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GENERALIZED CONFORMABLE FRACTIONAL OPERATORS

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Abstract. In [1] T. Abdeljawad has put an open problem, which is stated as: "Is it hard to fractionalize the conformable fractional calculus, either by iterating the conformable fractional derivative (Grunwald-Letnikov approach) or by iterating the conformable fractional integral of order $0 < \alpha \leq 1$ (Riemann approach)?. Notice that when $\alpha = 0$ we obtain Hadamard type fractional integrals".
In this article we claim that yes it is possible to iterate the conformable fractional integral of order $0 < \alpha \leq 1$ (Riemann approach), such that when $\alpha = 0$ we obtain Hadamard fractional integrals. First of all we prove Cauchy integral formula for repeated conformable fractional integral and proceed to define new generalized conformable fractional integral and derivative operators (left and right sided). We also prove some basic properties which are satisfied by these operators. These operators (integral and derivative) are the generalizations of Katugampola operators, Riemann-Liouville fractional operators, Hadamard fractional operators. We apply our results to a simple function. Also we consider a nonlinear fractional differential equation using this new formulation. We show that this equation is equivalent to a Volterra integral equation and demonstrate the existence and uniqueness of solution to the nonlinear problem. At the end, we give conclusion and point out an open problem.

1. Introduction and Preliminaries

The idea of fractional calculus has impelled a host of researchers towards it for a last few decades. Work has been carried out on it on a large scale and everyone has awakened its various aspects. The contributions of Euler, Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov, Hadamard and in the present century, Weyl, Riesz, Marchaud, Kober and Caputo are remarkable in this field [1–10]. Most of these researchers initially introduced fractional integrals, on the basis of which the associated fractional derivative and other related results were produced. Some of the most explored and commonly used definitions of fractional integrals are given below.
The right-sided Riemann-Liouville fractional integral operator of order $\beta > 0$ is given by [20]:

$$J^{\beta}_{p^+}\phi(r) = \frac{1}{\Gamma(\beta)} \int_p^r (r-w)^{\beta-1}\phi(w)dw, \quad \text{with} \quad r > p, \tag{1}$$

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which is based on the iteration of the Riemann integral operator $\int_p^r \phi(w)dw$. The Hadamard fractional integral introduced by J. Hadamard [13], for $\beta > 0$ is given by:

$$H_{p^+}^{\beta}\phi(r) = \frac{1}{\Gamma(\beta)} \int_p^r \left(\log \frac{r}{w}\right)^{\beta-1} \phi(w)\frac{dw}{w}, \quad \text{with} \quad r > p, \tag{2}$$

which is based on iterating the integral operator $\int_p^r \phi(w)\frac{dw}{w}$. Udita N. Katugampola has defined a generalized Katugampola integral operator [14], which for $\beta > 0$, $\tau \neq -1$ is given by:

$${}_p^{\tau}I_r^{\beta}\phi(r) = \frac{(\tau+1)^{1-\beta}}{\Gamma(\beta)} \int_p^r \left(r^{\tau+1} - w^{\tau+1}\right)^{\beta-1} \phi(w)w^{\tau}dw, \quad r > p \tag{3}$$

and is the generalization of both the above operators defined in (1) and (2). The operator in (3) is based on the iteration of the integral operator $\int_p^r \phi(w)w^{\tau}dw$. Concurrently with the operators (1), (2) and (3) their corresponding left-sided versions were also determined. Also by using these fractional integral operators, the associated fractional derivative operators were defined [13, 14, 20].
Khalil et al. [4] have discovered novel definitions of fractional derivative and connected integral which are given below.

Definition 1 ([4])**.** For $\phi : [0, \infty) \to \mathbb{R}$, the conformable fractional derivative of ϕ of order $\alpha \in (0, 1]$, at point $w \in (0, \infty)$ is defined as:

$$T_{\alpha}\phi(w) = \lim_{\epsilon \to 0} \frac{\phi(w + \epsilon w^{1-\alpha}) - \phi(w)}{\epsilon}, \tag{4}$$

for $w = 0$, it is defined as:

$$T_{\alpha}\phi(0) = \lim_{w \to 0^+} T_{\alpha}\phi(w).$$

If the conformable fractional derivative of ϕ of order α exists, then we say that ϕ is α-differentiable.

If ϕ is ordinary differentiable then the connection of conformable fractional derivative with the ordinary derivative for $w > 0$ is given by:

$$T_{\alpha}\phi(w) = w^{1-\alpha}\phi'(w), \tag{5}$$

where $\phi'(w)$ denotes the ordinary derivative of ϕ at the point w. It is simple to prove that a function could be α-differentiable at a point but not ordinary differentiable, see for detail [4]. This new definition is simple and satisfies almost all basic properties which the ordinary derivative does. These properties are given in the following theorem.

Theorem 1 ([4]). *Let $\alpha \in (0,1]$ and ϕ_1, ϕ_2 be α-differentiable functions at a point $w > 0$. Then for any μ_1, $\mu_2 \in \mathbb{R}$, we have:*

(1) $T_\alpha(\mu_1\phi_1 + \mu_2\phi_2) = \mu_1 T_\alpha(\phi_1) + \mu_2 T_\alpha(\phi_2)$.

(2) $T_\alpha(t^r) = rt^{r-\alpha}$, $\forall\, r \in \mathbb{R}$.

(3) $T_\alpha(c) = 0$, $\forall\, c \in \mathbb{R}$.

(4) $T_\alpha(\phi_1\phi_2) = \phi_2 T_\alpha(\phi_1) + \phi_1 T_\alpha(\phi_2)$.

(5) $T_\alpha\left(\dfrac{\phi_1}{\phi_2}\right) = \dfrac{\phi_2 T_\alpha(\phi_1) - \phi_1 T_\alpha(\phi_2)}{\phi_2^2}$.

The authors in [4] have also defined the conformable fractional integral of order $0 < \alpha \leq 1$ (about which an open problem was posed in [1]). This is defined as under.

Theorem 2 ([4]). *Let $\alpha \in (0,1]$, the conformable fractional integral of the continuous function $\phi : [p,q] \subseteq [0,\infty) \to \mathbb{R}$, of order α is defined as:*

$$I_\alpha(\phi)(w) = \int_p^q \phi(w) d_\alpha w := \int_p^q \phi(w) w^{\alpha-1} dw, \tag{6}$$

where the integral $\int_p^q dw$, on the right side represents the classical Riemann integral.

The inverse property is given in the following Theorem:

Theorem 3 ([4]). *For any continuous function ϕ in the domain of I_α, we have:*

$$T_\alpha I_\alpha \phi(r) = \phi(r).$$

In this paper, to give answer to the open problem given in [1] we define new generalized conformable fractional integral operators (right-sided and left-sided), by iterating conformable integral of order $\alpha \in (0,1]$. We prove semigroup property and linearity property for these operators. After introducing new fractional integral operators, we define the associated right-sided and left-sided generalized conformable fractional derivative operators. The semigroup property and linearity property are also proved for generalized conformable fractional derivative operator. By making use of these operators we also define Riemann-Liouville type conformable fractional operators. Our newly defined fractional operators are the generalizations of the Katugampola fractional operators, Riemann-Liouville fractional operators and Hadamard fractional integral operators. We apply our fractional differential operator to a simple function. Also we consider a nonlinear fractional differential equation using this new formulation. We show that this equation is equivalent to a Volterra integral equation and demonstrate the existence and uniqueness of solution to the nonlinear problem. At the end, we give conclusion and point out an open problem.

2. Generalized conformable fractional integral operators

Throughout this paper, we consider $\alpha \in (0,1]$, $\tau \in \mathbb{R}$ such that $\tau + \alpha \neq 0$. Also we take $0 \leq p < q$, $L_\alpha[p,q] = \{\phi(w) : \int_p^q \phi(w) d_\alpha w < \infty\}$. Define an operator ${}^{\tau}_{\alpha}K_{p^+} : L_\alpha[p,q] \to \mathbb{R}$

by:

$$ {}^{\tau}_{\alpha}K_{p^+}\phi(r) = \int_p^r \phi(w) w^{\tau} d_{\alpha}w, \quad r \in [p,q], $$

and ${}^{\tau}_{\alpha}K_{q^-} : L_{\alpha}[p,q] \to \mathbb{R}$ by:

$$ {}^{\tau}_{\alpha}K_{q^-}\phi(r) = \int_r^q \phi(w) w^{\tau} d_{\alpha}w, \quad r \in [p,q]. $$

Here $\int d_{\alpha}w$ represents the conformable fractional integral, which was defined in (6).
To define new generalized conformable fractional integral operators we need to prove the following result.

Theorem 4. ***(Cauchy Integral Formula for Repeated Conformable Integrals):*** *Let $\phi \in L_{\alpha}[p,q]$. Then the n times repeated right-sided and left-sided conformable fractional integrals are given by the single conformable fractional integrals*

$$ \begin{aligned} {}^{\tau}_{\alpha}K^{n}_{p^+}\phi(r) &= \int_p^r r_1^{\tau} \int_p^{r_1} r_2^{\tau} \int_p^{r_3} r_2^{\tau} \ldots \int_p^{r_{n-1}} \phi(r_n) r_n^{\tau} d_{\alpha}r_n \ldots d_{\alpha}r_2 d_{\alpha}r_1 \\ &= \frac{1}{(n-1)!} \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{n-1} \phi(w) w^{\tau} d_{\alpha}w \end{aligned} \tag{7} $$

and

$$ \begin{aligned} {}^{\tau}_{\alpha}K^{n}_{q^-}\phi(r) &= \int_r^q r_1^{\tau} \int_{r_1}^q r_2^{\tau} \int_{r_2}^q \ldots \int_{r_{n-1}}^q \phi(r_n) r_n^{\tau} d_{\alpha}r_n \ldots d_{\alpha}r_2 d_{\alpha}r_1 \\ &= \frac{1}{(n-1)!} \int_r^q \left(\frac{w^{\tau+\alpha} - r^{\tau+\alpha}}{\tau+\alpha} \right)^{n-1} \phi(w) w^{\tau} d_{\alpha}w, \end{aligned} \tag{8} $$

respectively.

Proof. First we prove (7). For $n = 1$, we have

$$ {}^{\tau}_{\alpha}K^{1}_{p^+}\phi(r) = \int_p^r \phi(w) w^{\tau} d_{\alpha}w, $$

which is just the definition of ${}^{\tau}_{\alpha}K_{p^+}$ and hence true.
Now we prove for $n = 2$. Let us define

$$ \varphi(r) = \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right) \phi(w) w^{\tau} d_{\alpha}w, \tag{9} $$

which is the right hand side of (7) when $n = 2$. So we want to show that $\varphi(r) = {}^{\tau}_{\alpha}K^2_{p^+}\phi(r)$. Since

$$\varphi(r) = \frac{1}{\tau+\alpha}\left[r^{\tau+\alpha}\int\limits_p^r \phi(w)w^\tau d_\alpha w - \int\limits_p^r w^{\tau+\alpha}\phi(w)w^\tau d_\alpha w\right]. \tag{10}$$

By taking the conformable derivative of both sides of (10) with respect to r, we get

$$\begin{aligned}
T_\alpha\varphi(r) = \frac{1}{\tau+\alpha}\Bigg[&r^{\tau+\alpha}T_\alpha\int\limits_p^r \phi(w)w^\tau d_\alpha w \\
&+\int\limits_p^r \phi(w)w^\tau d_\alpha w T_\alpha r^{\tau+\alpha} - T_\alpha\int\limits_p^r w^{\tau+\alpha}\phi(w)w^\tau d_\alpha w\Bigg] \\
= \frac{1}{\tau+\alpha}&\left[r^{\tau+\alpha}\phi(r)r^\tau + (\tau+\alpha)r^\tau\int\limits_p^r \phi(w)w^\tau d_\alpha w - r^{\tau+\alpha}\phi(r)r^\tau\right] \\
= r^\tau&\int\limits_p^r \phi(w)w^\tau d_\alpha w = r^\tau {}^{\tau}_{\alpha}K_{p^+}\phi(r).
\end{aligned}$$

Since (9) implies that $\varphi(p) = 0$, so

$$\varphi(r) = \varphi(r) - \varphi(p) = \int\limits_p^r T_\alpha\varphi(w)d_\alpha w = \int\limits_p^r {}^{\tau}_{\alpha}K_{p^+}\phi(w)w^\tau d_\alpha w = {}^{\tau}_{\alpha}K^2_{p^+}\phi(r).$$

Generally for any $n \in \mathbb{N}$, the proof is similar. First we expand the term $(r^{\tau+\alpha} - w^{\tau+\alpha})^{n-1}$ by the Binomial Theorem, and then write $\varphi(r)$ as written in (10) and take all the terms containing $r^{\tau+\alpha}$ outside the integral sign. The process is then similar as above, this shows that (7) is true for every positive integer n.
The identity (8) can be proved in the similar way by iterating the single integral ${}^{\tau}_{\alpha}K_{q^-}\phi(r) = \int\limits_r^q \phi(w)w^\tau d_\alpha w$, instead of ${}^{\tau}_{\alpha}K_{p^+}\phi(r) = \int\limits_p^r \phi(w)w^\tau d_\alpha w$. □

Construction of new integral operators:

Consider the results obtained in Theorem 4.

$${}^{\tau}_{\alpha}K^n_{p^+}\phi(r) = \frac{1}{(n-1)!}\int\limits_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha}\right)^{n-1}\phi(w)w^\tau d_\alpha w, \tag{11}$$

$${}^{\tau}_{\alpha}K^n_{q^-}\phi(r) = \frac{1}{(n-1)!}\int\limits_r^q \left(\frac{w^{\tau+\alpha} - r^{\tau+\alpha}}{\tau+\alpha}\right)^{n-1}\phi(w)w^\tau d_\alpha w. \tag{12}$$

In these cases we are applying the conformable integral operators ${}^{\tau}_{\alpha}K_{p^+}$ or ${}^{\tau}_{\alpha}K_{q^-}$, n times. Where n is restricted to be a positive integer. We generalize it and use a positive real number instead of positive integer n, which is what the fractional calculus requires, that is, the generalization of integer-order differentiation or n-fold integration. As in the case of previously defined fractional operators (integral or derivative), replacing the positive integer n by a positive real number β and using the gamma function, which is the generalization of the factorial function, we get new right-sided and left-sided generalized conformable fractional integral operators, which are defined below:

Definition 2. Let ϕ be a conformable integrable function on the interval $[p,q] \subseteq [0,\infty)$. The right-sided and left-sided generalized conformable fractional integrals ${}^{\tau}_{\alpha}K^{\beta}_{p^+}$ and ${}^{\tau}_{\alpha}K^{\beta}_{q^-}$ of order $\beta > 0$ with $\alpha \in (0,1]$, $\tau \in \mathbb{R}$, $\alpha + \tau \neq 0$ are defined by:

$$ {}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi(r) = \frac{1}{\Gamma(\beta)} \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} \phi(w) w^{\tau} d_{\alpha}w, \quad r > p $$

and

$$ {}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi(r) = \frac{1}{\Gamma(\beta)} \int_r^q \left(\frac{w^{\tau+\alpha} - r^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} \phi(w) w^{\tau} d_{\alpha}w, \quad q > r, $$

respectively, and ${}^{\tau}_{\alpha}K^{0}_{p^+}\phi(r) = {}^{\tau}_{\alpha}K^{0}_{q^-}\phi(r) = \phi(r)$. Here Γ denotes the well-known gamma function.

Remark 1. (1). For $\tau = 0$ in the above Definition 2, we get the Riemann Liouville type conformable fractional integral operators, which are given below:

$$ {}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi(r) = \frac{1}{\Gamma(\beta)} \int_p^r \left(\frac{r^{\alpha} - w^{\alpha}}{\alpha} \right)^{\beta-1} \phi(w) d_{\alpha}w, \quad r > p \tag{13} $$

and

$$ {}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi(r) = \frac{1}{\Gamma(\beta)} \int_r^q \left(\frac{w^{\alpha} - r^{\alpha}}{\alpha} \right)^{\beta-1} \phi(w) d_{\alpha}w, \quad q > r, \tag{14} $$

respectively, and ${}^{\tau}_{\alpha}K^{0}_{p^+}\phi(r) = {}^{\tau}_{\alpha}K^{0}_{q^-}\phi(r) = \phi(r)$. Here Γ denotes the well-known gamma function.
Note that the operators in (13) and (14) can also be obtained by taking the conformable integral operators $\int_p^r \phi(w) d_{\alpha}w$ and $\int_r^q \phi(w) d_{\alpha}w$ and iterating in the manner as done above in Theorem 4.
(2) Using the definition of conformable integral given in (6) and L'Hospital rule it is straightforward that when $\alpha \to 0$ in (13) and (14), we get the Hadamard fractional

integrals:

$$ {}_{p^+}K_0^\beta\phi(r) = \frac{1}{\Gamma(\beta)}\int_p^r \left(\log\frac{r}{w}\right)^{\beta-1}\phi(w)\frac{dw}{w}, \qquad r > p, $$

$$ {}_{q^-}K_0^\beta\phi(r) = \frac{1}{\Gamma(\beta)}\int_r^q \left(\log\frac{w}{r}\right)^{\beta-1}\phi(w)\frac{dw}{w}, \qquad r < q. $$

(3) For $\alpha = 1$, we get the Riemann-Liouville fractional integrals:

$$ {}_{p^+}K_1^\beta\phi(r) = \frac{1}{\Gamma(\beta)}\int_p^r (r-w)^{\beta-1}\phi(w)dw, \qquad r > p, $$

$$ {}_{q^-}K_1^\beta\phi(r) = \frac{1}{\Gamma(\beta)}\int_r^q (w-r)^{\beta-1}\phi(w)dw, \qquad r < q. $$

(4) For the case $\beta = 1$ in Definition 2, we get the conformable fractional integrals. And when $\alpha = \beta = 1$, $\tau = 0$ we get classical Riemann integrals.

Now we prove some basic properties for the obtained generalized operators.

Theorem 5. *Let $\phi \in L_\alpha[p,q]$, where $0 \le p < q$. For $\alpha \in (0,1]$, $\beta > 0$ we have*

$$ \lim_{\beta\to 0} {}^\tau_\alpha K^\beta_{p^+}\phi(r) = {}^\tau_\alpha K^0_{p^+}\phi(r) = \phi(r), \tag{15} $$

$$ \lim_{\beta\to 0} {}^\tau_\alpha K^\beta_{q^-}\phi(r) = {}^\tau_\alpha K^0_{q^-}\phi(r) = \phi(r). \tag{16} $$

Proof. Applying the relation (6), integration by parts and well-known property of Beta function:

$$ \begin{aligned} {}^\tau_\alpha K^\beta_{p^+}\phi(r) &= \frac{1}{\Gamma(\beta)}\int_p^r \left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta-1}\phi(w)w^\tau d_\alpha w \\ &= \frac{1}{\beta\Gamma(\beta)}\phi(p)\left(\frac{r^{\tau+\alpha}-p^{\tau+\alpha}}{\tau+\alpha}\right)^\beta + \frac{1}{\beta\Gamma(\beta)}\int_p^r T_\alpha\phi(w)\left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^\beta d_\alpha w \\ &= \frac{1}{\Gamma(\beta+1)}\phi(p)\left(\frac{r^{\tau+\alpha}-p^{\alpha}}{\tau+\alpha}\right)^\beta + \frac{1}{\Gamma(\beta+1)}\int_p^r T_\alpha\phi(w)\left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^\beta d_\alpha w. \end{aligned} $$

Taking limit as $\beta \to 0$, we have

$$ \lim_{\beta\to 0} {}^\tau_\alpha K^\beta_{p^+}\phi(r) = {}^\tau_\alpha K^0_{p^+}\phi(r) = \phi(p) + \int_p^r T_\alpha\phi(w)w^\tau d_\alpha w = \phi(r). $$

The proof is similar for (16). $\square$

Now we prove semigroup property for this newly defined operator. This property makes possible not only the definition of new integral, but also of new differentiation, by taking enough derivatives of ${}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi(r)$ and ${}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi(r)$, which we will discuss in next section.

Theorem 6. ***(Semigroup Property).*** *Let $\phi : [p,q] \subseteq [0,\infty) \to \mathbb{R}$ be a conformable integrable function. Then for β_1, $\beta_2 > 0$ and $\alpha \in (0,1]$ we have:*

$$\begin{aligned}{}^{\tau}_{\alpha}K^{\beta_1}_{p^+}{}^{\tau}_{\alpha}K^{\beta_2}_{p^+}\phi(r) &= {}^{\tau}_{\alpha}K^{\beta_1+\beta_2}_{p^+}\phi(r)\\ &= \frac{1}{\Gamma(\beta_1+\beta_2)}\int_p^r \left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta_1+\beta_2-1}\phi(w)w^{\tau}d_{\alpha}w. \end{aligned}\tag{17}$$

$$\begin{aligned}{}^{\tau}_{\alpha}K^{\beta_1}_{q^-}{}^{\tau}_{\alpha}K^{\beta_2}_{q^-}\phi(r) &= {}^{\tau}_{\alpha}K^{\beta_1+\beta_2}_{q^-}\phi(r)\\ &= \frac{1}{\Gamma(\beta_1+\beta_2)}\int_r^q \left(\frac{w^{\tau+\alpha}-r^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta_1+\beta_2-1}\phi(w)w^{\tau}d_{\alpha}w. \end{aligned}\tag{18}$$

Proof. Consider

$$\begin{aligned}&{}^{\tau}_{\alpha}K^{\beta_1}_{p^+}{}^{\tau}_{\alpha}K^{\beta_2}_{p^+}\phi(r)\\ &= \frac{1}{\Gamma(\beta_1)}\int_p^r \left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta_1-1} {}^{\tau}_{\alpha}K^{\beta_2}_{p^+}\phi(w)w^{\tau}d_{\alpha}w\\ &= \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_p^r \left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta_1-1} w^{\tau}\int_p^w \left(\frac{w^{\tau+\alpha}-s^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta_2-1}\phi(s)s^{\tau}d_{\alpha}s d_{\alpha}w\\ &= \frac{(\tau+\alpha)^{2-(\beta_1+\beta_2)}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_p^r \phi(s)s^{\tau}\int_s^r \left(r^{\tau+\alpha}-w^{\tau+\alpha}\right)^{\beta_1-1}\left(w^{\tau+\alpha}-s^{\tau+\alpha}\right)^{\beta_2-1} w^{\alpha-1+\tau}dw d_{\alpha}s,\end{aligned}$$

where in the last step we have exchanged the order of integration using Fubini's Theorem and applied the relation (6). Changing variables to l defined by, $w^{\tau+\alpha} = s^{\tau+\alpha} + (r^{\tau+\alpha} - s^{\tau+\alpha})l$, in the inner integral

$$\begin{aligned}&{}^{\tau}_{\alpha}K^{\beta_1}_{p^+}{}^{\tau}_{\alpha}K^{\beta_2}_{p^+}\phi(r)\\ &\quad= \frac{(\tau+\alpha)^{1-(\beta_1+\beta_2)}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_p^r \left(r^{\tau+\alpha}-s^{\tau+\alpha}\right)^{\beta_1+\beta_2-1}\phi(s)s^{\tau}\int_0^1 l^{\beta_2-1}\left(1-l\right)^{\beta_1-1}dl d_{\alpha}s\\ &\quad= \frac{1}{\Gamma(\beta_1+\beta_2)}\int_p^r \left(\frac{r^{\tau+\alpha}-s^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta_1+\beta_2-1}\phi(s)s^{\tau}d_{\alpha}s.\\ &\quad= {}^{\tau}_{\alpha}K^{\beta_1+\beta_2}_{p^+}\phi(r),\end{aligned}$$

where

$$\int_0^1 l^{\beta_2-1}(1-l)^{\beta_1-1}\,dl=\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)}$$

is the well known Euler Beta function.
The equation (18) can be proved in the same way. This completes the proof. □

If we consider a bounded interval $[p,q]$, such that $p\geq 0$. The operators ${}^{\tau}_{\alpha}K^{\beta}_{p^+}$ and ${}^{\tau}_{\alpha}K^{\beta}_{q^-}$ associate the function ${}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi(r)$ and ${}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi(r)$ to each conformable integrable function ϕ on $[p,q]$. Thus these are linear operators, which is proved in the following theorem.

Theorem 7. ***(Linearity).*** *The operators* ${}^{\tau}_{\alpha}K^{\beta}_{p^+}$ *and* ${}^{\tau}_{\alpha}K^{\beta}_{q^-}$ *are linear operators on* $L_\alpha[p,q]$*. That is, define*

$${}^{\tau}_{\alpha}K^{\beta}_{p^+},\ {}^{\tau}_{\alpha}K^{\beta}_{q^-}:L_\alpha[p,q]\to L_\alpha[p,q],$$

then

$$\begin{aligned}{}^{\tau}_{\alpha}K^{\beta}_{p^+}\left(\mu_1\phi_1+\mu_2\phi_2\right)&=\mu_1{}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi_1+\mu_2{}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi_2,\\ {}^{\tau}_{\alpha}K^{\beta}_{q^-}\left(\mu_1\phi_1+\mu_2\phi_2\right)&=\mu_1{}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi_1+\mu_2{}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi_2.\end{aligned}$$

For all $\phi_1,\ \phi_2\in L_\alpha[p,q]$ *and* $\mu_1,\ \mu_2\in\mathbb{R}$.

Proof. The proof is simple, consider

$$\begin{aligned}{}^{\tau}_{\alpha}K^{\beta}_{p^+}\left(\mu_1\phi_1+\mu_2\phi_2\right)(r)&=\frac{1}{\Gamma(\beta)}\int_p^r\left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta-1}\left(\mu_1\phi_1+\mu_2\phi_2\right)(w)w^{\tau}d_\alpha w\\ &=\frac{\mu_1}{\Gamma(\beta)}\int_p^r\left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta-1}\phi_1(w)w^{\tau}d_\alpha w\\ &\quad+\frac{\mu_2}{\Gamma(\beta)}\int_p^r\left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta-1}\phi_2(w)w^{\tau}d_\alpha w\\ &=\mu_1{}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi_1(r)+\mu_2{}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi_2(r).\end{aligned}$$

Similarly

$${}^{\tau}_{\alpha}K^{\beta}_{q^-}\left(\mu_1\phi_1+\mu_2\phi_2\right)(r)=\mu_1{}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi_1(r)+\mu_2{}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi_2(r).$$

□

In the following theorem we prove that the operators ${}^{\tau}_{\alpha}K^{\beta}_{p^+}$ and ${}^{\tau}_{\alpha}K^{\beta}_{q^-}$ are bounded on the space $L_\alpha[p,q]$.

Theorem 8. ***(Boundedness).*** *The operators ${}_{\alpha}^{\tau}K_{p^+}^{\beta}$ and ${}_{\alpha}^{\tau}K_{q^-}^{\beta}$ are bounded operators on $L_\alpha[p,q]$. That is, define*

$$ {}_{\alpha}^{\tau}K_{p^+}^{\beta},\ {}_{\alpha}^{\tau}K_{q^-}^{\beta} : L_\alpha[p,q] \to L_\alpha[p,q], $$

then

$$ \left\| {}_{\alpha}^{\tau}K_{p^+}^{\beta}\phi \right\| \leq M \left\|\phi\right\|_C, \ \left\| {}_{\alpha}^{\tau}K_{q^-}^{\beta}\phi \right\| \leq M \left\|\phi\right\|_C, \tag{19} $$

where $\|\phi\|_C = \max_{r\in[p,q]} |\phi(r)|$, $M = \frac{|(\tau+\alpha)^{-\beta}|}{\beta+1}(q^{\tau+\alpha}-p^{\tau+\alpha})^\beta$

Proof. The proof is simple, we consider

$$ \begin{aligned} \left\| {}_{\alpha}^{\tau}K_{p^+}^{\beta}\phi(r) \right\| &= \left\| \frac{1}{\Gamma(\beta)} \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} \phi(w) w^\tau d_\alpha w \right\| \\ &\leq \frac{|(\tau+\alpha)^{1-\beta}|}{\Gamma(\beta)} \|\phi\|_C \int_p^r \left(r^{\tau+\alpha} - w^{\tau+\alpha} \right)^{\beta-1} w^{\tau+\alpha-1} dw \\ &\leq \frac{|(\tau+\alpha)^{1-\beta}|}{\Gamma(\beta+1)} \|\phi\|_C \left(q^{\tau+\alpha} - p^{\tau+\alpha} \right)^\beta . \end{aligned} \tag{20} $$

In the case of the right generalized conformable fractional integral operator ${}_{\alpha}^{\tau}K_{q^-}^{\beta}$, the proof is similar. □

3. Generalized Conformable Fractional Derivative

Because the Riemann-Liouville approach to the generalized conformable fractional integrals began with an expression involving repeated conformable integration of a function. One can adopt the Grunwald-Letnikov approach to construct a fractional derivative operator firstly, on the basis of which the fractional integral operator can be defined. However, it is also possible to frame a definition for the generalized conformable fractional derivative using the definition already obtained above for the related integral. Now keeping the above integral operators under consideration, we define the right and left sided generalized conformable fractional derivative operators as follows:

Definition 3. Let ϕ be a conformable integrable function on the interval $[p,q]$. The right and left sided generalized conformable fractional derivative operators ${}_{\alpha}^{\tau}T_{p^+}^{\beta}$ and ${}_{\alpha}^{\tau}T_{q^-}^{\beta}$ of order $0<\beta<1$, $\alpha\in(0,1]$ with $p\geq 0$ are defined by:

$$ {}_{\alpha}^{\tau}T_{p^+}^{\beta}\phi(r) = \frac{r^{-\tau}}{\Gamma(1-\beta)} T_\alpha \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{-\beta} \phi(w) w^\tau d_\alpha w, \quad r>p $$

$$ {}_{\alpha}^{\tau}T_{q^-}^{\beta}\phi(r) = \frac{r^{-\tau}}{\Gamma(1-\beta)} T_\alpha \int_r^q \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{-\beta} \phi(w) w^\tau d_\alpha w, \quad q>r, $$

respectively, and ${}^{\tau}_{\alpha}T^{0}_{p^+}\phi(r) = {}^{\tau}_{\alpha}T^{0}_{q^-}\phi(r) = \phi(r)$. Here Γ denotes the gamma function and T_α denotes the conformable derivative of order α .

To proceed further we need to prove the following theorem which shows the relation between the fractional integral and derivative operators.

Theorem 9. *Let $\phi : [p,q] \subseteq [0,\infty) \to \mathbb{R}$ be a conformable integrable function. Then for $0 < \beta < 1$ and $\alpha \in (0,1]$ we have:*

$$ {}^{\tau}_{\alpha}T^{\beta}_{p^+}\phi(r) = \frac{\phi(p)(\tau+\alpha)^{\beta}\left(r^{\tau+\alpha} - p^{\tau+\alpha}\right)^{-\beta}}{\Gamma(1-\beta)} + {}^{\tau}_{\alpha}K^{1-\beta}_{p^+}\left(r^{1-(\tau+\alpha)}\frac{d}{dr}\phi(r)\right) \tag{21} $$

Proof. Let $u'(w) = w^{\tau+\alpha-1}\left(r^{\tau+\alpha} - w^{\tau+\alpha}\right)^{-\beta}$, $v(w) = \phi(w) - \phi(p)$.
Consider

$$ \int_p^r u'(w)v(w)dw = \int_p^r w^{\tau+\alpha-1}\left(r^{\tau+\alpha} - w^{\tau+\alpha}\right)^{-\beta}(\phi(w) - \phi(p))dw $$

$$ \Rightarrow \quad \frac{d}{dr}\int_p^r u'(w)v(w)dw = \frac{d}{dr}\int_p^r w^{\tau+\alpha-1}\left(r^{\tau+\alpha} - w^{\tau+\alpha}\right)^{-\beta}(\phi(w) - \phi(p))dw $$

multiplying both sides by $\frac{(\tau+\alpha)^{\beta}r^{1-(\tau+\alpha)}}{\Gamma(1-\beta)}$, we get

$$ \begin{aligned} & \frac{(\tau+\alpha)^{\beta}r^{1-(\tau+\alpha)}}{\Gamma(1-\beta)}\frac{d}{dr}\int_p^r u'(w)v(w)dw \\ = & \frac{(\tau+\alpha)^{\beta}r^{1-(\tau+\alpha)}}{\Gamma(1-\beta)}\frac{d}{dr}\int_p^r w^{\tau+\alpha-1}\left(r^{\tau+\alpha} - w^{\tau+\alpha}\right)^{-\beta}(\phi(w) - \phi(p))dw \\ = & \mathbf{I_1} + \mathbf{I_2}, \end{aligned} \tag{22} $$

where

$$ \begin{aligned} \mathbf{I_1} &= \frac{(\tau+\alpha)^{\beta}r^{1-(\tau+\alpha)}}{\Gamma(1-\beta)}\frac{d}{dr}\int_p^r w^{\tau}w^{\alpha-1}\left(r^{\tau+\alpha} - w^{\tau+\alpha}\right)^{-\beta}\phi(w)dw \\ &= \frac{r^{1-(\tau+\alpha)}}{\Gamma(1-\beta)}\frac{d}{dr}\int_p^r w^{\tau}\left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha}\right)^{-\beta}\phi(w)d_{\alpha}w \\ &= \frac{r^{1-(\tau+\alpha)}.r^{\alpha-1}}{\Gamma(1-\beta)}T_{\alpha}\int_p^r w^{\tau}\left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha}\right)^{-\beta}\phi(w)d_{\alpha}w = {}^{\tau}_{\alpha}T^{\beta}_{p^+}\phi(r). \end{aligned} \tag{23} $$

Also

$$\begin{aligned}
\mathbf{I_2} &= \frac{(\tau+\alpha)^{\beta} r^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dr} \int_p^r w^{\tau+\alpha-1} \left(r^{\tau+\alpha} - w^{\tau+\alpha}\right)^{-\beta} \phi(p) dw \\
&= \frac{r^{1-(\tau+\alpha)}(\tau+\alpha)^{\beta-1}}{\Gamma(1-\beta)} \frac{d}{dr} \int_p^r \left(-(\tau+\alpha) w^{\tau+\alpha-1}\right) \left(r^{\tau+\alpha} - w^{\tau+\alpha}\right)^{-\beta} \phi(p) dw \\
&= \frac{r^{1-(\tau+\alpha)}(\tau+\alpha)^{\beta-1}\phi(p)}{(\beta-1)\Gamma(1-\beta)} \frac{d}{dr} \left(r^{\tau+\alpha} - p^{\tau+\alpha}\right)^{-\beta+1} \\
&= \frac{-\phi(p)}{\Gamma(1-\beta)} \left(\frac{r^{\tau+\alpha} - p^{\tau+\alpha}}{\tau+\alpha}\right)^{-\beta}. \qquad (24)
\end{aligned}$$

putting values in (22), we get

$$\frac{(\tau+\alpha)^{\beta} r^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dr} \int_p^r u'(w) v(w) dw = {}^{\tau}_{\alpha}T^{\beta}_{p^+}\phi(r) - \frac{\phi(p)}{\Gamma(1-\beta)} \left(\frac{r^{\tau+\alpha} - p^{\tau+\alpha}}{\tau+\alpha}\right)^{-\beta}. \qquad (25)$$

Now considering the left side of (25) and differentiating the integral with respect to variable r, we get:

$$\begin{aligned}
\frac{(\tau+\alpha)^{\beta} r^{1-(\tau+\alpha)}}{\Gamma(1-\beta)} \frac{d}{dr} \int_p^r u'(w) v(w) dw &= \frac{(\tau+\alpha)^{\beta}}{\Gamma(1-\beta)} \int_p^r \left(r^{\tau+\alpha} - w^{\tau+\alpha}\right)^{-\beta} \phi'(w) dw. \\
&= {}^{\tau}_{\alpha}K^{1-\beta}_{p^+} \left(r^{1-(\tau+\alpha)} \frac{d}{dr}\phi(r)\right) \qquad (26)
\end{aligned}$$

From (25) and (26) we get the required result. □

We prove the inverse property for the defined generalized operators in the following theorem.

Theorem 10. ***(Inverse Property.)*** *For any continuous function ϕ in the domain of ${}^{\tau}_{\alpha}K^{\beta}_{p^+}$, ${}^{\tau}_{\alpha}K^{\beta}_{q^-}$, ${}^{\tau}_{\alpha}T^{\beta}_{p^+}$ and ${}^{\tau}_{\alpha}T^{\beta}_{q^-}$ we have*

$${}^{\tau}_{\alpha}T^{\beta}_{p^+} {}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi(r) = \phi(r), \;\; {}^{\tau}_{\alpha}T^{\beta}_{q^-} {}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi(r) = \phi(r). \qquad (27)$$

Similarly

$${}^{\tau}_{\alpha}K^{\beta}_{p^+} {}^{\tau}_{\alpha}T^{\beta}_{p^+}\phi(r) = \phi(r), \;\; {}^{\tau}_{\alpha}K^{\beta}_{q^-} {}^{\tau}_{\alpha}T^{\beta}_{q^-}\phi(r) = \phi(r). \qquad (28)$$

Proof. Consider

$$
\begin{aligned}
&{}^{\tau}_{\alpha}T^{\beta}_{p^+}\,{}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi(r) \\
&\quad = \frac{r^{-\tau}}{\Gamma(1-\beta)} T_\alpha \int_p^r \left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{-\beta} {}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi(w) w^\tau d_\alpha w \\
&\quad = \frac{r^{-\tau}}{\Gamma(1-\beta)\Gamma(\beta)} T_\alpha \int_p^r \left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{-\beta} w^\tau \int_p^w \left(\frac{w^{\tau+\alpha}-s^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta-1} \phi(s) s^\tau d_\alpha s d_\alpha w \\
&\quad = \frac{r^{-\tau}(\tau+\alpha)}{\Gamma(1-\beta)\Gamma(\beta)} T_\alpha \int_p^r \int_p^w \left(r^{\tau+\alpha}-w^{\tau+\alpha}\right)^{-\beta} \left(w^{\tau+\alpha}-s^{\tau+\alpha}\right)^{\beta-1} w^\tau \phi(s) s^\tau d_\alpha s d_\alpha w.
\end{aligned}
$$

Switching the order of integration and changing variables to u, define $w^{\tau+\alpha} = s^{\tau+\alpha} + (r^{\tau+\alpha} - s^{\tau+\alpha})u$,

$$
\begin{aligned}
{}^{\tau}_{\alpha}T^{\beta}_{p^+}\,{}^{\tau}_{\alpha}K^{\beta}_{p^+}\phi(r) &= \frac{r^{-\tau}}{\Gamma(1-\beta)\Gamma(\beta)} T_\alpha \int_p^r \int_0^1 u^{\beta-1} (1-u)^{-\beta} \phi(s) s^\tau du d_\alpha s \\
&= r^{-\tau} T_\alpha \int_p^r \phi(s) s^\tau d_\alpha s \\
&= \phi(r),
\end{aligned}
$$

where

$$\int_0^1 u^{\beta-1} (1-u)^{-\beta} du = B(\beta, 1-\beta) = \Gamma(\beta)\Gamma(1-\beta).$$

Similarly we can prove that

$${}^{\tau}_{\alpha}T^{\beta}_{q^-}\,{}^{\tau}_{\alpha}K^{\beta}_{q^-}\phi(r) = \phi(r).$$

To prove (28), we proceed as under:
Applying the operator ${}^{\tau}_{\alpha}K^{\beta}_{p^+}$ to both sides of (21) and using the relations (5) and (17), we

get:

$$
\begin{aligned}
& {}^{\tau}_{\alpha}K^{\beta}_{p^+}\,{}^{\tau}_{\alpha}T^{\beta}_{p^+}\phi(r) \\
= \; & \frac{\phi(p)(\tau+\alpha)^{\beta}}{\Gamma(1-\beta)}\,{}^{\tau}_{\alpha}K^{\beta}_{p^+}\left(r^{\tau+\alpha}-p^{\tau+\alpha}\right)^{-\beta} + {}^{\tau}_{\alpha}K^{\beta}_{p^+}\left({}^{\tau}_{\alpha}K^{1-\beta}_{p^+}\left(r^{-\tau}T_{\alpha}\phi(r)\right)\right) \\
= \; & \frac{\phi(p)(\tau+\alpha)^{\beta}}{\Gamma(1-\beta)}\frac{1}{\Gamma(\beta)}\int_{p}^{r}\left(\frac{r^{\tau+\alpha}-w^{\tau+\alpha}}{\tau+\alpha}\right)^{\beta-1}\left(w^{\tau+\alpha}-p^{\tau+\alpha}\right)^{-\beta}w^{\tau}d_{\alpha}w \\
& +{}^{\tau}_{\alpha}K_{p^+}\left(r^{-\tau}T_{\alpha}\phi(r)\right) \\
= \; & \frac{\phi(p)(\tau+\alpha)}{\Gamma(\beta)\Gamma(1-\beta)}\int_{p}^{r}\left(r^{\tau+\alpha}-w^{\tau+\alpha}\right)^{\beta-1}\left(w^{\tau+\alpha}-p^{\tau+\alpha}\right)^{-\beta}w^{\tau+\alpha-1}dw+\phi(r)-\phi(p) \\
= \; & \frac{\phi(p)(\tau+\alpha)}{\Gamma(\beta)\Gamma(1-\beta)}\int_{p}^{r}\left(r^{\tau+\alpha}-w^{\tau+\alpha}\right)^{\beta-1}\left(w^{\tau+\alpha}-p^{\tau+\alpha}\right)^{-\beta}w^{\tau+\alpha-1}dw+\phi(r)-\phi(p).
\end{aligned}
\tag{29}
$$

Changing variables to u defined by, $w^{\tau+\alpha}=p^{\tau+\alpha}+(r^{\tau+\alpha}-p^{\tau+\alpha})u$, in the inner integral, we get from (29):

$$
\begin{aligned}
{}^{\tau}_{\alpha}K^{\beta}_{p^+}\,{}^{\tau}_{\alpha}T^{\beta}_{p^+}\phi(r) &= \frac{\phi(p)}{\Gamma(\beta)\Gamma(1-\beta)}\int_{0}^{1}u^{-\beta}(1-u)^{\beta-1}du+\phi(r)-\phi(p) \\
&= \frac{\phi(p)}{\Gamma(\beta)\Gamma(1-\beta)}\Gamma(\beta)\Gamma(1-\beta)+\phi(r)-\phi(p) \\
&= \phi(r).
\end{aligned}
$$

Which is the required proof. □

Theorem 11. ***(Linearity.)*** *The generalized conformable fractional derivative operators are linear on its domain, that is:*

$$
\begin{aligned}
{}^{\tau}_{\alpha}T^{\beta}_{p^+}\left(\mu_1\phi_1+\mu_2\phi_2\right) &= \mu_1\,{}^{\tau}_{\alpha}T^{\beta}_{p^+}\phi_1+\mu_2\,{}^{\tau}_{\alpha}T^{\beta}_{p^+}\phi_2, \\
{}^{\tau}_{\alpha}T^{\beta}_{q^-}\left(\mu_1\phi_1+\mu_2\phi_2\right) &= \mu_1\,{}^{\tau}_{\alpha}T^{\beta}_{q^-}\phi_1+\mu_2\,{}^{\tau}_{\alpha}T^{\beta}_{q^-}\phi_2,
\end{aligned}
$$

for all $\phi_1,\,\phi_2\in L_{\alpha}[p,q]$ *and* $\mu_1,\,\mu_2\in\mathbb{R}$.

Proof. The proof is similar to the proof of Theorem 7. □

Theorem 12. ***(Semigroup Property).*** *Let* $\phi:[p,q]\subseteq[0,\infty)\to\mathbb{R}$ *be a conformable integrable function. Then for* $0<\beta_1<1$, $0<\beta_2<1$ *and* $\alpha\in(0,1]$ *we have:*

$$
\begin{aligned}
&{}^{\tau}_{\alpha}T^{\beta_1}_{p^+}\,{}^{\tau}_{\alpha}T^{\beta_2}_{p^+}\phi(r) = {}^{\tau}_{\alpha}T^{\beta_1+\beta_2}_{p^+}\phi(r) \\
&{}^{\tau}_{\alpha}T^{\beta_1}_{q^-}\,{}^{\tau}_{\alpha}T^{\beta_2}_{q^-}\phi(r) = {}^{\tau}_{\alpha}T^{\beta_1+\beta_2}_{q^-}\phi(r).
\end{aligned}
$$

Proof. The proof is similar to the proof of Theorem 6. □

We consider an example to illustrate the results. We shall find the generalized conformable fractional derivative of the power function and explore the response for different values of α, β, λ and τ.

Example. Consider the function $\phi(r) = r^{\lambda}$, $\lambda \in \mathbb{R}$, $r \geq 0$. Then for $0 < \beta < 1$, $\alpha \in (0, 1]$ we have

$$ {}_{\alpha}^{\tau}T_{0^+}^{\beta}\phi(r) = \frac{\Gamma(1+\frac{\lambda}{\tau+\alpha})(\tau+\alpha)^{\beta-1}}{\Gamma(1-\beta+\frac{\lambda}{\tau+\alpha})} r^{-\beta(\tau+\alpha)+\lambda}. $$

Proof. The proof is simple by taking $p = 0$ in the above Definition 3, we get:

$$ \begin{aligned} {}_{\alpha}^{\tau}T_{0^+}^{\beta} r^{\lambda} &= \frac{r^{-\tau}}{\Gamma(1-\beta)} T_{\alpha} \int_0^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{-\beta} w^{\lambda} w^{\tau} d_{\alpha} w \\ &= \frac{r^{-\tau}(\tau+\alpha)^{\beta}}{\Gamma(1-\beta)} T_{\alpha} \int_0^r \left(r^{\tau+\alpha} - w^{\tau+\alpha} \right)^{-\beta} w^{\lambda} w^{\tau+\alpha-1} dw. \end{aligned} \tag{30} $$

Making the substitution $w^{\tau+\alpha} = ur^{\tau+\alpha}$ in (30) and using the relations

$$ \int_0^1 u^{a-1} (1-u)^{b-1} du = B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \text{ and } \Gamma(t+1) = t\Gamma(t), $$

we get the required result. □

Remark 2. It is interesting to note that for $\alpha = 1$, $\beta = 1$ and $\tau = 0$, we obtain ${}_{1}^{0}T_{0^+}^{1} r^{\lambda} = \lambda r^{\lambda-1}$, as one would expect the ordinary derivative.

4. Applications to integral equations and fractional differential equations

As mentioned above that our newly obtained fractional operators generalize R-L operators, Hadamard operators, Katugampula operators, which have remarkable applications in various fields [5–8, 10, 11]. One of the possible applications has been given below (after this section) in the form of open problem related to image denoising. Moreover in the following results, we apply our operators to the field of integral equations and fractional differential equations. we observe that our integral operator is a kind of Volterra integral operator and this can be used as a solution of the non-linear problem given below. Further we prove the existence and uniqueness of that solution.

4.1. Equivalence Between the Generalized Non-Linear Problem and the Volterra Integral Equation.

Consider the non-linear fractional differential equation of order $\beta \in (0, 1)$.

$$ {}_{\alpha}^{\tau}T_{p^+}^{\beta}\phi(r) = f\left(r, \phi(r)\right), \quad r \in [p, q]. \tag{31} $$

Where $f : [p,q] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with respect to all its arguments. We seek condition that guarantee the existence and uniqueness of solution to the problem (31) in the set of functions

$$A = \left\{ \phi \in C([p,q]) : {}_{\alpha}^{\tau}T_{p^+}^{\beta}\phi(r) \in C([p,q]) \right\}.$$

First, let us observe that, for $\phi \in C([p,q])$, the problem (31) is equivalent to the problem of finding solution to the following Volterra integral equation

$$\phi(r) = \frac{1}{\Gamma(\beta)} \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} f\left(w, \phi(w)\right) w^{\tau} d_{\alpha}w. \tag{32}$$

Indeed, if $\phi \in C([p,q])$ satisfies (31), then applying operator ${}_{\alpha}^{\tau}K_{p^+}^{\beta}$ to the both sides of (31), and applying relation (28) we obtain equation (32). Conversely, taking $r \to p^+$ and applying operator ${}_{\alpha}^{\tau}T_{p^+}^{\beta}$ to both sides of (32), using the relation (27), we arrive to problem (31).

4.2. **Existence and Uniqueness of Solution for the Non-Linear Problem.** In the following theorem the existence and uniqueness of solution to the problem (31) are proved.

Theorem 13. *Let $f : [p,q] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and Lipschitz with respect to the second variable, i.e.,*

$$|f(r,x_1) - f(r,x_2)| < L|x_1 - x_2| \tag{33}$$

for all $r \in [p,q]$ and for all $x_1, x_2 \in \mathbb{R}$, $L > 0$. Then the problem (31) possesses a unique solution.

Proof. We start by showing that for the problem (31), there exists a unique solution $\phi \in C([p,q])$. Let us recall that the problem (31) is equivalent to the problem of finding solutions to the Volterra integral equation (32). This allows us to use the well known method for nonlinear Volterra integral equations, where first we prove existence and uniqueness of solutions on a subinterval of $[p,q]$.
Let us choose $p < r_1 < q$ to be such that the following condition is satisfied

$$0 < L\frac{(\tau+\alpha)^{-\beta}}{\Gamma(\beta+1)}(r_1^{\tau+\alpha} - p^{\tau+\alpha})^{\beta} < 1. \tag{34}$$

We shall prove the existence of a unique solution ϕ to (32) on the subinterval $[p, r_1] \subseteq [p,q]$. Let us define the following integral operator, $S : C[p,q] \to C[p,q]$, by:

$$S\phi(r) = \frac{1}{\Gamma(\beta)} \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} f\left(w, \phi(w)\right) w^{\tau} d_{\alpha}w. \tag{35}$$

Note that S is well defined and is a bounded operator as proved in Theorem 8. Using Theorem 8 and condition (33) we get:

$$\begin{aligned}
\|S\phi_1 - S\phi_2\|_{C([p,r_1])} &= \left\| \frac{1}{\Gamma(\beta)} \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} \left(f\left(w, \phi_1(w)\right) - f\left(w, \phi_2(w)\right) \right) w^\tau d_\alpha w \right\|_{C([p,r_1])} \\
&= \left\| {}^\tau_\alpha K^\beta_{p^+} \left(f(w, \phi_1(w)) - f(w, \phi_2(w)) \right) \right\|_{C([p,r_1])} \\
&\leq L \left\| {}^\tau_\alpha K^\beta_{p^+} \left(\phi_1(w) - \phi_2(w) \right) \right\|_{C([p,r_1])} \\
&\leq L \frac{|(\tau+\alpha)^{-\beta}|}{\beta+1} (q^{\tau+\alpha} - p^{\tau+\alpha})^\beta \left\| \phi_1(w) - \phi_2(w) \right\|_{C([p,r_1])},
\end{aligned}$$

and because condition (34) is satisfied, by the Banach fixed point theorem, there exists a unique solution $\phi^{*1} \in C([p, r_1])$ to the equation (32) on the interval $[p, r_1]$. Moreover, if we define the sequence $\phi^1_m(r) := S^m \phi_0(r)$, for $m = 1, 2, 3, ...$,

$$S^m \phi_0(r) = \frac{1}{\Gamma(\beta)} \int_p^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} f\left(w, S^{m-1}\phi_0(w)\right) w^\tau d_\alpha w \tag{36}$$

then, again, by the Banach fixed point theorem, we obtain the solution ϕ^{*1} as a limit of the sequence ϕ^1_m, i.e.,

$$\lim_{m\to\infty} \left\| \phi^1_m - \phi^{*1} \right\|_{C([p,r_1])} = 0. \tag{37}$$

Now, let us choose $r_2 = r_1 + h_1$, with $h_1 > 0$ such that $r_2 < q$ and

$$0 < L \frac{(\tau+\alpha)^{-\beta}}{\Gamma(\beta+1)} (r_2^{\tau+\alpha} - r_1^{\tau+\alpha})^\beta < 1. \tag{38}$$

Consider the interval $[r_1, r_2]$ and write equation (32) in the form of:

$$\begin{aligned}
\phi(r) &= \frac{1}{\Gamma(\beta)} \int_{r_1}^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} f\left(w, \phi(w)\right) w^\tau d_\alpha w \\
&+ \frac{1}{\Gamma(\beta)} \int_p^{r_1} \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} f\left(w, \phi(w)\right) w^\tau d_\alpha w.
\end{aligned} \tag{39}$$

Because on the interval $[p, r_1]$, equation (39) possesses a unique solution, we can rewrite (39) as follows:

$$\phi(r) = \phi_{01}(r) + \frac{1}{\Gamma(\beta)} \int_{r_1}^r \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} f\left(w, \phi(w)\right) w^\tau d_\alpha w \tag{40}$$

where

$$\phi_{01}(r) = \frac{1}{\Gamma(\beta)} \int_p^{r_1} \left(\frac{r^{\tau+\alpha} - w^{\tau+\alpha}}{\tau+\alpha} \right)^{\beta-1} f\,(w, \phi(w))\, w^\tau d_\alpha w. \tag{41}$$

By the same argument as before, we prove that there exists a unique solution $\phi^{*2} \in C([r_1, r_2])$ to equation (32) on $[r_1, r_2]$. Repeating the process as above, choosing $r_k = r_{k-1} + h_{k-1}$ such that $h_{k-1} > 0, t_k < q$,

$$0 < L \frac{(\tau+\alpha)^{-\beta}}{\Gamma(\beta+1)} (r_k^{\tau+\alpha} - r_{k-1}^{\tau+\alpha})^\beta < 1. \tag{42}$$

We see that equation (32) possesses a solution $\phi^{*k} \in C([\phi_{k-1}, \phi_k])$ on each interval $[\phi_{k-1}, \phi_k]$, $(k = 1, ..., l)$, where $p = \phi_0 < \phi_1 < \;\; < \phi_l = q$ and we conclude that for problem (31), there exists a unique solution $\phi \in C([p, q])$.
It remains to prove that $\phi \in A$, i.e., we need to show that ${}^\tau_\alpha T^\beta_{0^+}\phi(r) \in C([p, q])$. Recall that our solution ϕ can be approximated by the sequence $\phi_m(r) = S_m\phi_0(r)$, i.e.,

$$\lim_{m\to\infty} \|\phi_m - \phi\|_{C([p,q])} = 0 \tag{43}$$

with the choice of certain ϕ_m on each interval $[p, r_1], \ . \ . \ . \ , [r_{l-1}, q]$. Using (31) and the Lipschitz type condition (33) we have:

$$\left\| {}^\tau_\alpha T^\beta_{p^+}\phi_m - {}^\tau_\alpha T^\beta_{p^+}\phi \right\|_{C([p,q])} = \|f(r, \phi_m) - f(r, \phi)\|_{C([p,q])} \leq L\,\|\phi_m - \phi\|_{C([p,q])}$$

Then taking $m \to \infty$, we get

$$\lim_{m\to\infty} \left\| {}^\tau_\alpha T^\beta_{p^+}\phi_m - {}^\tau_\alpha T^\beta_{p^+}\phi \right\|_{C([p,q])} = 0$$

Since ${}^\tau_\alpha T^\beta_{p^+}\phi_m(r) = f(r, \phi_m(r))$ is continuous on $[p, q]$ we have that ${}^\tau_\alpha T^\beta_{p^+}$ belongs to the space $C([p, q])$. This completes the proof. □

5. Conclusions

According to the open problem asked by T. Abdeljawad in [1], "Is it possible to iterate the conformable integral of order $0 < \alpha \leq 1$", we conclude that yes it is possible to iterate the conformable integral of order $0 < \alpha \leq 1$. For this, when we take the conformable integral operator $\int_p^r d_\alpha t$ and apply the Riemann-Liouville approach, we get Riemann-Liouville type conformable fractional operators, where when $\alpha \to 0$ we get Hadamard fractional operators. Here, just for the sack of generalizations, we apply the Riemann-Liouville approach to the conformable integral operator $\int_p^r t^\tau d_\alpha t$ instead of $\int_p^r d_\alpha t$. As a result we get a more generalized conformable fractional operator, i.e the generalization of all the Katugampola operators, Riemann-Liouville fractional operators, Riemann-Liouville type conformable fractional operators, Hadamard fractional operators and so on.

Open Problem.

In [11], Hamid A. Jalab and Rabha W. Ibrahim have used a generalized Katugampola integral operator for image denoising. They have used this generalized operator of two parameters to construct fractional integral mask. Since our newly defined generalized conformable integral operators are of Katugampola type and contain three parameters. The questions arises that is it possible to follow the approach of Hamid A. Jalab and Rabha W. Ibrahim and use the generalized conformable fractional integral operators to construct fractional integral mask and use it for image denoising?.

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