \overline{c}, d, \overline{Q}) = -\infty$ for some $B \in \mathbf{B}, d \in \mathbf{d}$, (10) holds obviously.

If $f(\overline{A}, B, \underline{b}, \overline{c}, d, \overline{Q}) = \infty$ for some $B \in \mathbf{B}, d \in \mathbf{d}$, then the quadratic programming problem

$$\min \frac{1}{2} x^T \overline{Q} x + \overline{c}^T x \;\text{ subject to }\; \overline{A}x \leq \underline{b}, Bx = d,\; x \geq 0 \tag{11}$$

is infeasible. That is, there exist $B \in \mathbf{B},\; d \in \mathbf{d}$ such that the following system

$$\overline{A}x \leq \underline{b}, Bx = d,\; x \geq 0$$

is not solvable. By Theorem 2.4, we have the system

$$\overline{A}x \leq \underline{b}, B_{ye}x = d_y,\; x \geq 0$$

is not solvable for some $y \in \{\pm 1\}^k$. Thus we have

$$f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) = \infty$$

for some $y \in \{\pm 1\}^k$, which means (10) holds.

If $f(\overline{A}, B, \underline{b}, \overline{c}, d, \overline{Q})$ is finite for some $B \in \mathbf{B}, d \in \mathbf{d}$, then from Theorem 2.2 we know the Dorn dual problem to (11)

$$\max -\frac{1}{2} u^T \overline{Q} u - \underline{b}^T v - d^T w \;\text{ subject to }\; \overline{Q}u + \overline{A}^T v + B^T w + \overline{c} \geq 0, v \geq 0 \tag{12}$$

has an optimal solution $(u^*, v^*, w^*)^T$, then

$$\overline{Q}u^* + \overline{A}^T v^* + B^T w^* + \overline{c} \geq 0, v^* \geq 0 \tag{13}$$

and

$$f(\overline{A}, B, \underline{b}, \overline{c}, d, \overline{Q}) = -\frac{1}{2}(u^*)^T \overline{Q} u^* - \underline{b}^T v^* - d^T w^*. \tag{14}$$

Let $y = -\text{sgn } w^*$, then $y \in \{\pm 1\}^k$ and $|w^*| = -T_y w^*$. Then consider the quadratic program

$$\min \frac{1}{2} x^T \overline{Q} x + \overline{c}^T x \text{ subject to } \overline{A}x \leq \underline{b},\ B_{ye}x = d_y,\ x \geq 0 \tag{15}$$

and its Dorn dual problem

$$\max -\frac{1}{2} u^T \overline{Q} u - \underline{b}^T v - (d_y)^T w \text{ subject to } \overline{Q}u + \overline{A}^T v + (B_{ye})^T w + \overline{c} \geq 0, v \geq 0. \tag{16}$$

Also, form (13) we have

$$\begin{aligned} \overline{Q}u^* + \overline{A}^T v^* + (B_{ye})^T w^* + \overline{c} &= \overline{Q}u^* + \overline{A}^T v^* + (B_c - T_y B_\Delta)^T w^* + \overline{c} \\ &= \overline{Q}u^* + \overline{A}^T v^* + B_c^T w^* + B_\Delta |w^*| + \overline{c} \\ &\geq \overline{Q}u^* + \overline{A}^T v^* + (B_c + B - B_c)^T w^* + \overline{c} \\ &= \overline{Q}u^* + \overline{A}^T v^* + B^T w^* + \overline{c} \geq 0, \end{aligned}$$

thus, the quadratic program (16) is feasible. Now, if the primal program (15) is infeasible, then $f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) = \infty$, so $\overline{\varphi} = \infty$, thus (10) holds. If the program (15) is feasible, then from the feasibility of primal and dual program, together with the duality Theorem 2.1 and Theorem 2.2, the problem (16) has an optimal solution $(\hat{u}, \hat{v}, \hat{w})^T$ satisfying

$$f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) = -\frac{1}{2} \hat{u}^T \overline{Q} \hat{u} - \underline{b}^T \hat{v} - d_y^T \hat{w},$$

then we have

$$-\frac{1}{2}(u^*)^T \overline{Q} u^* - \underline{b}^T v^* - (d_y)^T w^* \leq -\frac{1}{2} \hat{u}^T \overline{Q} \hat{u} - \underline{b}^T \hat{v} - d_y^T \hat{w} \leq \overline{\varphi}$$

and from (14) we have

$$\begin{aligned} f(\overline{A}, B, \underline{b}, \overline{c}, d, \overline{Q}) &= -\frac{1}{2}(u^*)^T \overline{Q} u^* - \underline{b}^T v^* - d^T w^* \\ &= -\frac{1}{2}(u^*)^T \overline{Q} u^* - \underline{b}^T v^* - (d_c + d - d_c)^T w^* \\ &\leq -\frac{1}{2}(u^*)^T \overline{Q} u^* - \underline{b}^T v^* - (d_c^T w^* - d_\Delta^T |w^*|) \\ &= -\frac{1}{2}(u^*)^T \overline{Q} u^* - \underline{b}^T v^* - (d_c^T w^* + d_\Delta^T T_y w^*) \\ &= -\frac{1}{2}(u^*)^T \overline{Q} u^* - \underline{b}^T v^* - (d_y)^T w^* \leq \overline{\varphi}, \end{aligned}$$

so (10) holds. Clearly, we have proved that (10) holds for each $A \in \mathbf{A}, b \in \mathbf{b}$, which implies $\overline{f}(\overline{A}, \mathbf{B}, \underline{b}, \overline{c}, \mathbf{d}, \overline{Q}) \leq \overline{\varphi}$. Hence, by (i) and (ii), we can get $\overline{f}(\overline{A}, \mathbf{B}, \underline{b}, \overline{c}, \mathbf{d}, \overline{Q}) = \overline{\varphi}$, which completes the proof. $\square$

From Theorem 3.1 we are easy to obtain the following two corollaries.

Corollary 3.1. *Consider the IvQP*

$$\mathrm{Min}\{\frac{1}{2}x^T\mathbf{Q}x + \mathbf{c}^T x | \mathbf{A}x = \mathbf{b},\ x \geq 0\},$$

where Q is positive semifefinite for all $Q \in \mathbf{Q}$ and $\mathbf{b} \in \mathbb{R}^k$. We have

$$\begin{aligned}\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{Q}) &= \inf\{\frac{1}{2}x^T\underline{Q}x + \underline{c}^T x | \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0\},\\ \overline{f}(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{Q}) &= \sup_{y\in\{\pm1\}^k} f(A_{ye}, b_y, \overline{c}, \overline{Q}).\end{aligned}$$

Corollary 3.2. *Consider the IvQP*

$$\mathrm{Min}\{\frac{1}{2}x^T\mathbf{Q}x + \mathbf{c}^T x | \mathbf{A}x \leq \mathbf{b},\ x \geq 0\},$$

where Q is positive semifefinite for all $Q \in \mathbf{Q}$. We have

$$\begin{aligned}\underline{f}(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{Q}) &= \inf\{\frac{1}{2}x^T\underline{Q}x + \underline{c}^T x | \underline{A}x \leq \overline{b}, x \geq 0\},\\ \overline{f}(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{Q}) &= f(\overline{A}, \underline{b}, \overline{c}, \overline{Q}) = \inf\{\frac{1}{2}x^T\overline{Q}x + \overline{c}^T x | \overline{A}x \leq \underline{b}, x \geq 0\}.\end{aligned}$$

The special case of IvQP considered in Corollary 3.2 has been discussed in [20, 24]. The result of Corollary 3.2 is the same as those described in the [20, 24].

4 Relations between two formulas of the upper bound

We have presented the new formula of the upper bound in IvQP (3)

$$\overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q}) = \sup_{y\in\{\pm1\}^k} f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q})$$

in Theorem 3.1. Let

$$\overline{f}_1 = \sup_{y\in\{\pm1\}^k} f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q})$$

Consider the nonlinear program

$$\begin{aligned}&\max\ -\frac{1}{2}u^T\overline{Q}u - \underline{b}^T v - d_c^T w + d_\Delta^T|w|\\ &\text{subject to}\ \ \overline{Q}u + \overline{A}^T v + B_c^T w + B_\Delta^T|w| + \overline{c} \geq 0, v \geq 0.\end{aligned} \tag{17}$$

Let $\overline{f}_2$ be the optimal value of the problem (17). Hladík proved that $\overline{f}(\mathbf{A},\mathbf{B},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{Q}) = \overline{f}_2$, when the Dorn duality gap is zero [20]. Now, we prove that two formulas of upper bound are equivalent when the Dorn duality gap is zero.

For each certain y, the Dorn dual problem of (15) is

$$\begin{aligned}&\max\ -\frac{1}{2}u^T\overline{Q}u - \underline{b}^T v - (d_y)^T w\\ &\text{subject to}\ \ \overline{Q}u + \overline{A}^T v + (B_{ye})^T w + \overline{c} \geq 0, v \geq 0.\end{aligned} \tag{18}$$

First we give the following proposition, which will be used to obtain our result.

Proposition 4.1. *If $\overline{f}_2 = \infty$, then $\overline{f}_1 = \infty$.*

Proof: If $\overline{f}_2 = \infty$, that is (17) is unbounded, which means the feasible region of (17) is not empty. Let $(u_0, v_0, w_0)^T$ be an arbitrary feasible solution to problem (17), then it satisfies

$$\overline{Q}u_0 + \overline{A}^T v_0 + B_c^T w_0 + B_\Delta^T |w_0| + \overline{c} \geq 0, v_0 \geq 0. \tag{19}$$

Put $y_0 = -\text{sgn } w_0$, since $y_0 \in \{\pm 1\}^k$ and from (19) we can get

$$\begin{aligned} \overline{Q}u_0 + \overline{A}^T v_0 + (B_{y_0 e})^T w_0 + \overline{c} &= \overline{Q}u_0 + \overline{A}^T v_0 + B_c^T w_0 - B_\Delta^T T_{y_0} w_0 + \overline{c} \qquad (20) \\ &= \overline{Q}u_0 + \overline{A}^T v_0 + B_c^T w_0 + B_\Delta^T |w_0| + \overline{c} \geq 0, \end{aligned}$$

which means $(u_0, v_0, w_0)^T$ be a feasible solution to problem (18) corresponding to $y_0 = -\text{sgn } w_0 \in \{\pm 1\}^k$. Now we prove the proposition by contradiction.

① If $\overline{f}_1$ is finite, then there exists a real number r such that for each $y \in \{\pm 1\}^k$ there holds

$$f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) \leq r.$$

From Theorem 2.1, we can get for each $y \in \{\pm 1\}^k$

$$g(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) \leq f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) \leq r.$$

Thus, for any feasible solution $(u_0, v_0, w_0)^T$ to problem (17), its objective value satisfies

$$-\frac{1}{2}u_0^T \overline{Q} u_0 - \underline{b}^T v_0 - d_c^T w_0 + d_\Delta^T |w_0| = -\frac{1}{2}u_0^T \overline{Q} u_0 - \underline{b}^T v_0 - (d_{y_0})^T w_0 \leq g(\overline{A}, B_{y_0 e}, \underline{b}, \overline{c}, d_{y_0}, \overline{Q}) \leq r,$$

where $y_0 = -\text{sgn } w_0 \in \{\pm 1\}^k$.

So we have $\overline{f}_2 \leq r$, which is a contradiction.

② If $\overline{f}_1 = -\infty$, that is for each $y \in \{\pm 1\}^k$ we have

$$f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) = -\infty.$$

From Theorem 2.1, we get

$$g(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) \leq f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) = -\infty,$$

which means for each $y \in \{\pm 1\}^k$ the problem (18) is infeasible, which is a contradiction.

Thus, from the analysis of ① and ② we can get $\overline{f}_1 = \infty$. $\square$

Theorem 4.1. *If the Dorn duality gap of interval convex quadratic program (3) is zero, then $\overline{f}_1 = \overline{f}_2$.*

Proof: Obviously, from the Proposition 8 in Hladík [20] and Theorem 3.1 in section 3 we are easy to obtain $\overline{f}_1 = \overline{f}_2$. Here, we give a different proof, which does not use the results in [20] and Theorem 3.1.

(i) First we prove $\overline{f}_1 \leq \overline{f}_2$ by showing that

$$f_y = f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) \leq \overline{f}_2 \tag{21}$$

holds for each $y \in \{\pm 1\}^k$. Because the zero gap is guaranteed, then from the Dorn duality Theorem 2.2, we know

$$f_y = \varphi,$$

where φ is the optimal value of the problem (18). Moreover, for each $y \in \{\pm 1\}^k$ we have $d_y^T w \geq d_c^T w - d_\Delta^T |w|$ and $B_c^T w + B_\Delta^T |w| \geq (B_{ye})^T w$, then

$$\begin{aligned} -\frac{1}{2}u^T\overline{Q}u - \underline{b}^T v - d_y^T w &\leq -\frac{1}{2}u^T\overline{Q}u - \underline{b}^T v - (d_c^T w - d_\Delta^T |w|) \\ &= -\frac{1}{2}u^T\overline{Q}u - \underline{b}^T v - d_c^T w + d_\Delta^T |w|, \end{aligned}$$

which implies that

$$\max\ -\frac{1}{2}u^T\overline{Q}u - \underline{b}^T v - d_y^T w \leq \max\ -\frac{1}{2}u^T\overline{Q}u - \underline{b}^T v - d_c^T w + d_\Delta^T |w|. \tag{22}$$

And also, if

$$\overline{Q}u + \overline{A}^T v + (B_{ye})^T w + \overline{c} \geq 0, v \geq 0$$

we can get

$$\overline{Q}u + \overline{A}^T v + B_c^T w + B_\Delta^T |w| + \overline{c} \geq \overline{Q}u + \overline{A}^T v + (B_{ye})^T w + \overline{c} \geq 0, v \geq 0, \tag{23}$$

which means the feasible region of (17) contains those of (18). Thus, (22)and (23) implies $\varphi \leq \overline{f}_2$. Hence $f_y \leq \overline{f}_2$, for each $y \in \{\pm 1\}^k$. Thus, $\overline{f}_1 \leq \overline{f}_2$.

(ii) Second we prove $\overline{f}_2 \leq \overline{f}_1$.

① This is obvious if $\overline{f}_2 = -\infty$.

② If $\overline{f}_2 = \infty$, by Proposition 4.1, we are easy to obtain $\overline{f}_1 = \overline{f}_2$.

③ If $\overline{f}_2$ is finite, then the nonlinear program (17) has optimal solutions. Let $(u^*, v^*, w^*)^T$ be an optimal solution of the problem (17), then

$$\overline{f}_2 = -\frac{1}{2}(u^*)^T\overline{Q}u^* - \underline{b}^T v^* - d_c^T w^* + d_\Delta^T |w^*| \tag{24}$$

and

$$\overline{Q}u^* + \overline{A}^T v^* + B_c^T w^* + B_\Delta^T |w^*| + \overline{c} \geq 0, v^* \geq 0. \tag{25}$$

Let $y = -\text{sgn}\ w^*$, then $B_c^T w^* + B_\Delta^T |w^*| = (B_{ye})^T w^*$ and $d_y^T w^* = d_c^T w^* - d_\Delta^T |w^*|$, form (25) we have

$$\overline{Q}u^* + \overline{A}^T v^* + (B_{ye})^T w^* + \overline{c} \geq 0, v^* \geq 0,$$

which implies $(u^*, v^*, w^*)^T$ is an feasible solution to program (18). So we have

$$\begin{aligned} f_y &\geq -\frac{1}{2}(u^*)^T\overline{Q}u^* - \underline{b}^T v^* - d_y^T w^* \\ &= -\frac{1}{2}(u^*)^T\overline{Q}u^* - \underline{b}^T v^* - d_c^T w^* + d_\Delta^T |w^*| \\ &= \overline{f}_2, \end{aligned}$$

together with ① and ②, which proves $\overline{f}_2 \leq \overline{f}_1$.

Hence, from (i) and (ii) we obtain $\overline{f}_2 = \overline{f}_1$. $\square$

It is worth noting that the restrictive condition that the duality gap is zero, which is required to obtain $\overline{f}_2$, is removed in our formula $\overline{f}_1$. In most situations, it is not an easy task to verify whether this condition is satisfied when the data vary inside intervals.

5 Illustrative examples and Remarks

Note that the formula of $\overline{f}_2$ can equivalently be formulated as ([20])

$$\overline{f}_2 = \sup\ f_y \quad \text{subject to} \quad z \in \{\pm\}^k$$

where

$$f_y = \sup\{-\frac{1}{2}u^T\overline{Q}u - \underline{b}^T v - (d_c^T - d_\Delta^T T_y)w \,|\, \overline{Q}u + \overline{A}^T v + (B_c^T + B_\Delta^T T_y)w + \overline{c} \geq 0, v \geq 0, T_z w \geq 0\}.$$

Thus, both formulas of upper bound in [20] and this paper contain 2^k scenarios, but they cannot be transformed to each other by a simple one-one corresponding primal-dual relation.

Example 1 Consider the interval quadratic program

$$\begin{aligned}
\min\ [2,3]x_1^2 + 2x_2^2 - 2x_1x_2 + [-5,-3]x_1 + [1,2]x_2 \\
\text{subject to}\ \ [1,2]x_1 + x_2 \leq [2,4], \\
[2,3]x_1 + [-1,-0.5]x_2 \leq [3,4], \\
[4,5]x_1 + [-8,-7]x_2 = [1,1.5], \\
x_1, x_2 \geq 0.
\end{aligned}$$

This is the same example discussed in Hladík [20]. The corresponding interval matrices and vectors are

$$\mathbf{Q} = \begin{pmatrix} [4,6] & -2 \\ -2 & 4 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} [-5,-3] \\ [1,2] \end{pmatrix}, \mathbf{A} = \begin{pmatrix} [1,2] & 1 \\ [2,3] & [-1,-0.5] \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} [2,4] \\ [3,4] \end{pmatrix}, \mathbf{B} = \begin{pmatrix} [4,5] & [-8,-7] \end{pmatrix}, \mathbf{d} = \begin{pmatrix} [1,1.5] \end{pmatrix}.$$

We only discuss the solution procedure of the upper bound of the optimal value range, since the method for finding the lower bound is simple and it is the same in [20]. Note that the interval vector $\mathbf{d} \in \mathbb{IR}^1$ is one-dimensional and use Theorem 3.1, we determine the upper bound by computing the following two quadratic programs.

① Let $y = 1$, then

$$f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) = f(\overline{A}, \underline{B}, \underline{b}, \overline{c}, \overline{d}, \overline{Q})$$

so we have

$$\begin{aligned}
\min\ 3x_1^2 + 2x_2^2 - 2x_1x_2 - 3x_1 + 2x_2 \\
\text{subject to}\ \ 2x_1 + x_2 \leq 2, \\
3x_1 - 0.5x_2 \leq 3, \\
4x_1 - 8x_2 = 1.5, \\
x_1, x_2 \geq 0.
\end{aligned}$$

Solving this quadratic problem, the optimal value is $f(\overline{A}, \underline{B}, \underline{b}, \overline{c}, \overline{d}, \overline{Q}) \approx -0.7047$.

② Let $y = -1$, then

$$f(\overline{A}, B_{ye}, \underline{b}, \overline{c}, d_y, \overline{Q}) = f(\overline{A}, \overline{B}, \underline{b}, \overline{c}, \underline{d}, \overline{Q})$$

so we have

$$\begin{aligned} \min\ & 3x_1^2 + 2x_2^2 - 2x_1x_2 - 3x_1 + 2x_2 \\ \text{subject to}\ & 2x_1 + x_2 \leq 2, \\ & 3x_1 - 0.5x_2 \leq 3, \\ & 5x_1 - 7x_2 = 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

The optimal value of this problem is $f(\overline{A}, \overline{B}, \underline{b}, \overline{c}, \underline{d}, \overline{Q}) \approx -0.5217$. So the upper bound is

$$\overline{f}(\mathbf{A}, \mathbf{B}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{Q}) \approx max\{-0.7047, -0.5217\} = -0.5217.$$

Due to the duality gap is zero, then we compute the upper bound by the method proposed in [20] and obtain two corresponding optimal values $f_1 \approx -0.5217$ and $f_2 = -0.75$ (see example 9 in [20]), we also obtain

$$\overline{f}(\mathbf{A}, \mathbf{B}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{Q}) = max\{f_1, f_2\} \approx max\{-0.75, -0.5217\} = -0.5217.$$

Note that two sets $\{-0.75, -0.5217\}$ and $\{-0.7047, -0.5217\}$ are not the same. Hence, this example shows that although two different computing methods obtain the same upper bound, but the sets of optimal values of scenarios considered in two methods are not one to one corresponding.

Example 2 Consider the interval quadratic program

$$\begin{aligned} \min\ & 2x_1^2 + \frac{1}{2}x_2^2 - 2x_1x_2 + [-5, 1]x_1 + [-4, -3]x_2 \\ \text{subject to}\ & -2x_1 + [-2, 1]x_2 \leq [1, 4], \\ & [4, 5]x_1 - 2x_2 = 2, \\ & 6x_1 + [-4, -3]x_2 = 1.4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

The corresponding interval matrices and vectors are

$$\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} [-5, 1] \\ [-4, -3] \end{pmatrix}, \mathbf{A} = \begin{pmatrix} -2 & [-2, 1] \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} [1, 4] \end{pmatrix}, \mathbf{B} = \begin{pmatrix} [4, 5] & -2 \\ 6 & [-4, -3] \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 2 \\ 1.4 \end{pmatrix}.$$

The lower bound of the optimal value function can be determined by convex quadratic program

$$
\begin{aligned}
\min\ 2x_1^2 + \frac{1}{2}x_2^2 - 2x_1x_2 - 5x_1 - 4x_2 \\
\text{subject to}\quad -2x_1 - 2x_2 \le 4, \\
4x_1 - 2x_2 \le 2, \\
5x_1 - 2x_2 \ge 2, \\
6x_1 - 4x_2 \le 1.4, \\
6x_1 - 3x_2 \ge 1.4, \\
x_1, x_2 \ge 0.
\end{aligned}
$$

It can be shown that $\underline{f} = -\infty$ (unbounded).

We first compute the upper bound by Theorem 3.1. This problem can be decomposed into four convex quadratic programs

$$
\begin{aligned}
\min\ 2x_1^2 + \frac{1}{2}x_2^2 - 2x_1x_2 + x_1 - 3x_2 \\
\text{subject to}\quad -2x_1 + x_2 \le 1, \\
4x_1 - 2x_2 = 2, \\
6x_1 - 3x_2 = 1.4, \\
x_1, x_2 \ge 0,
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\min\ 2x_1^2 + \frac{1}{2}x_2^2 - 2x_1x_2 + x_1 - 3x_2 \\
\text{subject to}\quad -2x_1 + x_2 \le 1, \\
4x_1 - 2x_2 = 2, \\
6x_1 - 4x_2 = 1.4, \\
x_1, x_2 \ge 0.
\end{aligned}
$$

and

$$
\begin{aligned}
\min\ 2x_1^2 + \frac{1}{2}x_2^2 - 2x_1x_2 + x_1 - 3x_2 \\
\text{subject to}\quad -2x_1 + x_2 \le 1, \\
5x_1 - 2x_2 = 2, \\
6x_1 - 3x_2 = 1.4, \\
x_1, x_2 \ge 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\min\ 2x_1^2 + \frac{1}{2}x_2^2 - 2x_1x_2 + x_1 - 3x_2 \\
\text{subject to}\quad -2x_1 + x_2 \le 1, \\
5x_1 - 2x_2 = 2, \\
6x_1 - 4x_2 = 1.4, \\
x_1, x_2 \ge 0.
\end{aligned}
$$

We can get the program (26) is infeasible and hence the optimal value is ∞. The optimal values of the three remaining problems are $-3, \ -3.8244, \ -0.9972$, respectively. Thus, $\overline{f} = \infty$.

Now we compute this example by the method proposed in [20]. Consider four problems

$$\begin{aligned} f_1 = \sup \ & -2u_1^2 - \frac{1}{2}u_2^2 + 2u_1u_2 - v - 2w_1 - 1.4w_2 \\ \text{subject to } \ & 4u_1 - 2u_2 - 2v + 5w_1 + 6w_2 + 1 \geq 0, \\ & -2u_1 + u_2 + v - 2w_1 - 3w_2 - 3 \geq 0, \\ & v, w_1, w_2 \geq 0. \end{aligned}$$

and

$$\begin{aligned} f_2 = \sup \ & -2u_1^2 - \frac{1}{2}u_2^2 + 2u_1u_2 - v - 2w_1 - 1.4w_2 \\ \text{subject to } \ & 4u_1 - 2u_2 - 2v + 5w_1 + 6w_2 + 1 \geq 0, \\ & -2u_1 + u_2 + v - 2w_1 - 4w_2 - 3 \geq 0, \\ & v, w_1, -w_2 \geq 0. \end{aligned}$$

and

$$\begin{aligned} f_3 = \sup \ & -2u_1^2 - \frac{1}{2}u_2^2 + 2u_1u_2 - v - 2w_1 - 1.4w_2 \\ \text{subject to } \ & 4u_1 - 2u_2 - 2v + 4w_1 + 6w_2 + 1 \geq 0, \\ & -2u_1 + u_2 + v - 2w_1 - 3w_2 - 3 \geq 0, \\ & v, -w_1, w_2 \geq 0. \end{aligned}$$

and

$$\begin{aligned} f_4 = \sup \ & -2u_1^2 - \frac{1}{2}u_2^2 + 2u_1u_2 - v - 2w_1 - 1.4w_2 \\ \text{subject to } \ & 4u_1 - 2u_2 - 2v + 4w_1 + 6w_2 + 1 \geq 0, \\ & -2u_1 + u_2 + v - 2w_1 - 4w_2 - 3 \geq 0, \\ & v, -w_1, -w_2 \geq 0. \end{aligned}$$

It can be shown that four corresponding optimal values are $f_1 = -22.5$, $f_2 = 40.6848$, $f_3 = -\infty$ and $f_4 = 0.6650$, respectively. Thus we have $\overline{f}_2 = 40.6848$. Note that the condition of zero duality gap is not satisfied in this example, so we only know that $\overline{f} > \overline{f}_2$. $\square$

It is worth pointing out that, all the known methods for computing the upper bound, though important theoretically, are applicable only in a small dimension. This is not surprising in view of NP-hardness of the problem. Thus, an interesting subject worthy of further research is to derive suitable methods for upper bound approximations for interval quadratic program. An efficient method for finding the approximation for upper bound of the interval linear program has been developed in [17].

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