Accepted Manuscript

Stability approach to the fractional variational iteration method used for the dynamic analysis of viscoelastic beams

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PII: S0377-0427(18)30376-5
DOI: https://doi.org/10.1016/j.cam.2018.06.024
Reference: CAM 11754

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 4 January 2018
Revised date : 19 June 2018

Please cite this article as: O. Martin, Stability approach to the fractional variational iteration method used for the dynamic analysis of viscoelastic beams, *Journal of Computational and Applied Mathematics* (2018), https://doi.org/10.1016/j.cam.2018.06.024

Highlights (for review)

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- An algorithm based on the modified VIM and Laplace transformation techniques is presented.
- Approach of the variable change in fractional integral calculus.
- Stability study of a fractional nonlinear operator defined by VIM.
- Using the properties of convolution product in Banach spaces $L^p(\mathbf{R^n})$ and the fixed point theory are determined the stability intervals.
- Numerical examples show the influence of fractional derivative and the loading on the structure response.

Stability approach to the fractional variational iteration method used for the dynamic analysis of viscoelastic beams

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Abstract. Non-integer derivatives are frequently used to describe the constitutive behavior of viscoelastic materials. The dynamic analysis of a simply supported viscoelastic beam for a fractional Zener model is performed with the help of a modified variational iteration method. The structure is subjected to two loading scenarios: a uniformly distributed transverse load and a periodic concentrated force at the center of the beam.
Using the properties of convolution product in Banach spaces $L^p(\mathbf{R^n})$ and the fixed point theory, the stability temporal intervals of a nonlinear operator for a given fractional order v are determined. The graphical representations present in the numerical examples show how the existence of fractional derivative in the selected rheological model influences the dynamic response of the structure compared to the classical Zener model. The results of this study prove that the presented algorithm is an efficient tool for solving the fractional integro - differential equations.

Key words: viscoelastic beam, fractional calculus, fixed point theory, Galerkin method, variational iteration method, Laplace transform, stability of a nonlinear operator.

1. Introduction

In the design of the complex structures, viscoelastic components are frequently present due to their abilities to dampen out the vibrations. Based on the experimental data, since 1938 Gemant [1] has proposed the application of fractional calculus in the study of these mechanical structural systems exhibiting internal dissipation. Original results concerning the application of fractional derivative to the structural dynamics problems can be found in the works [2-17]. The following papers [18-26] have proved that the fractional integral - partial differential equations are valuable tools for modeling physical phenomena. During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations, such as: variational iteration method [27–28], homotopy perturbation method [29-30], Adomian's decomposition method [31] and finite element method [32-34].

The most important advantage of the fractional derivative approach in applications is their non-local property. It is well known that the integer order differential operator is a local operator, but the fractional order differential operator is a non-local operator. This means that the next state of the system depends not only on its current state, but also on all of its historical states. This is more realistic and it is one reason why the fractional calculus has become more and more popular.

In this paper, the governing equation for a simply supported viscoelastic beam under two loading scenarios: a uniformly distributed transverse load and a periodic concentrated force applied at the center of the beam is presented using Euler – Bernoulli theory. This equation is accompanied by a constitutive law defined in a hereditary integral form. In order to obtain the quasi-static exact solution (i.e. the solution ignoring inertia effects) the correspondence principle for the classical Zener model [35-37] will be used. This principle relates mathematically the solution of a linear, viscoelastic boundary value problem to an analogous problem of an elastic body of the same geometry and under the same initial boundary conditions. We mention that not all problems can be solved by this principle, but only those for which the boundary conditions do not vary with the time. Then, the governing equation is achieved using the fractional Zener model, [38-39], which is obtained by generalizing the results of classical Zener model. Written in terms of the transverse deflection, the integro-differential equation is solved with a mixed algorithm based on the Galerkin's method. For the spatial domain, the shape functions that satisfy the boundary conditions are presented and for the time domain will be used a modified variational iteration method, the changing of variables in fractional integral, Laplace transforms, Bessel functions and binomial series expansion.

The fixed point theory is often used in the nonlinear analysis, with an expansive evolution in the last decades. The basic result from the metrical fixed point theory is the Contraction Principle [44] that is used in many domains of the applied mathematics. The fractional nonlinear equation of the amplitude of oscillations was solved using a variational iterative method, accompanied by the techniques of the fixed point theory and the stability studies. The concept of stability is fundamental in the study of the differential equations, the difference equations and the dynamical systems, [45]-[47]. The results obtained by Harder and Hicks [45] have been applied by Saadati, Vaezpour and Rhoades, [46] to the integral equations and in this work for a fractional integro-differential equation, using a strict contractive type operator.

Studies of the quasi-static case, classical and fractional models are accompanied by the graphical representations of the solutions of the governing equation. It is also discussed the role of the fractional derivative order in modifying the properties that correspond to the classical models.

2 Problem formulation

Let us consider a simply supported isotropic beam of length L and transverse section $\bar{S}$ subjected to the loading $\bar{p}(t)$. In Euler's theory [16], the geometrical equation is given by

$$\varepsilon(x,t) = z\,\frac{\partial^2 w(x,t)}{\partial x^2}\,, \tag{1}$$

where x and z represent the axial and transverse coordinates, t is the time, $w(x,t)$ – the deflection and ε- the strain. A constitutive law in the hereditary integral form, [35], [37] is proposed:

$$\sigma(x,t) = G(0)\varepsilon(x,t) - \int_0^t \frac{dG(t-\tau)}{\mathrm{d}\tau}\,\varepsilon(x,\tau)\,d\tau\,, \tag{2}$$

where $G(t)$ is the relaxation modulus for the beam material and σ the stress corresponding to the strain ε.

According to d'Alembert's principle, the governing equation is of the following form

$$\frac{\partial^2 M(x,t)}{\partial x^2} = \bar{p}(t) - \rho\bar{S}\,\frac{\partial^2 w(x,t)}{\partial t^2} \tag{3}$$

where ρ is the density of the material. The bending moment $M\,(x,t)$ is written as

$$M(x,t)=\iint_{\bar{S}}\sigma(x,t)\,z\,dydz\,. \tag{4}$$

Using (2), the bending moment in a beam of the constant cross section is equal to

$$M(x,t)=G(0)\,I\,\frac{\partial^2 w(x,t)}{\partial x^2}-I\int_0^t\frac{dG(t-\tau)}{d\tau}\frac{\partial^2 w(x,\tau)}{\partial x^2}d\tau \tag{5}$$

where I is the area moment of inertia defined by $\iint_{\bar{S}} z^2\,dydz$.

Thus, after the substitution of (5) in (3), the following motion equation in terms of the transverse deflection is obtained, [39]:

$$\rho\bar{S}\,\frac{\partial^2 w(x,t)}{\partial t^2}+G(0)I\,\frac{\partial^4 w(x,t)}{\partial x^4}-I\int_0^t\frac{dG(t-\tau)}{d\tau}\frac{\partial^4 w(x,\tau)}{\partial x^4}d\tau=\bar{p}(t)$$

(6)

Solving the equation (6), w is found in the classical case for boundary conditions corresponding to a simply supported beam:

$$w(0,t)=w(L,t)=0\quad\text{and}\quad M(0,t)=M(L,t)=0 \tag{7}$$

and the initial conditions:

$$w(x,0)=0,\ \left.\frac{\partial w}{\partial x}\right|_{t=0}=0,\ x\in[0,L]$$

(8)

3. Rheological model

For the beginning it is chosen a classical Zener model, which consists of Hooke element in serial connection with a Kelvin-Voigt element (a spring and dashpot in parallel). The constitutive equation for any section x is of the following form, [39]:

$$\sigma(t)+\frac{\eta}{k_1+k_2}\dot{\sigma}(t)=\frac{k_1k_2}{k_1+k_2}\varepsilon(t)+\frac{k_1\eta}{k_1+k_2}\dot{\varepsilon}(t) \tag{9}$$

where k_1, k_2 are the elastic modulus of the springs and η is the coefficient of viscosity of the dashpot. Now consider that the material of the beam is in its relaxation phase. Hence, under the constant strain $\varepsilon=\varepsilon_0$, the stress will decrease and the solution of differential equation (9) for the condition: $\sigma(0)=k_1\,\varepsilon_0$, is

$$\sigma(t) = \frac{k_1 k_2}{k_1 + k_2}\left(1 + \frac{k_1}{k_2} e^{-\frac{t}{\tau_a}}\right)\varepsilon_0 = G(t)\,\varepsilon_0\,, \tag{10}$$

where
$$\tau_a = \frac{\eta}{k_1 + k_2} \tag{11}$$

is the relaxation time. Therefore, the relaxation modulus is defined as

$$G(t) = \frac{k_1 k_2}{k_1 + k_2}\left(1 + \frac{k_1}{k_2} e^{-\frac{t}{\tau_a}}\right). \tag{12}$$

Bagley and Torvik gave a physical justification for the concept of fractional derivatives in conjunction with viscoelasticity [3]. Using the fractional calculus, the classical viscoelastic model (9) will lead to the constitutive equation:

$$\sigma(t) + \frac{\eta}{k_1 + k_2}\frac{d^{\nu}\sigma(t)}{dt^{\nu}} = \frac{k_1 k_2}{k_1 + k_2}\,\varepsilon(t) + \frac{k_1 \eta}{k_1 + k_2}\frac{d^{\nu}\varepsilon(t)}{dt^{\nu}}\,, \tag{13}$$

where $\nu \in (0,1)$ is the order of fractional derivative (Riemann – Liouville definition, see e.g. [3]). The relaxation modulus (12) becomes of the form, [22]:

$$G(t) = \frac{k_1 k_2}{k_1 + k_2}\left(1 + \frac{k_1}{k_2} E_{\nu}\left(-\left(\frac{t}{\tau_a}\right)^{\nu}\right)\right), \tag{14}$$

where Mittag – Leffler function is defined by

$$E_{\nu}\left(-\left(\frac{t}{\tau_a}\right)^{\nu}\right) = \sum_{k=0}^{\infty}(-1)^k \frac{(t/\tau_a)^{\nu k}}{\Gamma(\nu k + 1)},\ \ 0 < \nu < 1,\ \tau_a > 0\,. \qquad +$$

For ν = 1 (classical Zener model) this function reduces to exp ($-t/\tau_a$). In the paper [39], the dependence of the relaxation modulus on the time t for various values of ν in the range $0 < \nu < 1$ is presented
The equation of motion (6) obtained under the assumption of Bernoulli - Euler theory, neglecting rotary inertia, shear deformation and using a fractional viscoelastic Zener material model of ν fractional order is now of the form:

$$\rho\bar{S}\,\frac{\partial^2 w(x,t)}{\partial t^2} + G(0) I\,\frac{\partial^4 w(x,t)}{\partial x^4} - I\int_0^t \frac{d^{\nu} G(t-\tau)}{d\tau^{\nu}}\frac{\partial^4 w(x,\tau)}{\partial x^4}\,d\tau = \bar{p}(t) \tag{15}$$

4. Galerkin analysis

To solve the equation (15) for a simply supported beam subjected to the loading $\bar{p}(t)$, the transverse deflection w is defined as

$$w_n(x,t) = \sum_{i=1}^{n} a_i(t)\varphi_i(x) \tag{16}$$

where $\varphi_i(x)$ is the i^{th} shape function and $a_i(t)$ is the corresponding time-dependent amplitude. The shape functions are chosen to be linearly independent, orthonormal, so

$$\int_0^L \varphi_i(x)\varphi_j(x)\,dx = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{17}$$

and must satisfy all boundary conditions for the convergence of Galerkin's method. Substituting the expansion (16) into (6), the following residual function will result

$$\bar{R}_n(x,t) = \rho\bar{S}\left(\sum_{i=1}^{n} a_i''(t)\varphi_i(x)\right) + G(0)I\left(\sum_{i=1}^{n} a_i(t)\varphi_i^{(4)}(x)\right) -$$

$$- I\int_0^t \frac{d^{\nu}G(t-\tau)}{d\tau^{\nu}}\left(\sum_{i=1}^{n} a_i(\tau)\varphi_i^{(4)}(x)\right)d\tau - \bar{p}(t) \tag{18}$$

The shape functions are chosen of the form

$$\varphi_i(x) = \sqrt{\frac{2}{L}}\sin\frac{i\pi x}{L},\ \ i = 1, 2, ..., \ n \tag{19}$$

for which

$$\varphi_i^{(4)}(x) = \left(\frac{i\pi}{L}\right)^4 \varphi_i(x) = \bar{\eta}_i\varphi_i(x) \tag{20}$$

The Galerkin's method requires that the residual to be orthogonal to each of the shape functions, so

$$\iint_{\Omega} \bar{R}_n(x,t)\varphi_j(x)\,dx\,dt = 0,\ j = 1, 2, \ldots, n \tag{21}$$

where $\overline{\Omega} = [0, L]\times[0, t]$.This leads to n equations verified by the functions $a_j(t)$:

$$\rho\bar{S}a_j''(t) - \bar{\eta}_j I\int_0^t \frac{d^{\nu}G(t-\tau)}{d\tau^{\nu}} a_j(\tau)\,d\tau + \bar{\eta}_j IG(0)a_j(t) = \bar{p}(t)\int_0^L \varphi_j(x)dx \tag{22}$$

or

$$a_j''(t) - \omega_j\int_0^t \frac{d^{\nu}G(t-\tau)}{d\tau^{\nu}} a_j(\tau)\,d\tau + \bar{\sigma}_j a_j(t) = P_j(t) \tag{23}$$

where

$$\bar{\eta}_j = \left(\frac{j\pi}{L}\right)^4,\ \omega_j = \frac{\bar{\eta}_j I}{\rho \bar{S}},\ \bar{\sigma}_j = \frac{\bar{\eta}_j I G(0)}{\rho \bar{S}},\ P_j(t) = \frac{\bar{p}(t)}{\rho \bar{S}} \cdot \frac{2\sqrt{2L}}{j\pi},\ j\text{ - odd number}.$$

The initial conditions lead to

$$\frac{d^k a_j}{dt^k}\Big|_{t=0} = \int_0^L \frac{\partial^k w_n(x,0)}{\partial t^k} \varphi_j(x)dx,\ k = 0,1 \tag{24}$$

Solving the problem (23) – (24), the functions $a_j(t)$ are determined independently from each other. Finally, the value of the transverse deflection $w(x,t)$ will be found by (16).

5. The variational iteration method (VIM)

To obtain the solution of the integral-differential equation (23) will rewrite it using the linear operator $\bar{L}$ and the nonlinear operator $\bar{N}$:

$$\bar{L}a_j(t) + \bar{N}a_j(t) - P_j(t) = 0 \tag{25}$$

where

$$\bar{L}a_j(t) = a_j''(t) + \bar{\sigma}_j a_j(t)$$

$$\bar{N}a_j(t) = -\omega_j \int_0^t \frac{d^\nu G(t-\tau)}{d\tau^\nu}\, a_j(\tau)d\tau$$

According to the variational iteration method (VIM) [27], the correction functional in t direction is defined as:

$$a_{j,m+1}(t) = a_{j,m}(t) + \int_0^t \lambda(\xi)(\bar{L}a_{j,m}(\xi) + \bar{N}\tilde{a}_{j,m}(\xi) - P_j(\xi))d\xi \tag{26}$$

To obtain the successive approximations $a_{j,m+1}$, we will determine the Lagrange multiplier λ, using the variational techniques. The function $\tilde{a}_{j,m}$ has a restricted variation in t direction, which means $\delta\tilde{a}_{j,m} = 0$. Remind that $\delta a_{j,m}(0) = 0, \delta a'_{j,m}(0) = 0.$ Therefore, it will get, via the integration by parts, as

$$\delta a_{j,m+1}(t) = \delta a_{j,m}(t) + \delta\int_0^t \lambda(\xi)\left[a''_{j,m}(\xi) - \omega_j \int_0^\xi \frac{d^\nu G(\xi-\tau)}{d\tau^\nu}\tilde{a}_{j,m}(\tau)d\tau + \bar{\sigma}_j a_{j,m}(\xi) - P_j(\xi)\right]d\xi =$$

$$= \left(1 - \lambda'(\xi)\Big|_{\xi=t}\right)\delta a_{j,m}(t) + \lambda(\xi)\Big|_{\xi=t}\delta a'_{j,m}(t) + \int_0^t \left[\left(\lambda''(\xi) + \bar{\sigma}_j\lambda(\xi)\right)\delta a_{j,m}(\xi)\right]d\xi = 0 \tag{27}$$

So, λ(ξ) satisfies the following conditions

$$\lambda(\xi)\Big|_{\xi=t}=0, \quad 1-\lambda'(\xi)\Big|_{\xi=t}=0, \quad \lambda''(\xi)+\bar{\sigma}_j\lambda(\xi)=0 \tag{28}$$

and the Lagrange multiplier will be identified as

$$\lambda(\xi)=\frac{1}{\sqrt{\bar{\sigma}_j}}\sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right). \tag{29}$$

The iteration formula (26) becomes

$$a_{j,m+1}(t)=a_{j,m}(t)+\frac{1}{\sqrt{\bar{\sigma}_j}}\int_0^t \sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)\left[a''_{j,m}(\xi)-\omega_j\int_0^{\xi}\frac{d^{\nu}G(\xi-\tau)}{d\tau^{\nu}}a_{j,m}(\tau)\,d\tau+\bar{\sigma}_j a_{j,m}(\xi)-P_j(\xi)\right]d\xi$$

$$m=0,\ 1,\ldots \tag{30}$$

with the initial conditions:

$$\frac{d^k a_{j,m}(t)}{dt^k}\Big|_{t=0}=0, \ \ m,k=0,1.$$

Applying the integration by parts twice for the first term of the integral and (28), we will get

$$\begin{aligned}a_{j,m+1}(t)&=a_{j,m}(t)+\frac{1}{\sqrt{\bar{\sigma}_j}}\sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)a'_{j,m}(\xi)\Big|_0^t-\cos\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)a_{j,m}(\xi)\Big|_0^t-\\&-\sqrt{\bar{\sigma}_j}\int_0^t a_{j,m}(\xi)\sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)d\xi-\frac{\omega_j}{\sqrt{\bar{\sigma}_j}}\int_0^t\sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)\left(\int_0^{\xi}\frac{d^{\nu}G(\xi-\tau)}{d\tau^{\nu}}a_{j,m}(\tau)\,d\tau\right)d\xi+\\&+\sqrt{\bar{\sigma}_j}\int_0^t a_{j,m}(\xi)\sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)d\xi-\frac{1}{\sqrt{\bar{\sigma}_j}}\int_0^t P_j(\xi)\sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)d\xi\,.\end{aligned}$$

Finally, for odd values of j, (30) becomes

$$a_{j,m+1}(t)=-\frac{\omega_j}{\sqrt{\bar{\sigma}_j}}\int_0^t\sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)\left(\int_0^{\xi}\frac{d^{\nu}G(\xi-\tau)}{d\tau^{\nu}}a_{j,m}(\tau)\,d\tau\right)d\xi-\frac{1}{\sqrt{\bar{\sigma}_j}}\int_0^t P_j(\xi)\sin\left(\sqrt{\bar{\sigma}_j}\,(\xi-t)\right)d\xi$$

.

$$\tag{31}$$

In the case of nonlinear problems, the iterative formula (31) defines a slow process, requiring a longer period of time, even if we use a modern computing technique. On the other hand, the sequence $\{a_{j,m}(t)\}_m$ defined by (31) converges to the exact solution of the equation (23), if the initial approximation $a_{j,0}(t)$ verifies the initial conditions. In this paper, it is proposed to remove these inconveniences by associating Laplace transform techniques to MIV.

Let us now consider that the sequence $\{a_{j,m}(t)\}_m$ converges to the solution $a_j(t)$ of the equation (23). Therefore, the iterative process (31) ceases when $a_{j,m}(t) = a_{j,m+1}(t)$ with an error less than a value $\tilde{\varepsilon}$ very small and the solution $a_j(t)$ will be equal to

$$a_j(t) = a_{j,m}(t) = a_{j,m+1}(t) \tag{32}$$

So, (31) will be of the form

$$a_j(t) = -\frac{\omega_j}{\sqrt{\overline{\sigma}_j}} \int_0^t \sin\left(\sqrt{\overline{\sigma}_j}(\xi - t)\right)\left(\int_0^\xi \frac{d^\nu G(\xi-\tau)}{d\tau^\nu} a_j(\tau)d\tau\right)d\xi - \frac{1}{\sqrt{\overline{\sigma}_j}}\int_0^t P_j(\xi)\sin\left(\sqrt{\overline{\sigma}_j}(\xi - t)\right)d\xi \tag{33}$$

Theorem 1. If the function $a_j : [0, T] \to \mathbf{R}$, $D = [0, T]$, $a_j \in C^2(D)$ satisfies the integral equation (33), then it is the solution of the integro - differential equation (23).

■ Let

$$q_j(\xi) = \int_0^\xi \frac{d^\nu G(\xi-\tau)}{d\tau^\nu} a_j(\tau)d\tau . \tag{34}$$

Then, the derivatives of the function $a_j(t)$ defined by (33)

$$a_j(t) = \frac{-\omega_j}{\sqrt{\overline{\sigma}_j}} \int_0^t \sin\left(\sqrt{\overline{\sigma}_j}(\xi - t)\right) q_j(\xi)d\xi - \frac{1}{\sqrt{\overline{\sigma}_j}}\int_0^t P_j(\xi)\sin(\sqrt{\overline{\sigma}_j}(\xi - t))\,d\xi \tag{35}$$

will be

$$a_j'(t) = \omega_j \int_0^t \cos(\sqrt{\overline{\sigma}_j}(\xi - t)) q_j(\xi)d\xi + \int_0^t P_j(\xi)\cos(\sqrt{\overline{\sigma}_j}(\xi - t))\,d\xi$$

$$a_j''(t) = \omega_j\sqrt{\overline{\sigma}_j}\int_0^t \sin(\sqrt{\overline{\sigma}_j}(\xi - t)) q_j(\xi)d\xi + \omega_j q_j(t) + \sqrt{\overline{\sigma}_j}\int_0^t P_j(\xi)\sin(\sqrt{\overline{\sigma}_j}(\xi - t))d\xi + P_j(t) .$$

Introducing this last derivative on the left side of the equation (23) yields that

$$\omega_j\sqrt{\overline{\sigma}_j}\int_0^t \sin(\sqrt{\overline{\sigma}_j}(\xi - t)) q_j(\xi)d\xi + \omega_j q_j(t) + \sqrt{\overline{\sigma}_j}\int_0^t P_j(\xi)\sin(\sqrt{\overline{\sigma}_j}(\xi - t))d\xi + P_j(t) - \omega_j q_j(t) +$$

$$+\bar{\sigma}_j\left[\frac{-\omega_j}{\sqrt{\bar{\sigma}_j}}\int_0^t \sin\left(\sqrt{\bar{\sigma}_j}(\xi-t)\right)q_j(\xi)d\xi-\frac{1}{\sqrt{\bar{\sigma}_j}}\int_0^t P_j(\xi)\sin(\sqrt{\bar{\sigma}_j}(\xi-t))d\xi\right]=P_j(t).$$

Thus, the theorem is proved. ■

8. Solving of the time-domain equations

Using the Laplace transform techniques and Convolution Theorem for the equation (35), will find that

$$A_j(s)=-\frac{\omega_j}{\sqrt{\bar{\sigma}_j}}\frac{-\sqrt{\bar{\sigma}_j}}{s^2+\bar{\sigma}_j}Q(s)+\frac{\tilde{P}_j(s)}{\sqrt{\bar{\sigma}_j}}\frac{\sqrt{\bar{\sigma}_j}}{s^2+\bar{\sigma}_j}, \tag{36}$$

where $A_j(s)$, $Q(s)$ and $\tilde{P}_j(s)$ are the Laplace transforms $(\xrightarrow{L})$ of $a_j(t)$, $q(t)$ and $P_j(t)$, respectively.

To obtain $Q(s)$, a change of variables: $\theta(\tau)=\xi-\tau$ is defined in the integral $q(\xi)$. The substitution of θ to (34) leads to

$$\frac{d^\nu G(\theta(\tau))}{d\tau^\nu}=\frac{d^\nu G(\theta)}{d\theta^\nu}\frac{d^\nu \theta}{d\tau^\nu}$$

Using the Riemann – Liouville fractional derivative operator, [3]

$$D_{RL}^\nu(\theta(\tau))=\frac{1}{\Gamma(1-\nu)}\frac{d}{d\tau}\int_0^\tau \frac{\theta(\tau-u)}{u^\nu}du,\ \nu\in(0,1),$$

where $\Gamma(x)$ is Gamma function [40], we obtain that

$$r(\tau)=\frac{d^\nu\theta(\tau)}{d\tau^\nu}=\frac{1}{\Gamma(1-\nu)}\frac{d}{d\tau}\int_0^\tau\frac{\xi+u-\tau}{u^\nu}du=\frac{1}{\Gamma(1-\nu)}\left(\int_0^\tau\frac{(-1)}{u^\nu}du+\frac{\xi}{\tau^\nu}\right)=\frac{1}{\Gamma(1-\nu)}\left(\frac{\xi}{\tau^\nu}-\frac{\tau}{(1-\nu)\tau^\nu}\right).$$

Replacing τ by θ, the fractional derivative above becomes

$$r(\theta)=\frac{\theta-\nu\xi}{(\xi-\theta)^\nu\,\Gamma(2-\nu)},\quad 0\le\theta<\xi\le t,\ \nu\in(0,1) \tag{37}$$

and

$$q(\xi)=\int_0^\xi\frac{d^\nu G(\theta)}{d\theta^\nu}a_j(\xi-\theta)r(\theta)d\theta=\int_0^\xi\Pi(\theta)r(\theta)d\theta. \tag{38}$$

To study the convergence of the improper integral (38), let $\Pi(\theta)r(\theta)$ be integrable on the interval $[0, \xi-\bar{\varepsilon}]$ and be unbounded in $(\xi-\bar{\varepsilon}, \xi)$, where $\bar{\varepsilon} = \xi(1-\nu)$ > 0. Then, we will find

$$\lim_{\varepsilon\to 0}\int_0^{\xi-\bar{\varepsilon}}\Pi(\theta)\,r(\theta)d\theta = \lim_{\nu\to 1}\int_0^{\xi-\xi(1-\nu)}\Pi(\theta)\,r(\theta)d\theta = \lim_{\nu\to 1}\int_0^{\xi\nu}\Pi(\theta)\,r(\theta)d\theta$$

According to the mean value theorem, [43], for the function $r(\theta)\le 0$ on the interval (0, $\nu\xi$]:

if $\Pi:[0,\,\nu\xi]\to\mathbf{R}$ is continuous and $r(\theta)$ is an integrable function that does not change sign on $(0,\,\nu\xi)$, then there exists $\tilde{\theta}$ in this interval such that

$$\lim_{\nu\to 1}\int_0^{\xi\nu}\Pi(\theta)\,r(\theta)d\theta = \lim_{\nu\to 1} r(\tilde{\theta})\int_0^{\xi\,\nu}\Pi(\theta)\,d\theta = \lim_{\nu\to 1}\frac{\tilde{\theta}-\nu\xi}{(\xi-\tilde{\theta})^{\nu}\,\Gamma(2-\nu)}\int_0^{\xi\,\nu}\Pi(\theta)\,d\theta = -\int_0^{\xi}\Pi(\theta)\,d\theta$$

Hence, the integral $q_j(\xi)$ can be rewritten in the form

$$q_j(\xi) = -\int_0^{\xi}\frac{d^{\nu}G(\theta)}{d\theta^{\nu}}a_j(\xi-\theta)\,d\theta, \tag{39}$$

To obtain the Laplace transform of $G(t)$ consider now the *Correspondence principle* that can be formally stated by the following equations, [22]:

$$\begin{array}{lllllll} t & \xrightarrow{\ L\ } & \dfrac{1}{s^2} & \Rightarrow & \dfrac{t^{\nu}}{\Gamma(1+\nu)} & \xrightarrow{\ L\ } & \dfrac{1}{s^{\nu+1}} \\ e^{-t/\tau} & \xrightarrow{\ L\ } & \dfrac{1}{s}\cdot\dfrac{\tau s}{1+\tau s} & \Rightarrow & E_{\nu}\left(-\left(\dfrac{t}{\tau}\right)^{\nu}\right) & \xrightarrow{\ L\ } & \dfrac{1}{s}\cdot\dfrac{\tau^{\nu}s^{\nu}}{1+\tau^{\nu}s^{\nu}} \end{array} \tag{40}$$

Recalling that

$$G(t) = \mu\left(1+\frac{k_1}{k_2}E_{\nu}(-(t/\tau_a)^{\nu})\right),\quad \mu = \frac{k_1k_2}{k_1+k_2} \tag{41}$$

$$\tau_2^{\nu} = \frac{\eta}{k_2},\quad \tau_a^{\nu} = \frac{\eta}{k_1+k_2},\quad \frac{k_1}{k_2} = \frac{\tau_2^{\nu}}{\tau_a^{\nu}} - 1, \tag{42}$$

it is obtained

$$G(t) = \mu\left(1+\left(\frac{\tau_2^{\nu}}{\tau_a^{\nu}}-1\right)E_{\nu}\left(-\left(\frac{t}{\tau_a}\right)^{\nu}\right)\right). \tag{43}$$

Noting with $\tilde{G}(s)$ - Laplace transform of $G(t)$ and using the formula:

$$\frac{d^{\nu} f(t)}{dt^{\nu}} \xrightarrow{L} s^{\nu} F(s) - \sum_{k=1}^{n} f^{(k-1)}(0) s^{\nu-k} ,$$

(44)

$F(s)$ being the Laplace transform of $f(t)$, we find that

$$L\left(\frac{d^{\nu} G(\theta)}{d\theta^{\nu}}\right) = -(s^{\nu} \tilde{G}(s) - s^{\nu-1} G(0)) =$$

$$= -\left(\mu \frac{s^{\nu}}{s}\left(1 + \left(\frac{\tau_2^{\nu}}{\tau_a^{\nu}} - 1\right)\frac{\tau_a^{\nu} s^{\nu}}{1+\tau_a^{\nu} s^{\nu}}\right) - s^{\nu-1} G(0)\right) = -s^{\nu-1}\left(\mu \frac{1+\tau_2^{\nu} s^{\nu}}{1+\tau_a^{\nu} s^{\nu}} - G(0)\right). \quad (45)$$

Inserting this result into (36) we find the following equation:

$$A_j(s) = -s^{\nu-1} \frac{\bar{\sigma}_j}{G(0)(s^2 + \bar{\sigma}_j)}\left(\mu \frac{1+\tau_2^{\nu} s^{\nu}}{1+\tau_a^{\nu} s^{\nu}} - G(0)\right) A_j(s) + \frac{\tilde{P}_j(s)}{s^2 + \bar{\sigma}_j}. \quad (46)$$

Solving (46), we will get

$$A_j(s) = \frac{\tilde{P}_j(s)}{(s^2 + \bar{\sigma}_j)\left(1 + \frac{\bar{\sigma}_j s^{\nu-1}}{s^2 + \bar{\sigma}_j}\left(\frac{\mu}{G(0)} \cdot \frac{1+\tau_2^{\nu} s^{\nu}}{1+\tau_a^{\nu} s^{\nu}} - 1\right)\right)} . \quad (47)$$

Once determined $A_j(s)$, this will be used to calculate the original $a_j(t)$ for the initial conditions (24). Finally, an approximate value of the transverse deflection $w(x, t)$ will be found by (16).

9. Numerical examples

9.1.

Consider a simply supported beam subjected to the uniformly distributed load $\bar{p}(t) = p_0$ = 16 N/m, which is applied as a creep load at t = 0 (the load is applied suddenly at t = 0 and then held constant). The length of the beam is L = 4 m, the width b = 0.08 m and the height h = 0.23 m. These input data lead to the moment of inertia of the rectangular section: $I = bh^3/12 = 8 \cdot 10^{-5}\,\mathrm{m}^4$. The material is taken to have the density of 1565 kg/m^3. Rheological model will be a fractional Zener model with the relaxation modulus expressed by (14), where, [14], [39]:

$$k_1 = G(0) = 9.8 \cdot 10^7 \text{ N/m}^2, \quad k_2 = 2.45 \cdot 10^7 \text{ N/m}^2, \quad \eta = 2.74 \cdot 10^8 \text{ N-sec/ m}^2$$

$$\overline{\eta}_j = \left(\frac{j\pi}{L}\right)^4 = 0.38 j^4, \; \omega_j = \frac{\overline{\eta}_j I}{\rho \overline{S}} = \frac{0.38 \cdot j^4 \cdot 8 \cdot 10^{-5}}{1565 \cdot 0.0184} = 0.106 \cdot 10^{-5} j^4 \tag{48}$$

$$\overline{\sigma}_j = \omega_j G(0) = 0.106 \cdot 10^{-5} j^4 \cdot 9.8 \cdot 10^7 = 104 j^4 \;\;, \;\; P_j(t) = P_{j0} = \frac{p_0}{\rho \overline{S}} \frac{2\sqrt{2L}}{j\pi} = \frac{0.0625\, p_0}{j} = \frac{1}{j}.$$

Quasi - static analysis

For $\nu = 1$, the relaxation modulus (12) becomes:

$$G(t) = 1.96 \cdot 10^7 + 7.84 \cdot 10^7 e^{-t/2.24} \;\; \text{N/m}^2 \tag{49}$$

with t in seconds. Then, the corresponding creep compliance $D(t)$ is of the form, [35]:

$$D(t) = 0.51 \cdot 10^{-7} - 0.408 \cdot 10^{-7} e^{-\frac{t}{11.2}} \;. \tag{50}$$

In the quasi-static case, an exact solution can be computed using *Correspondence principle*, [39]. In view of this, the transverse displacements are calculated by the formula

$$w(x,t) = \overline{w}(x) D(t), \tag{51}$$

where $D(t)$ is defined in (50) and $\overline{w}$ is the solution for a similar elastic structure that has the elasticity modulus equal to 1, [16], [37]:

$$\overline{w}(x) = \frac{p_0 x (x^3 - 2l x^2 + l^3)}{24 I}.$$

(52)

Fractional dynamic model

Using the definitions (41) – (42):

$$\mu = 1.96 \cdot 10^7, \; \tau_2^\nu = 11.2, \quad \tau_a^\nu = 2.24, \quad \frac{k_1}{k_2} = \frac{\tau_2^\nu}{\tau_a^\nu} - 1, \tag{53}$$

the formula (47) leads to the values:

$$A_j(s) = \frac{\tilde{P}_{j0}}{(s^2 + \overline{\sigma}_j)\left(1 + \frac{\overline{\sigma}_j s^{\nu-1}}{s^2 + \overline{\sigma}_j}\left(\frac{\mu}{G(0)} \cdot \frac{1 + \tau_2^\nu s^\nu}{1 + \tau_a^\nu s^\nu} - 1\right)\right)} =$$

$$= \frac{1}{j} \frac{1}{s(s^2 + \overline{\sigma}_j)\left(1 + \frac{104 j^4 s^{\nu - 1}}{s^2 + 104 j^4}\left(\frac{1.96 \cdot 10^7}{9.8 \cdot 10^7} \cdot \frac{1 + 11.2 s^\nu}{1 + 2.24 s^\nu} - 1\right)\right)} =$$

$$= \frac{1}{j\,s}\frac{(1+2.24\,s^{\nu})\cdot 0.45 s^{1-\nu}}{(2.24 s^{\nu+2}+2.24\cdot 104\,j^4 s^{\nu}+s^2+104\,j^4-83\,j^4 s^{\nu-1})\cdot 0.45 s^{1-\nu}} = \tag{54}$$

$$= \frac{1}{j\,s}\frac{(1+0.45\,s^{-\nu})\,s}{s^3+104\,j^4 s+0.45 s^{3-\nu}+104\,j^4\cdot 0.45 s^{1-\nu}-83\cdot 0.45\,j^4} = \frac{1}{j}C_j(s)\,,$$

where $\widetilde{P}_{j0}$ is the Laplace transform of P_{j0}.
Using the notations: β = 0.45, $b = 104\,j^4$ and Maclaurin expansion

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x|<1, \tag{55}$$

the function $C_j(s)$ can be written as

$$C_j(s) = \frac{1+\beta s^{-\nu}}{s^3+104\,s\,j^4-83\beta\,j^4+\beta s^{3-\nu}+104\beta\,j^4 s^{1-\nu}} =$$

$$= \frac{1+\beta s^{-\nu}}{s(s^2+b)(1+\beta s^{-\nu})-83\beta\,j^4} = \frac{1}{s(s^2+b)}\sum_{i=0}^{\infty}\frac{(83\beta\,j^4)^i}{s^i(s^2+b)^i(1+\beta s^{-\nu})^i} = \tag{56}$$

$$= \sum_{i=0}^{\infty}\frac{(83\beta\,j^4)^i}{s^{i+1}(s^2+b)^{i+1}(1+\beta s^{-\nu})^i} = \frac{1}{s(s^2+b)} + \sum_{i=1}^{\infty} F_i(s)U_i(s),$$

where $F_i(s)$ and $U_i(s)$ are the Laplace transforms of the functions $f_i(t)$ and $u_i(t)$ that will be determined below. The above series is convergent if

$$\left|\frac{83\cdot 0.45\cdot j^4)}{s(s^2+104\,j^4)(1+0.45 s^{-\nu})}\right| < 1, \tag{57}$$

where variable s of the Laplace transform $A_j(s)$ is a complex number $s = \alpha + i\,\omega$, $\alpha, \omega \in \mathbf{R}$.
For $\alpha > \beta$, β = 0.45 and $|\omega| < \alpha$, the condition (57) is met

$$\frac{83\cdot 0.45\cdot j^4\cdot |s^{\nu}|}{|s|\,|s^2+104\,j^4|\,|s^{\nu}+0.45|} < \frac{83\cdot 0.45\cdot j^4\cdot |s^{\nu}|}{0.45\cdot 104\cdot j^4\cdot |s^{\nu}|} = \frac{83}{104} < 1\,. \tag{58}$$

To find the original function of $A_j(s)$, we will use the Convolution theorem for i = 1, 2,…

$$a_j(t) = \frac{1}{j}\left[I_0(t) + \sum_{i=1}^{\infty} \int_0^t f_i(\tau) u_i(t-\tau)\, d\tau \right], \tag{59}$$

where

$$\frac{1}{s^2(s^2+b)} \xrightarrow{L^{-1}} I_0(t) = \frac{1}{b}(1-\cos(\sqrt{b}t)\,. \tag{60}$$

Now we choose

$$F_i(s) = \frac{(83\beta j^4)^i}{(s^2+b)^{i+1}}\,. \tag{61}$$

According to the Laplace transform theory, there is the following formula, [42]:

$$\frac{1}{(s^2+a^2)^k} \xrightarrow{L^{-1}} \frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-1/2} J_{k-1/2}(at), \tag{62}$$

where $J_\alpha(z)$ is the Bessel function of first kind, $\alpha \in \mathbf{R}$. So, the corresponding original function is

$$f_i(t) = \frac{(83\beta j)^i \sqrt{\pi}}{\Gamma(i+1)}\left(\frac{t}{2\sqrt{b}}\right)^{i+1/2} J_{i+1/2}(\sqrt{b}t) \tag{63}$$

Next, choosing

$$U_i(s) = \frac{s^{-i-1}}{(1+\beta s^{-\nu})^i},\ i = 1,2,\ldots \tag{64}$$

will be used the Laplace transform formula for Mittag – Leffler function, $E^{\lambda}_{\mu,\gamma}(z)$, which was given by Tomovski et al., [28]:

$$L\left[x^{\gamma-1}E^{\lambda}_{\mu,\gamma}(\omega x^{\mu})\right] = \frac{s^{\lambda\mu-\gamma}}{(s^{\mu}-\omega)^{\lambda}},\quad \left|\frac{\omega}{s^{\mu}}\right| < 1,\ \lambda,\mu,\omega \in \mathbf{C}. \tag{65}$$

Therefore, we will get

$$U_i(s) = \frac{s^{\nu i-i-1}}{(s^{\nu}+\beta)^i} \xrightarrow{L^{-1}} u_i(t) = t^i E^i_{\nu,i+1}(-\beta t^{\nu}) = t^i \sum_{m=0}^{\infty} \frac{\Gamma(i+m)(-1)^m \beta^m t^{\nu m}}{\Gamma(i)\Gamma(m\nu+i+1)m!}\,. \tag{66}$$

Knowing $f_i(t)$ and $u_i(t)$ will be determined $a_j(t)$ by (59). Then, using (16) for $n = 3$ will get the deflection w of beam at point $x = L/2$ at time t.

Therefore, for the same numerical example, the form of the Laplace transform $A_i(s)$ obtained with the fractional VIM coincides with that obtained in [39] with VIM, in which was determined the convergence interval: $[0, t_c] = [0, 5.5]$ seconds.

Classical dynamic analysis

If in (54) it is considered that $\nu = 1$, then $A_j(s)$ will be of the form

$$A_j(s) = \frac{1}{j} \cdot \frac{s + 0.45}{s(s^3 + 0.45s^2 + 104j^4 s + 9.43j^4)}\ . \tag{67}$$

It should be clear from (67) that as j increases, the importance of the j-th shape will decrease. According to the results presented in the paper [35], if a total of five terms: A_1, A_3, …, A_9 is used for Galerkin - Laplace transform analysis, then the solution (16) can be considered exact.

In Figure 1 are plotted: uex(t) - the transverse displacements in the quasi-static analysis, the exact deflection curves wex (x,t) obtained in the classical case with the method presented in [35] and with the above algorithm, WL100(t_i). So, $\nu = 1$, $x = 2$m, $t_i = 5.5i/100$, $i = 1, 2, ..., 200$, $t \in [0, 5.5]$ seconds

Figure 2 shows the numerical solutions $w(x,t)$ found by the algorithm above for various values of ν, which are denoted by WL30 ($\nu = 0.3$), WL50 ($\nu = 0.5$), WL70 ($\nu = 0.7$), WL100 ($\nu = 1$) in the midpoint of the beam, $x = 2$ m and $0 < t \leq 5.5$ seconds with $t_i = 5.5i/100$, $i = 1, 2, ..., 100$. The graphical representations highlight the convergence of the fractional solutions to the classical solution, $\nu = 1$.

It can be seen that these curves oscillate around of the exact quasi-static solution, uex and the amplitude of oscillations decrease with the increasing of the time due to the presence of the viscoelastic damping.

Notice that the dynamic response of the beam for the fractional mechanical model to the corresponding curve of the classical model ($\nu = 1$) as the time tends to zero.

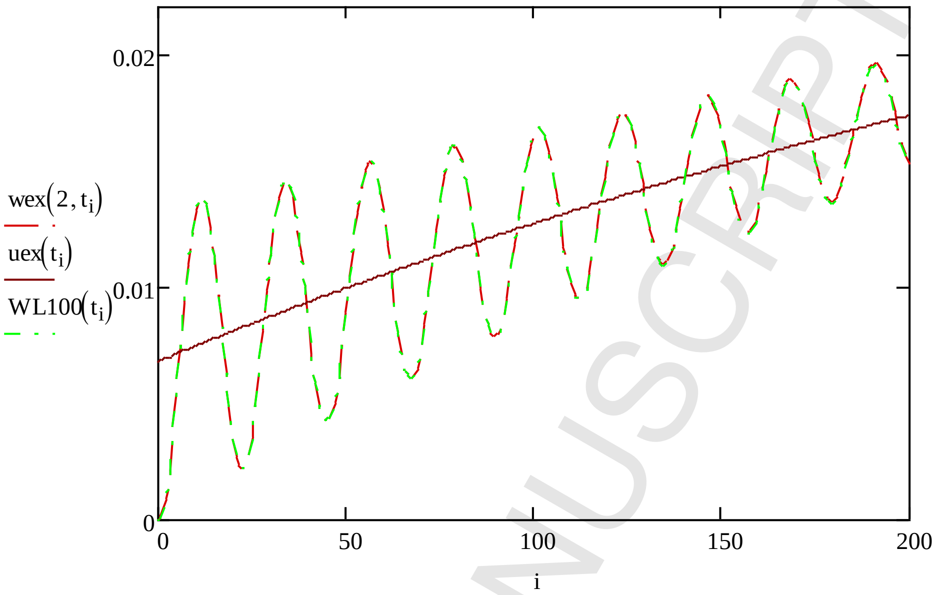

Fig. 1. uex(t) - the transverse displacements in the quasi-static analysis, the exact deflection curves *w* obtained with the method presented in [35], denoted by wex(2, t_i) and with the above algorithm, WL100(t_i)

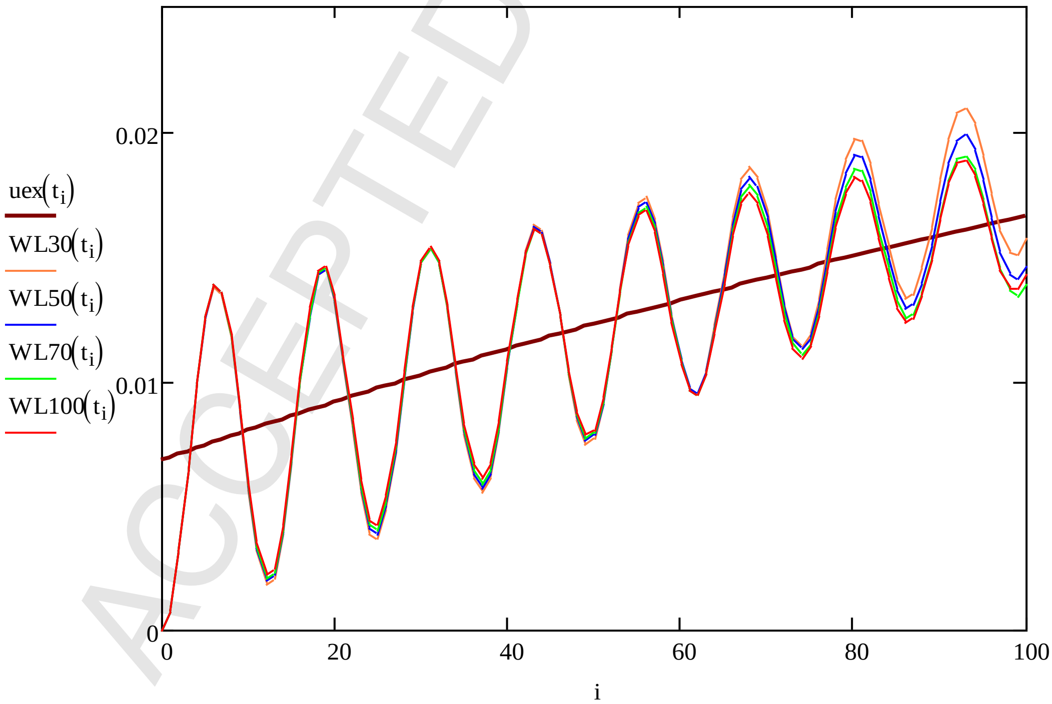

Fig. 2. Numerical solutions $w_n(x,t)$ for various values of ν, which are denoted by WL30 (ν = 0.3), WL50 (ν = 0.5), WL70 (ν = 0.7), WL100 (ν = 1) at the midpoint of beam, x = 2 m and $0 < t \leq 5.5$ seconds with $t_i = 5.5i/100$, i = 1, 2, ..., 100.

9. 2.

In the second loading case, the simply supported beam is subjected to a periodic concentrated force $\overline{p}(t) = p_0 \sin(\pi t) = 16 \sin(\pi t)$ N applied at the center of the beam. Using the same input data (48) is defined

$$P_j(t) = P_{j0} \sin(\pi t) = \frac{1}{j} \sin(\pi t) \quad \Rightarrow \quad \tilde{P}_j(s) = \frac{1}{j} \cdot \frac{\pi}{s^2 + \pi^2}. \tag{73}$$

According to the formula (54), the Laplace transform $A_j(s)$ of $a_j(t)$ becomes

$$A_j(s) = \frac{\tilde{P}_{j0}}{(s^2 + \overline{\sigma}_j)\left(1 + \dfrac{\overline{\sigma}_j s^{\nu-1}}{s^2 + \overline{\sigma}_j}\left(\dfrac{\mu}{G(0)} \cdot \dfrac{1 + \tau_2^\nu s^\nu}{1 + \tau_a^\nu s^\nu} - 1\right)\right)} =$$

$$= \frac{1}{j} \frac{1}{s(s^2 + 104\, j^4)\left(1 + \dfrac{104\, j^4 s^{\nu-1}}{s^2 + 104\, j^4}\left(\dfrac{1.96 \cdot 10^7}{9.8 \cdot 10^7} \cdot \dfrac{1 + 11.2\, s^\nu}{1 + 2.24\, s^\nu} - 1\right)\right)} \cdot \frac{s}{s^2 + \pi^2} \cdot \pi =$$

$$= A_{j0}(s) \cdot \frac{s}{s^2 + \pi^2} \cdot \pi \,, \tag{74}$$

where the values $a_j(t)$ and $A_j(s)$ obtained in the first example are noted with $a_{j0}(t)$ and $A_{j0}(s)$, respectively. Therefore, the Convolution theorem leads to

$$A_j(s) = A_{j0}(s) \cdot L(\cos(\pi t) \cdot \pi \quad \Rightarrow \quad a_j(t) = \pi \int_0^t \cos(\pi(t-\tau))\, a_{j0}(\tau) d\tau \tag{75}$$

and the transverse displacements at the center of the beam are obtained of the form

$$w_5(2,t) = \sum_{k=0}^{2} a_{2k+1}(t)\, \varphi_{2k+1}(x) \tag{76}$$

with an approximation of 10^{-7}.

In Figure 3 are plotted: the exact deflection curves wexs(t) obtained in the classical case with the method presented in [35] and with the above algorithm, WLS100(t). So, ν = 1, x = 2m, $\tau_i = 5.5i/200$, i = 1, 2, ..., 200, $t \in [0, 5.5]$ seconds.

Figure 4 is a graphical representation of the solutions $w(x,t)$ found by the algorithm above for various values of v, which are denoted by WLS30 (ν = 0.3), WLS70 (ν = 0.7) and WLS100 (ν = 1) in the midpoint of the beam, x = 2 m and $0 < t \leq 5.5$ seconds with $t_i = 5.5i/100$, i = 1, 2, ..., 100.

Figure 5 shows the exact classical solution wexs(t) with the method presented in [35] and the Convolution theorem for this loading case, in the temporal interval [0, 20] seconds. It can be notice that the oscillating process becomes uniformly after the first 8 seconds.

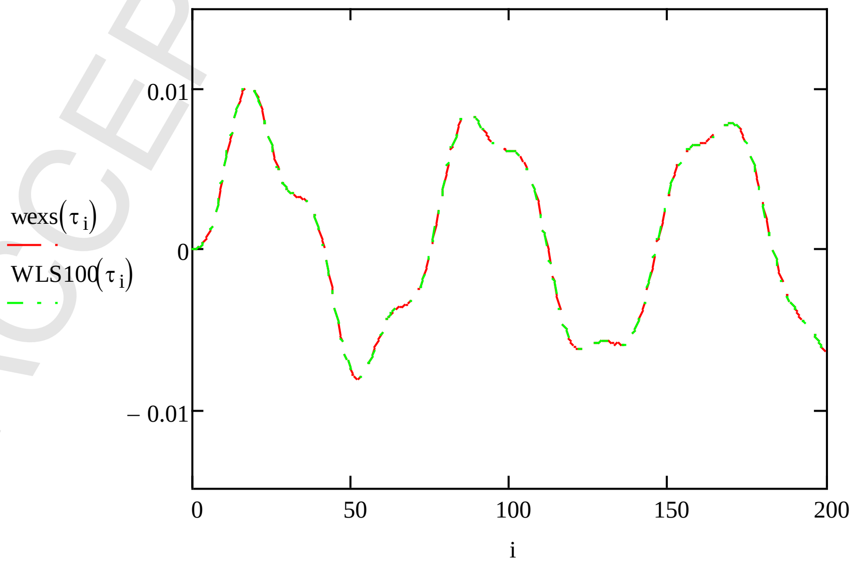

Fig. 3. The exact deflection wexs(t_i) obtained with the method presented in [35] and WLS100(t_i) found with the above algorithm ($\nu = 1$), at the midpoint of beam ($x = 2$ m)for $0 < t \leq 5.5$ seconds with $t_i = 5.5i/200$, $i = 1, 2, ..., 200$.

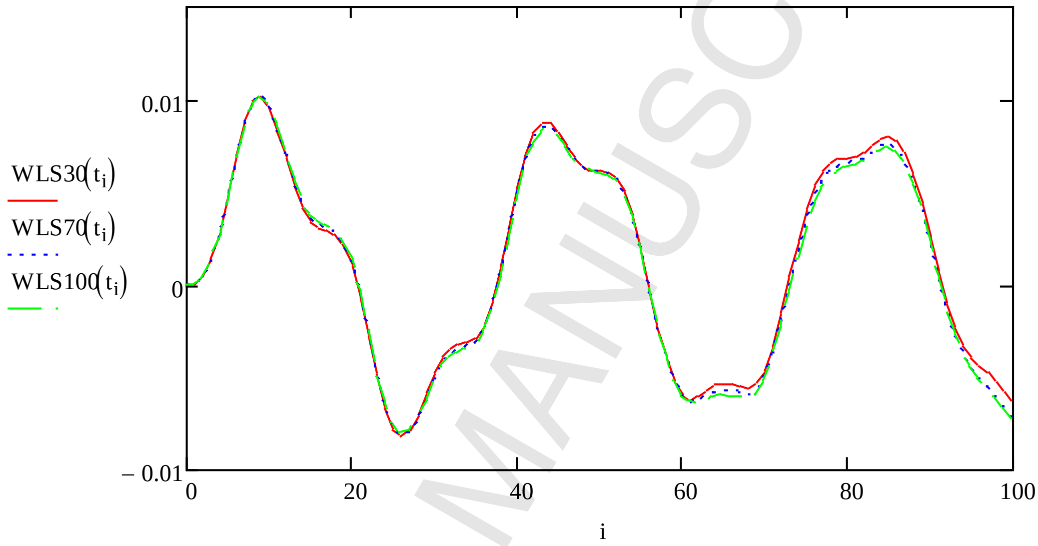

Fig. 4. Numerical solutions $w(x,t)$ for various values of ν, which are denoted by WL30 ($\nu = 0.3$), WL70 ($\nu = 0.7$), WL100 ($\nu = 1$) at the midpoint of beam, $x = 2$ m and $0 < t \leq 5.5$ seconds with $t_i = 5.5i/100$, $i = 1, 2, ..., 100$.

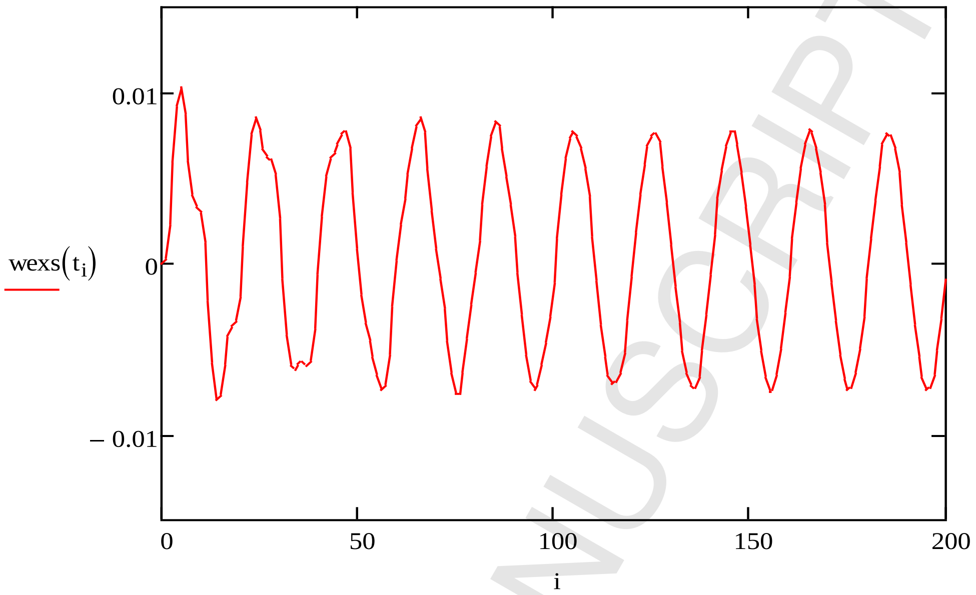

Fig. 5. The exact classical transverse displacements wexs(t_i) at the midpoint of beam, x = 2 m
and $0 < t \leq 20$ seconds with $t_i = 20i/200$, i = 1, 2, ..., 200

10. Study of VIM stability

The form (54) of $A_j(s)$, obtained in this paper with the fractional variational iterative method, coincides with that obtained in the paper [39], in which was determined the time interval $X = [0, t_c] = [0, 5.5]$ seconds, corresponding to the convergence conditions (58), (65).
It will further approach the stability to VIM in this convergence interval.

Let $X = L^1([0, t_c])$ be the Banach space of the continuous, real-valued functions on $[0, t_c]$ with the norm

$$\left\| a_j(t) \right\|_1 = \int_0^t \left| a_j(\tau) \right| d\tau \tag{77}$$

and T a self-map of X, such that

$$a_{j,m+1}(t) = T(a_{j,m}(t)), \tag{78}$$

where, in accordance with the relationship (31), is defined

$$T(a_{j,m}(t)) = \frac{\omega_j}{\sqrt{\sigma_j}} \int_0^t \sin\left(\sqrt{\sigma_j}(t-\xi)\right) q_{j,m}(\xi)\, d\xi - \frac{1}{\sqrt{\sigma_j}} \int_0^t P_j(\xi) \sin\left(\sqrt{\sigma_j}(\xi - t)\right) d\xi,$$

(79)

$$q_{j,m}(\xi)=\int_0^{\xi}\frac{d^{\nu}G(\xi-\tau)}{d\tau^{\nu}}a_{j,m}(\tau)\,d\tau\,.$$

Let $S1(t)=\sin(\sqrt{\overline{\sigma}_j}\,t)$. Using the definition of convolution product that turns the Banach space $L^1(\mathbf{R}^n)$ into a commutative and associative algebra, (79) can be written as

$$T(a_{j,m}(t))=\frac{\omega_j}{\sqrt{\overline{\sigma}_j}}\left(\left(S1*\frac{d^{\nu}G}{dt^{\nu}}\right)*a_{j,m}\right)(t)-\frac{1}{\sqrt{\overline{\sigma}_j}}\int_0^t P_j(\xi)\sin\left(\sqrt{\overline{\sigma}_j}\,(\xi-t)\right)d\xi\,.$$

(80)

It was proved in Theorem 1 that, if there is a fixed point for *T*, then this is the solution of the integro-differential equation (23). Intuitively, a fixed point iteration procedure is numerically stable if small modifications in the initial data or in the data that are involved in the computation process will produce a small influence on the computed value of the fixed point. Using the strict contractive type operators, Saadati, Vaezpour and Rhoades, [46], we have obtained important results on the stability of the nonlinear operators associated with the integral equations, continuing thus, the studies of Harder and Hicks, [45].

It will show that *T* is stable, according to the following theorem, [46]

Theorem 2. Let $(X,\ \|\cdot\|)$ be a Banach space and *T* a self-map of *X* satisfying

$$\|Tx-Ty\|\le L\|x-Tx\|+\alpha\|x-y\| \tag{81}$$

for all $x, y \in X$, where the linear operator $L \ge 0$, $0 \le \alpha < 1$. If *T* has a fixed point, then *T* is stable.

In view of (78) – (79), the linear operator $L = 0$. For the beginning, it will demonstrate that the nonlinear mapping *T* has a fixed point.

With the help of Schwarz inequality and the translation invariance of measure in $L^2([0,\ t_c])$, the following inequalities will be obtained:

$$\begin{aligned}
&\left\|a_{j,m+1}(t)-a_{j,n+1}(t)\right\|_1=\left\|T(a_{j,m}(t))-a_{j,n}(t)\right\|_1\le\\
&\le\frac{\omega_j}{\sqrt{\overline{\sigma}_j}}\left\|\left(\left(S1*\frac{d^{\nu}G}{dt^{\nu}}\right)*(a_{j,m}-a_{j,n})\right)(t)\right\|_1=\\
&=\frac{\omega_j}{\sqrt{\overline{\sigma}_j}}\left\|\left(\hat{S}*(a_{j,m}-a_{j,n})\right)(t))\right\|_1\le\frac{\omega_j}{\sqrt{\overline{\sigma}_j}}\int_0^{t_C}\int_0^{t}\left|\hat{S}(t-\tau)\right|\left|a_{j,m}(\tau)-a_{j,n}(\tau)\right|d\tau dt\le\\
&\le\frac{\omega_j}{\sqrt{\overline{\sigma}_j}}\left\|tr_{\tau}\hat{S}(t)\right\|_2\left\|a_{j,m}(t)-a_{j,n}(t)\right\|_2\le\frac{\omega_j}{\sqrt{\overline{\sigma}_j}}\left\|\hat{S}(t)\right\|_2\left\|a_{j,m}(t)-a_{j,n}(t)\right\|_1,
\end{aligned} \tag{82}$$

where $\|\psi\|_{L^q} \le \|\psi\|_{L^p}$, if $p \le q$ and $tr_\tau \hat{S}(t) = \hat{S}(t-\tau), \left\| tr_\tau \hat{S}(t) \right\|_p = \left\| \hat{S}(t) \right\|_p$, [44].

Further, if $\sin(\sqrt{\bar{\sigma}_j}\, t) \in L^2([0, t_c])$ and $\dfrac{d^\nu G(t)}{dt^\nu} \in L^1([0,\ t_c])$, then, [44],

$$\left\| \hat{S}(t) \right\|_2 = \left\| \left(S1 * \frac{d^\nu G}{dt^\nu} \right)(t) \right\|_2 \le \left\| \sin\left(\sqrt{\bar{\sigma}_j}\, t\right) \right\|_2 \left\| \frac{d^\nu G(t)}{dt^\nu} \right\|_1 \tag{83}$$

and (82) becomes

$$\begin{aligned} \left\| a_{j,m+1}(t) - a_{j,n+1}(t) \right\|_1 &\le \left\| \left(S1 * \frac{d^\nu G}{dt^\nu} \right)(t) \right\|_2 \left\| a_{j,m}(t) - a_{j,n}(t) \right\|_1 \le \\ &\le \frac{\omega_j}{\bar{\sigma}_j} \left\| \sqrt{\bar{\sigma}_j}\, \sin\left(\sqrt{\bar{\sigma}_j}\, t\right) \right\|_2 \left\| \frac{d^\nu G(t)}{dt^\nu} \right\|_1 \left\| a_{j,m}(t) - a_{j,n}(t) \right\|_1 = \\ &= \frac{1}{k_1} \cdot N1 \cdot k_1 \cdot N2 \cdot \left\| a_{j,m}(t) - a_{j,n}(t) \right\|_1 = \hat{K} \left\| a_{j,m}(t) - a_{j,n}(t) \right\|_1 . \end{aligned} \tag{84}$$

The approach of stability to VIM will be done for two time intervals: [0, 1] and then, [1, t_c].
First, we will note $\Omega = \sqrt{\bar{\sigma}_j}$ and find the values of the first two norms:

$$N1 = \left\| \sqrt{\bar{\sigma}_j}\, \sin\left(\sqrt{\bar{\sigma}_j}\, t\right) \right\|_2$$

$$k_1 N2 = \left\| \frac{d^\nu G(t)}{dt^\nu} \right\|_1$$

on an interval [$t0$, tf]. This will be included in turn in the two above temporal intervals and will be chosen such that $\hat{K} < 1$. Therefore, the nonlinear mapping T will have a fixed point in this case.

a. $$N1 = \left\| \sqrt{\bar{\sigma}_j}\, \sin\left(\sqrt{\bar{\sigma}_j}\, t\right) \right\|_2 = \left(\int_{t0}^{tf} \Omega \sin^2(\Omega t)\, dt \right)^{1/2} = \left(\int_{\Omega t0}^{\Omega tf} \sin^2 u\, du \right)^{1/2} \tag{85}$$

Since the difference between the values of the a_1 and a_3 is of the order 10^{-5}, the stability study of the iterative method will be achieved for $j = 1$.

b. The relaxation modulus is of the form, (14):

$$G(t) = \beta_1 + \beta_2 E_\nu\left(-\frac{t^\nu}{\tau_a^\nu} \right),\ \beta_1 = \frac{k_1 k_2}{k_1 + k_2},\ \ \beta_2 = \frac{k_1^2}{k_1 + k_2},$$

$$E_\nu\left(-t^\nu/\tau_a^\nu\right)=\sum_{i=0}^{\infty}\frac{(-1)^i t^{i\nu}/(\tau_a^\nu)^i}{\Gamma(i\nu+1)}$$

and recall, [47], that

$$\frac{d^\nu t^k}{dt^\nu}=\frac{\Gamma(k+1)}{\Gamma(k-\nu+1)}t^{k-\nu},\quad \frac{d^\nu 1}{dt^\nu}=\frac{1}{\Gamma(1-\nu)}t^{-\nu}.$$

Then

$$\frac{d^\nu E_\nu\left(-t^\nu/\tau_a^\nu\right)}{dt^\nu}=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k\nu+1)(\tau_a^\nu)^k}\cdot\frac{\Gamma(k\nu+1)t^{\nu k-\nu}}{\Gamma(k\nu-\nu+1)}=$$

$$=\frac{t^{-\nu}}{\Gamma(1-\nu)}+\sum_{k=1}^{\infty}\frac{(-1)^k t^{\nu k}}{(\tau_a^\nu)^k}\cdot\frac{t^{-\nu}}{\Gamma((k-1)\nu+1)}=\frac{t^{-\nu}}{\Gamma(1-\nu)}+\sum_{l=0}^{\infty}\frac{(-1)^{l+1}t^{\nu(l+1)}}{(\tau_a^\nu)^{l+1}}\cdot\frac{t^{-\nu}}{\Gamma(l\nu+1)}=$$

$$=\frac{t^{-\nu}}{\Gamma(1-\nu)}-\frac{1}{\tau_a^\nu}\sum_{l=0}^{\infty}\left(-\frac{t^\nu}{\tau_a^\nu}\right)^l\cdot\frac{1}{\Gamma(l\nu+1)}.$$

(86)

These formulas lead to the fractional derivative of the relaxation modulus as

$$\frac{d^\nu G(t)}{dt^\nu}=\beta_1\frac{t^{-\nu}}{\Gamma(1-\nu)}+\beta_2\left[\frac{t^{-\nu}}{\Gamma(1-\nu)}-\frac{1}{\tau_a^\nu}\sum_{l=0}^{\infty}\frac{(-t^\nu)^l}{(\tau_a^\nu)^l\,\Gamma(\nu l+1)}\right]=$$

$$=(\beta_1+\beta_2)\frac{t^{-\nu}}{\Gamma(1-\nu)}-\frac{\beta_2}{\tau_a^\nu}\sum_{l=0}^{\infty}\frac{(-t^\nu)^l}{(\tau_a^\nu)^l\,\Gamma(\nu l+1)}=$$

$$=k_1\left(\frac{t^{-\nu}}{\Gamma(1-\nu)}-\frac{k_1}{(k_1+k_2)\tau_a^\nu}E_\nu\left(-t^\nu/\tau_a^\nu\right)\right). \tag{87}$$

The Figure 6 is the graphical representation of the function:

$$E(\nu,t)=\frac{k_1}{(k_1+k_2)\tau_a^\nu}E_\nu\left(-t^\nu/\tau_a^\nu\right)=\frac{9.8\cdot 10^7}{(9.8+2.45)\cdot 10^7\cdot 2.24}E_\nu\left(-t^\nu/2.24\right)$$

(88)

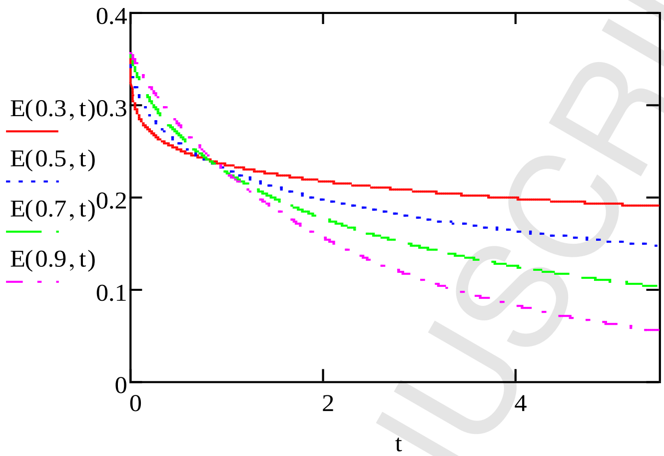

Fig.6. Variation of $E(\nu, t)$ in the temporal interval of convergence [0, 5.5] seconds

So,

$$k_1 \cdot N2 = \left\| \frac{d^{\nu} G(t)}{dt^{\nu}} \right\|_1 = \int_{t0}^{tf} \left| k_1 \left(\frac{t^{-\nu}}{\Gamma(1-\nu)} - E(\nu,t) \right) \right| dt \tag{89}$$

and $\hat{K}$ becomes of the form:

$$\hat{K} = \frac{1}{k_1} \left\| \sqrt{\bar{\sigma}_j} \sin\left(\sqrt{\bar{\sigma}_j} t\right) \right\|_2 \int_{t0}^{tf} \left| k_1 \left(\frac{t^{-\nu}}{\Gamma(1-\nu)} - E(\nu,t) \right) \right| dt = \tag{90}$$

$$= \left\| \sqrt{\bar{\sigma}_j} \sin\left(\sqrt{\bar{\sigma}_j} t\right) \right\|_2 \int_{t0}^{tf} \left| \frac{t^{-\nu}}{\Gamma(1-\nu)} - E(\nu,t) \right| dt = N1 \cdot N2 \, .$$

The choice of the intervals [$t0$, tf] and the results of the calculations above are presented in Table 1.

Table 1

Intervals		[0, 1]	[1, t_c]
ν = 0.6	[$t0$, tf]	[0.5, 1]	[1, 3]
	$N1$	1.458	3.271

	$N2$	0.632	0.29
	$\hat{K}$	0.919	0.968
ν = 0.7	$[t0, tf\,]$	[0.1, 1]	[1, 4.5]
	$N1$	2.136	4.174
	$N2$	0.45	0.226
	$\hat{K}$	0.961	0.942
ν = 0.8	$[t0, tf\,]$	[0.05, 1]	[1, 5.5]
	$N1$	2.193	4.837
	$N2$	0.415	0.138
	$\hat{K}$	0.91	0.668
ν = 0.9	$[t0, tf\,]$	[0.05, 1]	[1, 5.5]
	$N1$	2.193	4.837
	$N2$	0.225	0.093
	$\hat{K}$	0.493	0.45

To calculate the norm $N2$, we take into account that it is defined by an improper integral on the interval [0, 1]. Also, for the function $E\ (\nu,\ t)$ was used the minimum value of this, which corresponds to the maximum value of tf for a given ν, as in Table 2:

Table 2

$E(0.6, 3)$	$E(0.7, 4.5)$	$E(0.8, 5.5)$	$E(0.9, 5.5)$
0.163	0.117	0.08	0.055

Therefore, for the chosen time intervals $[t0, tf\,]$ the values of $\hat{K} < 1$. Putting $L = 0$ and $\alpha = \hat{K}$, the condition (81) is verified in Theorem 2 for the nonlinear mapping T, hence T is stable.

11. Conclusions

A computational scheme, based on the modified VIM and on the techniques of the Laplace transformation, is presented for the dynamic analysis of the beam with Bernoulli – Euler theory, in which the damping characteristics are described in terms of the fractional derivatives of order $\nu \in (0, 1)$.

Using the fractional Zener model, obtained by generalizing the appropriate classical model, is found the motion equation in terms of the transverse deflections for a simply supported viscoelastic beam subjected to the uniformly distributed loading and a periodic concentrated force applied at the center of the beam.

To solve this equation, the iterative process that is specific to the VIM is replaced by an efficient algorithm based on the Correspondence principle proposed by Mainardi, [22], for the fractional calculus.

Also, it is avoided the definition of initial approximation of the solution, which is a difficult step for the nonlinear problems.

The validity of the proposed method is verified by the numerical results and graphical representations for the quasi – static and classical dynamic response of the beam, and then, for the dynamic study of a fractional Zener model. A very good consistency is found between the exact solution w obtained for $\nu = 1$ with Galerkin – Laplace transform method, [35] and that obtained by the current algorithm in the temporal convergence interval [0, 5.5] seconds. Because in this time interval the amplitude of the oscillations has maximum values, the results obtained are very important for the engineering design.

The study of time-dependent vertical response $w(L/2,t)$, presented in the graphs above, shows that the differences obtained between the classic analysis and fractional dynamic analysis depend on the load type of the beam. In the case of the periodic concentrated force at $x = L/2$, the curves corresponding to the classical calculus and the fractional calculus are very close, but these differ significantly from one value to another of ν for a uniformly distributed transverse load.

In the dynamic analysis of the fractional viscoelastic structures it is important to approach the problem of stability of a nonlinear operator T, because the small modifications in the initial data or in the data that are involved in the computation process will produce a small influence on the computed value of the solution. It is defined in (78) a fractional nonlinear operator T by VIM and proved in Theorem 1 that, if $a_j(t)$ is a fixed point for T, then $a_j(t)$ verified the integral - differential equation (23). Next, using the properties of convolution product in Banach spaces $L^p(\mathbf{R^n})$, the fixed point theory and the results presented in the paper [46] are determined the stability intervals included in the convergence intervals that correspond to a given fractional order ν.

In the last decade, the viscoelastic structures are analyzed using the finite element method which, unlike the method presented in this paper, is based on the discretization of time interval that leads to the errors caused by the choice of the time step.

The results of this study, which approaches the fractional derivative, may allow researchers to choose a suitable mathematical model that will precisely fit with a particular experimental model and can be easily extended to the dynamic analysis of the more complex viscoelastic structures.

Acknowledgements

This research did not receive any specific grant from funding agencies in the public, commercial, or not for-profit sectors.

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