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# Identification of two classes of planar septic Pythagorean hodograph curves

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## Abstract

Polynomial Pythagorean hodograph (PH) curves have a polynomial arc length function and rational offsets, which distinguish PH curves from general polynomial parametric curves. However, these algebraic properties can hardly be used directly for the identification of PH curves. In order to determine whether or not a given septic planar polynomial curve is a PH curve, Zheng *et al.* (2016) studied a class of septic curves' geometric properties and proposed an efficient algorithm. In this paper, we further complete their work on Bézier control polygons of septic PH curves. We point out that there are three classes of septic PH curves according to different factorizations of their derivatives. Except the first class which has been studied, geometric characteristics of the other two classes are proposed. By introducing auxiliary points, the results are in terms of angles and lengths of legs of their Bézier control polygons. Moreover, we give feasible methods for the construction of auxiliary points, including various degenerate cases.

*Key words:* Bézier curve; Pythagorean hodograph; Control polygon; Arc length; Geometric characteristic

## 1. Introduction

Polynomial parametric curves are the most popular tools for modeling in computer aided design. Pythagorean hodograph (PH) curves [1] are an important subclass of these curves. As incorporating special algebraic structures in their derivatives, PH curves have polynomial arc-length functions and rational offset curves, which distinguish them from general polynomial curves. These properties also make them an elegant solution of a variety of difficult issues arising in lots of practical applications, such as NC milling, real-time motion control, railway design, shape blending, etc.

Many recent works have used PH curves as a specific tool for solving various problems, where

Hermite interpolation problem is usually involved. Hence, there are a lot of researches on the construction of PH curves in Hermite interpolation problem with diverse boundary conditions. Farouki and Neff [2] proposed a method using PH quintics for the problem of  $C^1$  Hermite interpolation. Because there are always four distinct PH quintics for any given  $C^1$  Hermite data [2], Moon *et al.* [3] discussed rules for selection of the "good" result. Meek and Walton [4] considered the problem of  $G^1$  Hermite interpolation using PH cubics. Byrtus and Bastl [5] extends the work of Meek and Walton [4] and presented a thorough analysis of the number and the quality of the interpolants. Kosinka and Jüttler [6] further addressed the problem in Minkowski space. Jüttler introduced a  $G^2[C^1]$  Hermite interpolation using PH septics, and gave a method for approximation of an arbitrary curve with PH septics. By characteriz-

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ing PH curves by roots of their hodographs in the complex representation, Kong *et al.* [7] solved  $C^1$  Hermite interpolation problem using a special type of PH curves. Farouki [8] addressed the problem of designing smoothly rounded right-angle corners with PH curves of odd degrees.

On the other hand, users in "reverse engineering" want to identify whether or not a given polynomial curve, specified by Bézier control points, is a PH curve [9]. Necessary and sufficient conditions for any planar parametric curve to be a PH curve were proposed based on an algebraic structure concerning Pythagorean triples in unique factorization domains [2,10]. Although control polygons provide an intuitive and efficient way to manipulate "free-form" curves and surfaces, "internal structure" of PH curves are not directly available from their control polygons.

Farouki [1] proposed an intuitive geometric condition on the Bézier control polygon for a cubic curve to be a PH curve, which are conditions in terms of lengths and inner angles of its control polygon legs. To obtain more geometric freedom for practical design problems, the study on geometric constraints on the control polygon is proceeded to higher order PH curves [11–14] and indirect-PH curves [15]. Furthermore, theories on PH curves are generalized to higher dimensional curves [16–20] and rational polynomial curves [21,22]. More details can be found in remarkable publications [23–25] and references therein.

However, the work on septic PH curves is still not completed, Zheng *et al.* [13] proposed geometric characteristics for only one class of septic PH curves. As presented in the following sections, septic PH curves can be classified into three classes, referred as class I, class II, and class III septic PH curves, according to different factorizations of their derivatives in complex form. Septic PH curves can be regular or irregular curves. Class I septic PH curves, which has been studied by Zheng *et al.* [13], are all regular curves, while there may be cusps on a class II or class III curve. Meanwhile, geometric characteristics of these three classes might be quite different. In this paper, characteristics of the latter two classes of septic PH curves are studied, respectively. The strategy to achieve these results is to represent a planar curve by a complex polynomial in Bernstein form. The complex expression allows for a clear and uniform analysis of geometric characteristics of septic PH curves.

The remainder of this paper is organized as fol-

lows. Section 2 introduces some fundamental aspects of PH curves in Bernstein form. In Section 3 and Section 4, geometric characteristics of two classes of septic PH curves are presented, respectively. Section 5 gives an example for the construction of  $G^2$  rounded corner curves using these two classes of septic PH curves. Finally, in Section 6 we conclude the paper.

## 2. Preliminaries

Let  $\mathbb{R}$  and  $\mathbb{C}$  denote the sets of all real numbers and complex numbers, respectively. Throughout this paper, we will be denoting a complex number by a single bold character, *e.g.*,  $\mathbf{z}$ . Let  $\|\cdot\|$  denote the usual complex norm on  $\mathbb{C}$ . Following [11,23], we use the complex representation of  $\mathbb{R}^2$  because it will facilitate the computation in later analysis of planar PH curves. Thus, the expressions such as a planar point  $(x, y)$  and the complex number  $x + iy$  are interchangeable. Similarly, a planar parametric curve  $\mathbf{P}(t) = (x(t), y(t))$  can be identified with a complex-valued function  $\mathbf{P}(t) = x(t) + iy(t)$  and vice versa.

A complex number can be considered as a planar vector, for example, let  $\mathbf{z}_i = r_i e^{i\theta_i}$ ,  $i = 0, 1, 2$ , where  $r_i = \|\mathbf{z}_i\|$  is the norm of  $\mathbf{z}_i$ , and  $\theta_i = \arg \mathbf{z}_i$  is the argument of  $\mathbf{z}_i$ . If they satisfy the condition  $\mathbf{z}_1^2 = \mathbf{z}_0 \mathbf{z}_2$ , then we may separate the equation into two equivalent conditions on lengths and arguments, respectively, that is

$$\begin{aligned} r_1^2 &= r_0 r_2, \\ 2\theta_1 &= \theta_0 + \theta_2, \end{aligned}$$

which do not depend on computation of complex numbers. Let  $\mathbf{P}_i$ ,  $i = 0, 1, 2$ , be three points in the complex plane, we use the notation  $(\mathbf{P}_0 - \mathbf{P}_1)^2 = (\mathbf{P}_0 - \mathbf{P}_1) \cdot (\mathbf{P}_0 - \mathbf{P}_1)$ , and  $(\mathbf{P}_0 - \mathbf{P}_1) : (\mathbf{P}_1 - \mathbf{P}_2) = \frac{\mathbf{P}_0 - \mathbf{P}_1}{\mathbf{P}_1 - \mathbf{P}_2}$  in the following discussion.

Let  $B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$ ,  $i = 0, \dots, n$ , be Bernstein polynomials. A septic Bézier curve is defined by

$$\mathbf{P}(t) = \sum_{i=0}^7 B_i^7(t) \mathbf{P}_i, \quad 0 \leq t \leq 1, \quad (1)$$

where  $\mathbf{P}_i$ ,  $i = 0, \dots, 7$ , are control points. Such curves enjoy a lot of elegant properties, *e.g.*, convex hull property, geometric invariance, variational diminishing property, etc [26].

The polygon formed by consecutively connecting the control points  $\mathbf{P}_i$ ,  $i = 0, \dots, 7$ , with lines is called the *control polygon*. Let  $\Delta \mathbf{P}_i$  denote the first forward-difference of the  $i$ -th control point, *i.e.*,

129  $\Delta P_i = P_{i+1} - P_i$ , then the first derivative of the  
130 curve (1) can be represented as

$$P'(t) = 7 \sum_{i=0}^6 B_i^6(t) \Delta P_i. \quad (2)$$

131 Let  $x$  and  $y$  be real polynomials with respect to  
132 a parameter  $t \in [0, 1]$ , a planar curve  $P(t) = x(t) +$   
133  $\mathbf{i}y(t)$  is called a *Pythagorean hodograph (PH)* curve  
134 if and only if its hodograph  $P'(t) = x'(t) + \mathbf{i}y'(t)$   
135 satisfies the Pythagorean condition

$$x'^2(t) + y'^2(t) = \sigma^2(t),$$

136 for some real polynomial  $\sigma(t)$  [17]. Equivalently, a  
137 planar curve  $P(t)$  is a PH curve if and only if

$$P'(t) = w(t)[u(t) + \mathbf{i}v(t)]^2, \quad (3)$$

138 for some real polynomials  $u(t)$ ,  $v(t)$ ,  $w(t)$ , where  $u(t)$   
139 and  $v(t)$  are relatively prime [12].

140 For septic PH curves, we know that the degrees  
141 of  $u(t) + \mathbf{i}v(t)$  and  $w(t)$  are either 3 and 0, or 4 and  
142 1, or 2 and 2. Here we assume the real polynomials  
143  $u(t)$  and  $v(t)$  are relatively prime, thus the three  
144 cases of septic PH curves are mutually exclusive. In  
145 following discussion, we call them class I, class II,  
146 and class III septic PH curves, respectively. Class I  
147 septic PH curves are always regular, and they have  
148 the maximum number of **inflections** [10]. Geometric  
149 characteristics of class I septic PH curves have been  
150 studied by Zheng *et al.* [13], we extend their work  
151 by considering the latter two classes in this paper.

152 Following the sufficient and necessary condition  
153 of (3), we can write  $P'(t)$  in terms of Bernstein poly-  
154 nomials, where

$$w(t) = \sum_{i=0}^2 a_i B_i^2(t), \quad (4)$$

$$u(t) + \mathbf{i}v(t) = \sum_{i=0}^2 z_i B_i^2(t),$$

155 for class II septic PH curves, or

$$w(t) = \sum_{i=0}^4 a_i B_i^4(t), \quad (5)$$

$$u(t) + \mathbf{i}v(t) = \sum_{i=0}^1 z_i B_i^1(t),$$

156 for class III septic PH curves, where  $a_i \in \mathbb{R}$ ,  $z_j \in \mathbb{C}$ ,  
157  $i = 0, \dots, 4$ ,  $j = 0, 1, 2$ .

158 Notably, a class II or III septic PH curve may be  
159 an irregular curve, *i.e.*, there may be cusps on a class  
160 II or III septic curve. For a class II septic PH curve  
161  $P(t)$ , let  $\xi_i$ ,  $i = 0, 1$ , be roots of the real polynomial  
162  $w(t)$ , then they are either conjugate complex num-  
163 bers, or both real numbers. If they are conjugate

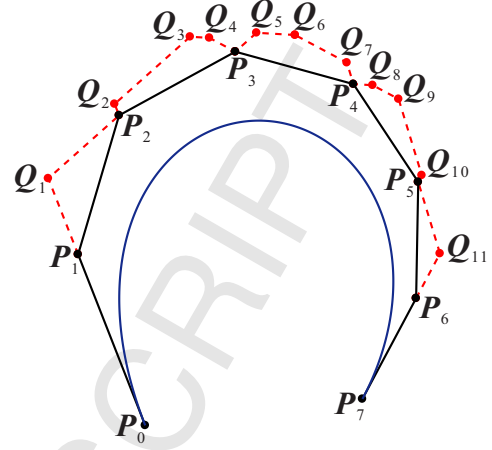


Fig. 1. Bézier control polygon and auxiliary points of a class II septic PH curve. We observe that  $P_0P_1 \parallel P_1Q_1 \parallel P_2Q_2$ ,  $Q_1P_2 \parallel Q_2Q_3 \parallel P_3Q_5$ ,  $Q_3Q_4 \parallel Q_5Q_6 \parallel P_4Q_8$ ,  $Q_4P_3 \parallel Q_6Q_7 \parallel Q_8Q_9$ ,  $Q_7P_4 \parallel Q_9Q_{10} \parallel P_5Q_{11}$ ,  $Q_{10}P_5 \parallel Q_{11}P_6 \parallel P_6P_7$ .

164 complex numbers, then the curve is regular, other-  
165 wise, the real numbers  $\xi_i$ ,  $i = 0, 1$ , are the param-  
166 eter values of the cusps. If  $w(t) = 1$  for a given septic  
167 PH curve, that is  $a_0 = a_1 = a_2 = 1$ , then the curve  
168 is actually a class I quintic PH curve [14]. In other  
169 words, if we do degree elevation, class I quintic PH  
170 curves can be classified as class II septic PH  
171 curves.

172 Similarly, there is even number of cusps for a class  
173 III septic PH curve but no more than four. If  $w(t) =$   
174 1, that is  $a_0 = a_1 = a_2 = a_3 = a_4 = 1$ , then it  
175 is a cubic PH curve. Moreover, if  $w$  is a polynomial  
176 of degree 2, then the curve is a class II quintic PH  
177 curve [14]. In other words, PH cubics and class II  
178 PH quintics can be classified as class III septic PH  
curves.

### 179 3. Class II septic PH curve

180 **Theorem 1** A planar septic curve with Bézier con-  
181 trol points  $P_i$ ,  $i = 0, \dots, 7$ , is a class II septic PH  
182 curve if and only if there are points  $Q_i$ ,  $i = 1, \dots, 11$ ,  
183 such that

$$\begin{aligned} & \Delta P_0 : 3(Q_1 - P_1) : 15(Q_2 - P_2) \\ & = 15(P_5 - Q_{10}) : 12(P_6 - Q_{11}) : \Delta P_6 \\ & = 3\Delta Q_3 : 2\Delta Q_5 : 3(Q_8 - P_4) \\ & = 3(P_3 - Q_4) : 2\Delta Q_6 : 3\Delta Q_8 \\ & = 12(P_2 - Q_1) : 15\Delta Q_2 : 40(Q_5 - P_3) \\ & = 40(P_4 - Q_7) : 15\Delta Q_9 : 12(Q_{11} - P_5), \end{aligned} \quad (6)$$

184 and they further satisfy

$$\begin{aligned}
 3(P_2 - Q_1)^2 &= 5(P_1 - P_0) \cdot (P_3 - Q_4), \\
 15(Q_3 - Q_2)^2 &= 32(Q_1 - P_1) \cdot (Q_7 - Q_6), \\
 4(Q_5 - P_3)^2 &= 9(Q_2 - P_2) \cdot (Q_9 - Q_8), \\
 15(Q_4 - Q_3)^2 &= 4(P_1 - P_0) \cdot (Q_{10} - P_5), \\
 25(Q_6 - Q_5)^2 &= 9(Q_1 - P_1) \cdot (P_6 - Q_{11}), \\
 15(P_4 - Q_8)^2 &= 4(Q_2 - P_2) \cdot (P_7 - P_6), \\
 4(Q_7 - P_4)^2 &= 9(P_3 - Q_4) \cdot (Q_{10} - P_5), \\
 15(Q_{10} - Q_9)^2 &= 32(Q_7 - Q_6) \cdot (P_6 - Q_{11}), \\
 3(Q_{11} - P_5)^2 &= 5(Q_9 - Q_8) \cdot (P_7 - P_6).
 \end{aligned} \tag{7}$$

**PROOF.** For a planar septic PH curve with Bézier control points  $P_i$ ,  $i = 0, \dots, 7$ , because there are  $a_i \in \mathbb{R}$ ,  $z_i \in \mathbb{C}$ ,  $i = 0, 1, 2$ , such that the polynomials  $w(t)$  and  $u(t) + iv(t)$  satisfy (4), we may expand (3) as

$$\begin{aligned}
 P'(t) &= a_0 z_0^2 (1-t)^6 \\
 &+ (4a_0 z_0 z_1 + 2a_1 z_0^2) (1-t)^5 t \\
 &+ [a_0 (4z_1^2 + 2z_0 z_2) + 8a_1 z_0 z_1 + a_2 z_0^2] (1-t)^4 t^2 \\
 &+ [4a_0 z_1 z_2 + 2a_1 (4z_1^2 + 2z_0 z_2) + 4a_2 z_0 z_1] (1-t)^3 t^3 \\
 &+ [a_0 z_2^2 + 8a_1 z_1 z_2 + a_2 (4z_1^2 + 2z_0 z_2)] (1-t)^2 t^4 \\
 &+ (2a_1 z_2^2 + 4a_2 z_1 z_2) (1-t) t^5 \\
 &+ a_2 z_2^2 t^6.
 \end{aligned}$$

By matching the coefficients of Bernstein polynomials with (2), we get following decompositions of the control polygon legs,

$$\begin{aligned}
 7\Delta P_0 &= a_0 z_0^2, \\
 42\Delta P_1 &= 4a_0 z_0 z_1 + 2a_1 z_0^2, \\
 105\Delta P_2 &= a_0 (4z_1^2 + 2z_0 z_2) + 8a_1 z_0 z_1 + a_2 z_0^2, \\
 140\Delta P_3 &= 4a_0 z_1 z_2 + 2a_1 (4z_1^2 + 2z_0 z_2) + 4a_2 z_0 z_1, \\
 105\Delta P_4 &= a_0 z_2^2 + 8a_1 z_1 z_2 + a_2 (4z_1^2 + 2z_0 z_2), \\
 42\Delta P_5 &= 2a_1 z_2^2 + 4a_2 z_1 z_2, \\
 7\Delta P_6 &= a_2 z_2^2.
 \end{aligned} \tag{8}$$

Therefore, we may introduce auxiliary points  $Q_i$ ,  $i = 1, \dots, 11$ , as follows,

$$\begin{aligned}
 Q_1 &= P_1 + \frac{a_1}{21} z_0^2 = P_2 - \frac{2a_0}{21} z_0 z_1, \\
 Q_2 &= P_2 + \frac{a_2}{105} z_0^2, \\
 Q_3 &= Q_2 + \frac{8a_1}{105} z_0 z_1, \\
 Q_4 &= Q_3 + \frac{2a_0}{105} z_0 z_2 = P_3 - \frac{4a_0}{105} z_1^2, \\
 Q_5 &= P_3 + \frac{a_2}{35} z_0 z_1, \\
 Q_6 &= Q_5 + \frac{a_1}{35} z_0 z_2,
 \end{aligned}$$

$$\begin{aligned}
 Q_7 &= Q_6 + \frac{2a_1}{35} z_1^2 = P_4 - \frac{a_0}{35} z_1 z_2, \\
 Q_8 &= P_4 + \frac{2a_2}{105} z_0 z_2, \\
 Q_9 &= Q_8 + \frac{4a_2}{105} z_1^2, \\
 Q_{10} &= Q_9 + \frac{8a_1}{105} z_1 z_2 = P_5 - \frac{a_0}{105} z_2^2, \\
 Q_{11} &= P_5 + \frac{2a_2}{21} z_1 z_2 = P_6 - \frac{a_1}{21} z_2^2,
 \end{aligned}$$

thus it is clear that (6) and (7) hold, and we have proved the necessity.

On the contrary, if there are points  $Q_i$ ,  $i = 1, \dots, 11$ , for a given septic Bézier curve, such that (6) and (7) hold, then we have to find  $a_i$  and  $z_i$ ,  $i = 0, 1, 2$ , to make the equation (4) hold. If  $\Delta P_6 \neq 0$  for the given Bézier curve, then we let  $a_2 = 1$ , and

$$\begin{aligned}
 z_0 &= \sqrt{105(Q_2 - P_2)}, \\
 z_1 &= \frac{\sqrt{105\Delta Q_8}}{2}, \\
 z_2 &= \sqrt{7\Delta P_6}, \\
 a_0 &= \frac{\Delta P_0}{15(Q_2 - P_2)}, \\
 a_1 &= \frac{Q_1 - P_1}{5(Q_2 - P_2)}.
 \end{aligned}$$

Using these values we can verify (8). Therefore, the equations of (4) hold and the curve is a class II septic PH curve.

However, it is possible that  $\Delta P_6 = 0$ , which means  $P_6$  and  $P_7$  coincide, and it gives  $a_2 = 0$ . In this case, we may let  $a_0 = 1$  if  $\Delta P_0 \neq 0$ , otherwise, we have  $a_1 = 1$ . Thus, the complex coefficients  $z_i$ ,  $i = 0, 1, 2$ , can be found according to the expressions of  $Q_i$ ,  $i = 1, \dots, 11$ .

Theorem 1 gives a necessary and sufficient condition for class II septic PH curves. Notably, although we use complex representation in (7), the conditions can be easily separated into constraints on angles and lengths of control polygon legs and auxiliary legs. Now we give feasible methods to identify whether or not a given septic curve is a class II septic PH curve, the key is the construction of all the auxiliary points. Regardless of the symmetric property of Bézier control polygon, we discuss five different cases according to positions of the first three control points.

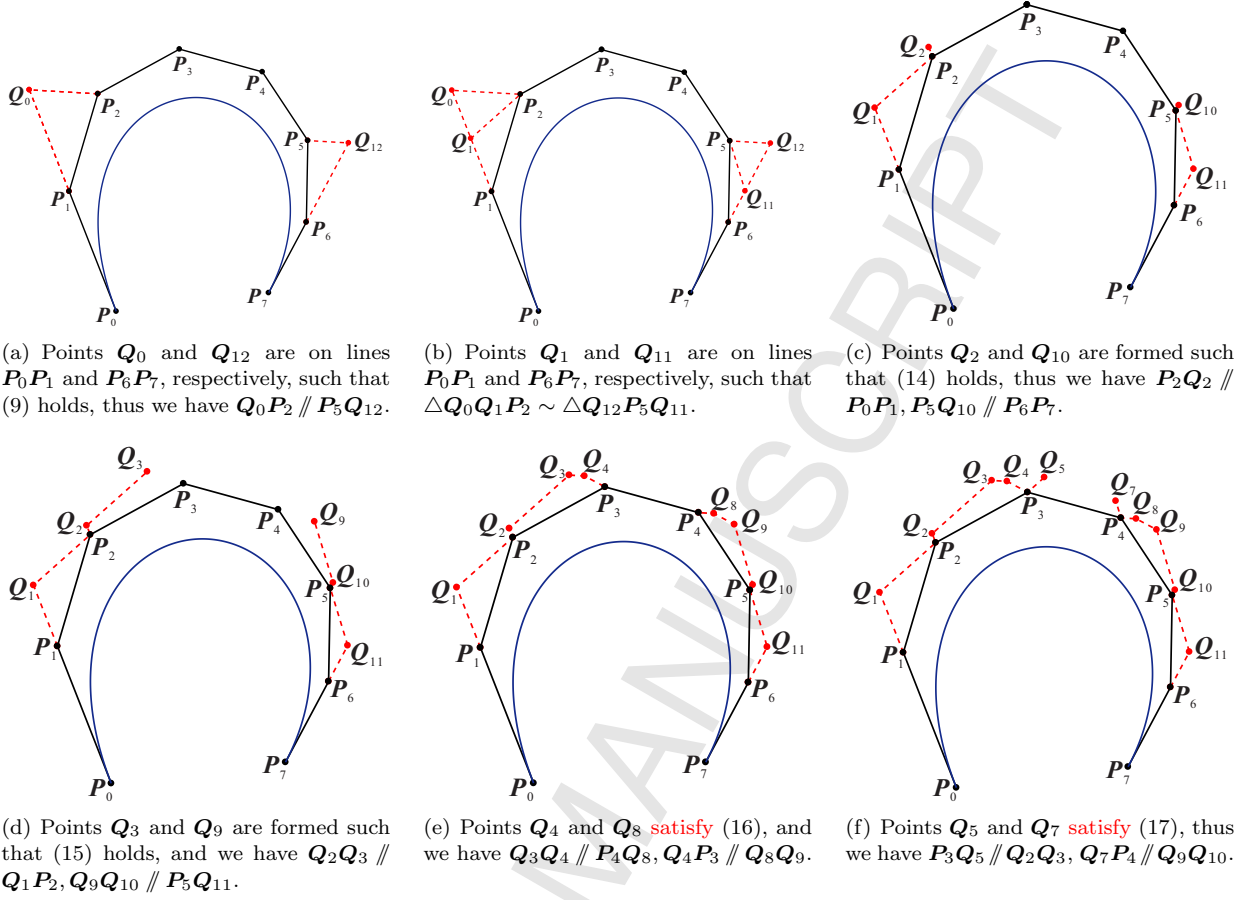


Fig. 2. The procedure of construction of auxiliary points for a class II septic PH curve.

3.1. Case 1: there are eight distinct control points, and points  $P_0, P_1, P_2$  are not co-linear

Generally, there are eight distinct Bézier control points for a septic polynomial curve. If a given curve  $P(t)$  is a septic PH curve, and the first three control points are not co-linear, then we have  $a_0 \neq 0, z_1 \neq 0$ . Because the symmetric property of Bézier control polygon, we also assume the control points  $P_5, P_6, P_7$  are not co-linear, thus we have  $a_2 \neq 0$ . Without loss of generality, we set  $a_2 = 1$ .

Let  $Q_0$  and  $Q_{12}$  be points on lines  $P_0P_1$  and  $P_6P_7$ , respectively, such that

$$\arg \frac{P_2 - Q_0}{\Delta P_0} = \arg \frac{\Delta P_6}{P_2 - Q_0}, \quad (9)$$

$$Q_0P_2 \parallel P_5Q_{12},$$

as shown in Fig. 2(a).

Because the complex numbers  $z_0^2$  and  $z_2^2$  determine the directions of  $\Delta P_0$  and  $\Delta P_6$ , respectively, the complex number  $z_0z_2$  determines the direction-

s of  $P_2 - Q_0$  and  $Q_{12} - P_5$ . Moreover, according to (8), there are points  $Q_1$  and  $Q_{11}$  on lines  $P_0P_1$  and  $P_6P_7$ , respectively, such that

$$Q_1 - P_1 = \frac{a_1}{21} z_0^2,$$

$$P_2 - Q_1 = \frac{2a_0}{21} z_0 z_1,$$

$$Q_{11} - P_5 = \frac{2a_2}{21} z_1 z_2,$$

$$P_6 - Q_{11} = \frac{a_1}{21} z_2^2.$$

Now we may derive  $\triangle Q_0Q_1P_2 \sim \triangle Q_{11}P_5Q_{12}$  and

$$\frac{\|\Delta Q_0\|}{\|P_5 - Q_{12}\|} = \frac{\|Q_0 - P_2\|}{\|Q_{11} - P_5\|},$$

$$\frac{\|\Delta Q_0\|}{\|P_5 - Q_{12}\|} = \frac{\|P_2 - Q_1\|}{\|Q_{11} - P_5\|}, \quad (10)$$

as shown in Fig. 2(b).

Let

$$\begin{aligned}
 k_0 &= \frac{Q_0 - P_1}{\Delta P_0}, \\
 k_1 &= \frac{P_6 - Q_{12}}{\Delta P_6}, \\
 c_0 &= \pm \frac{\|P_2 - Q_0\| \cdot \|Q_{12} - P_5\|}{\|\Delta P_0\| \cdot \|\Delta P_6\|}, \\
 c_1 &= \pm \frac{\|P_2 - Q_0\|^2}{\|\Delta P_0\| \cdot \|\Delta P_6\|},
 \end{aligned} \tag{11}$$

thus we have

$$\begin{aligned}
 \Delta Q_0 &= -(k_0 - \frac{a_1}{3a_0})\Delta P_0, \\
 \Delta Q_{11} &= -(k_1 - \frac{a_1}{3})\Delta P_6.
 \end{aligned}$$

According to the construction of the auxiliary points  $Q_i$ ,  $i = 1, \dots, 11$ , it is clear that

$$\left( \frac{\|P_2 - Q_{11}\|}{\|Q_{11} - P_5\|} \right)^2 = a_0^2 \left( \frac{\|\Delta P_0\|}{\|\Delta P_6\|} \right)^2,$$

so following (10) we can obtain

$$\begin{aligned}
 a_1^2 - 3(k_0 a_0 + k_1) a_1 + 9a_0(k_0 k_1 + c_0) &= 0, \\
 a_0(3k_0 - a_1)^2 + 9c_1 &= 0.
 \end{aligned} \tag{12}$$

By substituting the second equation into the first, we get a quartic equation with respect to  $a_1$ ,

$$\begin{aligned}
 a_1^4 - 9k_1 a_1^3 + 27k_1^2 a_1^2 - 27(k_0 c_1 + k_1^3) a_1 \\
 + 81c_1(k_0 k_1 + c_0) &= 0,
 \end{aligned}$$

which means the system (12) can be solved numerically. Notably, there are actually four equations due to different selection of signs of  $c_0$  and  $c_1$ , that is there are at most sixteen solutions for (12).

Now we give a method for the construction of auxiliary points as follows.

**Step 1** Let  $Q_0$  and  $Q_{12}$  be points on lines  $P_0 P_1$  and  $P_6 P_7$ , respectively, such that (9) holds, as shown in Fig. 2(a).

**Step 2** Let  $a_0, a_1$  be a pair of real solutions of the system (12). For each pair of  $a_0$  and  $a_1$ , let  $Q_1$  and  $Q_6$  be points on lines  $P_0 P_1$  and  $P_5 P_6$ , respectively, such that

$$\begin{aligned}
 Q_1 &= P_1 + \frac{a_1}{3a_0} \Delta P_0, \\
 Q_{11} &= P_6 - \frac{a_1}{3} \Delta P_6,
 \end{aligned} \tag{13}$$

thus the triangles  $\triangle Q_0 Q_1 P_2$  and  $\triangle Q_{12} P_5 Q_{11}$  are similar, as shown in Fig. 2(b).

**Step 3** Let  $Q_2$  and  $Q_{10}$  be points such that  $P_2 Q_2 \parallel P_0 P_1$ ,  $Q_{10} P_5 \parallel P_6 P_7$ , and they further satisfy

$$\begin{aligned}
 Q_2 &= P_2 + \frac{1}{15a_0} \Delta P_0, \\
 Q_{10} &= P_5 - \frac{a_0}{15} \Delta P_6,
 \end{aligned} \tag{14}$$

as shown in Fig. 2(c).

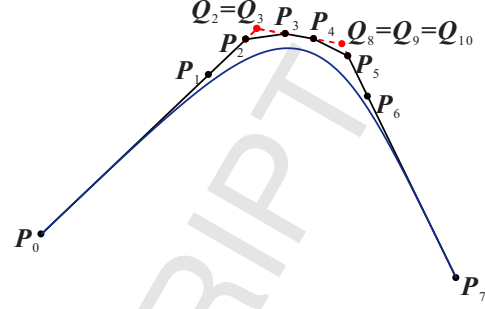


Fig. 4. Bézier control polygon and auxiliary points of a class II septic PH curve, where points  $P_0, P_1, P_2$  are co-linear.

**Step 4** Let  $Q_3$  and  $Q_9$  be points such that  $Q_2 Q_3 \parallel Q_1 P_2$ ,  $Q_9 Q_{10} \parallel P_5 Q_{11}$ , and they further satisfy

$$\begin{aligned}
 Q_3 &= Q_2 + \frac{4}{5a_0} (P_2 - Q_1), \\
 Q_9 &= Q_{10} - \frac{4a_1}{5} (Q_{11} - P_5),
 \end{aligned} \tag{15}$$

as shown in Fig. 2(d).

**Step 5** As shown in Fig. 2(e), let  $Q_4$  and  $Q_8$  be points such that  $Q_3 Q_4 \parallel P_4 Q_8$ ,  $Q_4 P_3 \parallel Q_8 Q_9$ , and they further satisfy

$$\begin{aligned}
 \arg \frac{\Delta Q_3}{\Delta P_0} &= \arg \frac{\Delta P_6}{\Delta Q_3}, \\
 \arg \frac{P_3 - Q_4}{\Delta Q_2} &= \arg \frac{\Delta Q_2}{\Delta P_0}.
 \end{aligned} \tag{16}$$

**Step 6** Let  $Q_5$  and  $Q_7$  be points such that  $P_3 Q_5 \parallel Q_2 Q_3$ ,  $Q_7 P_4 \parallel Q_9 Q_{10}$ , and they further satisfy

$$\begin{aligned}
 Q_5 &= P_3 + \frac{3}{8a_1} \Delta Q_2, \\
 Q_7 &= P_4 - \frac{3a_0}{8a_1} \Delta Q_9,
 \end{aligned} \tag{17}$$

as shown in Fig. 2(f).

**Step 7** Finally, let  $Q_6$  be a point such that  $Q_5 Q_6 \parallel Q_3 Q_4$ ,  $Q_6 Q_7 \parallel Q_8 Q_9$ , as shown in Fig. 1.

Notably, the relation of the angles are implied in construction of the auxiliary points, so identification of a class II septic PH curve can be performed by testing the system of (7) using only the norms of the vectors.

### 3.2. Case 2: points $P_0, P_1, P_2$ are co-linear but distinct points

In this case, we can immediately derive  $a_0 \neq 0$ ,  $a_1 \neq 0$ ,  $z_1 = 0$ , thus the system (8) becomes

$$\begin{aligned}
 7\Delta P_0 &= a_0 z_0^2, \\
 42\Delta P_1 &= 2a_1 z_0^2,
 \end{aligned}$$

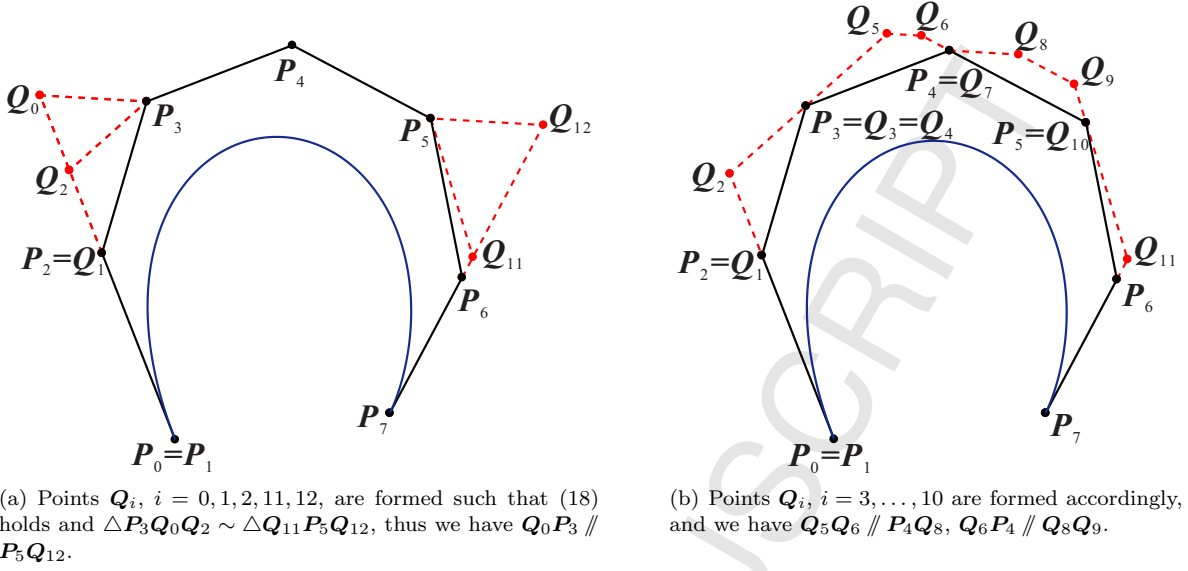


Fig. 3. Construction of auxiliary points for a class II septic PH curve, where  $P_0 = P_1$ .

$$\begin{aligned} 105\Delta P_2 &= 2a_0 z_0 z_2 + z_0^2, \\ 140\Delta P_3 &= 4a_1 z_0 z_2, \\ 105\Delta P_4 &= a_0 z_2^2 + 2z_0 z_2, \\ 42\Delta P_5 &= 2a_1 z_2^2, \\ 7\Delta P_6 &= z_2^2. \end{aligned}$$

$$\begin{aligned} 140\Delta P_3 &= 2a_1(4z_1^2 + 2z_0 z_2) + 4z_0 z_1, \\ 105\Delta P_4 &= 8a_1 z_1 z_2 + 4z_1^2 + 2z_0 z_2, \\ 42\Delta P_5 &= 2a_1 z_2^2 + 4z_1 z_2, \\ 7\Delta P_6 &= z_2^2. \end{aligned}$$

Therefore, we know points  $P_5$ ,  $P_6$ ,  $P_7$  are also co-linear, see Fig. 4 for an example.

Note that  $\Delta P_3$  has the direction of  $z_0 z_2$ ,  $\Delta P_2$  can be decomposed according to the vectors  $z_0 z_2$  and  $z_0^2$ ,  $\Delta P_4$  can be decomposed according to the vectors  $z_0 z_2$  and  $z_2^2$ . Thus, we know that  $Q_3$  is the intersecting point of lines  $P_0 P_1$  and  $P_3 P_4$ , and  $Q_8$  is the intersecting point of lines  $P_3 P_4$  and  $P_6 P_7$ .

Moreover, it is clear that  $Q_1 = P_2$ ,  $Q_2 = Q_3$ ,  $Q_4 = Q_5 = P_3$ ,  $Q_9 = Q_{10} = Q_8$ ,  $Q_{11} = P_5$ . Finally, since  $a_0 = \frac{\Delta P_0}{15(Q_2 - P_2)}$  and  $a_1 = \frac{Q_1 - P_1}{5(Q_2 - P_2)}$ , we may get  $Q_6 = Q_7$  be a point on the line  $P_3 P_4$  such that  $(Q_6 - P_3) : (P_4 - Q_6) = a_1 : a_0$ .

### 3.3. Case 3: points $P_0$ and $P_1$ coincide

Here we suppose  $P_0 = P_1$  and  $P_6 \neq P_7$  (the case  $P_0 = P_1$  and  $P_6 = P_7$  will be discussed later), as shown in Fig. 3. In this case, we can get  $a_0 = 0$ , thus the system (8) becomes

$$\begin{aligned} 7\Delta P_0 &= 0, \\ 42\Delta P_1 &= 2a_1 z_0^2, \\ 105\Delta P_2 &= 8a_1 z_0 z_1 + z_0^2, \end{aligned}$$

It is obvious that the point  $Q_1$  coincides with the control point  $P_2$ , and we further get  $Q_3 = Q_4 = P_3$ ,  $Q_7 = P_4$ , and  $Q_{10} = P_5$ . Let  $Q_0$  and  $Q_{12}$  be points on lines  $P_1 P_2$  and  $P_6 P_7$ , respectively, such that

$$\arg \frac{P_3 - Q_0}{\Delta P_1} = \arg \frac{\Delta P_6}{P_3 - Q_0}, \quad (18)$$

$$Q_0 P_3 \parallel P_5 Q_{12}.$$

Since  $\Delta P_1$  and  $\Delta P_6$  have the directions of  $z_0^2$  and  $z_2^2$ , respectively, both  $P_3 - Q_0$  and  $Q_{12} - P_5$  have the direction of  $z_0 z_2$ . Moreover, because  $P_3 - Q_2 = \frac{8a_1}{105} z_0 z_1$  and  $Q_{11} - P_5 = \frac{2}{21} z_1 z_2$ , the triangles  $\triangle P_3 Q_0 Q_2$  and  $\triangle Q_{11} P_5 Q_{12}$  are similar, and we have

$$\begin{aligned} \frac{\|P_3 - Q_0\|}{\|\Delta Q_{11}\|} &= \frac{\|Q_0 - Q_2\|}{\|Q_{12} - P_5\|}, \\ \frac{\|P_3 - Q_0\|}{\|\Delta Q_{11}\|} &= \frac{\|P_3 - Q_2\|}{\|Q_{11} - P_5\|}, \end{aligned} \quad (19)$$

as shown in Fig. 3(a). Moreover, following the construction of all the auxiliary points, we know

$$\left( \frac{\|P_3 - Q_2\|}{\|Q_{11} - P_5\|} \right)^2 = \frac{16|a_1| \|\Delta P_1\|}{25 \|\Delta P_6\|}.$$

Let

$$k_0 = \frac{Q_0 - P_2}{\Delta P_1},$$

$$k_1 = \frac{P_6 - Q_{12}}{\Delta P_6},$$

$$c_0 = \pm \frac{\|P_3 - Q_0\| \cdot \|Q_{12} - P_5\|}{\|\Delta P_1\| \cdot \|\Delta P_6\|},$$

$$c_1 = \pm \frac{\|P_3 - Q_0\|^2}{\|\Delta P_1\| \cdot \|\Delta P_6\|},$$

thus we can compute

$$Q_0 - Q_2 = (k_0 - \frac{1}{5a_1})\Delta P_1,$$

$$\Delta Q_{11} = -(k_1 - \frac{a_1}{3})\Delta P_6,$$

and derive a system of equations with respect to  $a_1$  following (19),

$$5k_0a_1^2 - (15k_0k_1 + 1 + 15c_0)a_1 + 3k_1 = 0,$$

$$16a_1(3k_1 - a_1)^2 + 75c_1 = 0.$$

By solving the system, we get

$$a_1 = \frac{48k_1(1 - 15c_0) + 1875k_0^2c_1 - 720k_0k_1^2}{16(1 - 15c_0)^2 - 240k_0k_1}. \quad (20)$$

Now we conclude our method to construct all the auxiliary points as follows.

Step 1 Let  $Q_0$  and  $Q_{12}$  be points on lines  $P_1P_2$  and  $P_6P_7$ , respectively, such that (18) holds.

Step 2 Compute  $a_1$  following (20), let

$$Q_2 = P_2 + \frac{1}{5a_1}\Delta P_1,$$

$$Q_{11} = P_6 - \frac{a_1}{3}\Delta P_6,$$

as shown in Fig. 3(a).

Step 3 Let  $Q_5$  and  $Q_9$  be points such that

$$Q_5 = P_3 + \frac{3}{8a_1}(P_3 - Q_2),$$

$$Q_9 = P_5 - \frac{4a_1}{5}(Q_{11} - P_5).$$

Step 4 Let  $Q_6$  and  $Q_8$  be points such that  $Q_5Q_6 \parallel P_4Q_8 \parallel Q_0P_3$ ,  $Q_6P_4 \parallel Q_8Q_9$ , and they further satisfy

$$\arg \frac{P_4 - Q_6}{Q_5 - P_3} = \arg \frac{Q_5 - P_3}{\Delta P_1},$$

as shown in Fig. 3(b).

#### 3.4. Case 4: points $P_0, P_1, P_2$ coincide

In this case, we have  $a_0 = a_1 = 0$ , and the system (8) becomes

$$7\Delta P_0 = 0,$$

$$42\Delta P_1 = 0,$$

$$105\Delta P_2 = z_0^2,$$

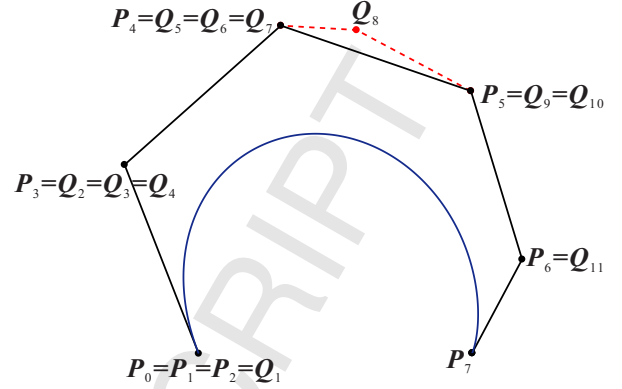


Fig. 5. Bézier control polygon and auxiliary points of a class II septic PH curve, where  $P_0 = P_1 = P_2$ .

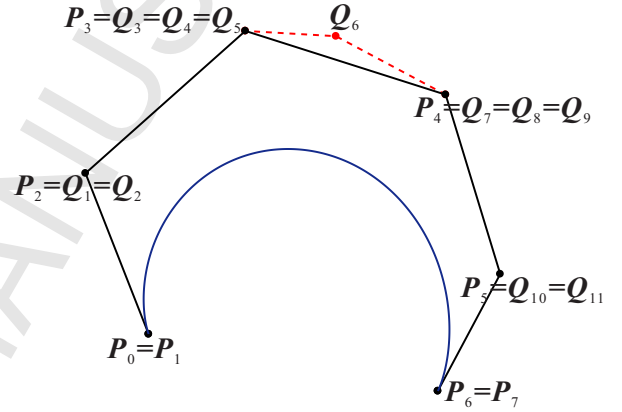


Fig. 6. Bézier control polygon and auxiliary points of a class II septic PH curve, where  $P_0 = P_1$ , and  $P_6 = P_7$ .

$$140\Delta P_3 = 4z_0z_1,$$

$$105\Delta P_4 = 4z_1^2 + 2z_0z_2,$$

$$42\Delta P_5 = 4z_1z_2,$$

$$7\Delta P_6 = z_2^2.$$

Therefore, we immediately get

$$Q_1 = P_1,$$

$$Q_2 = Q_3 = Q_4 = P_3,$$

$$Q_5 = Q_6 = Q_7 = P_4,$$

$$Q_9 = Q_{10} = P_5,$$

$$Q_{11} = P_6,$$

as shown in Fig. 5. Furthermore, let  $Q_8$  be a point such that

$$\arg \frac{\Delta P_5}{Q_8 - P_3} = \arg \frac{Q_8 - P_3}{\Delta P_1},$$

$$\arg \frac{P_5 - Q_8}{\Delta P_3} = \arg \frac{\Delta P_3}{\Delta P_2}.$$

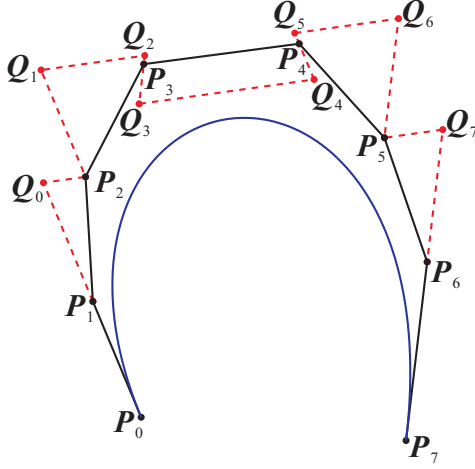


Fig. 7. Bézier control polygon and auxiliary points of a class III septic PH curve. There are eight distinct control points, and we have  $P_0P_1 \parallel P_1Q_0 \parallel P_2Q_1 \parallel Q_1Q_2 \parallel P_2P_3 \parallel P_3Q_2 \parallel Q_2Q_3 \parallel P_3P_4 \parallel P_4Q_3 \parallel Q_3Q_4 \parallel P_4P_5 \parallel P_5Q_4 \parallel Q_4Q_5 \parallel P_5P_6 \parallel P_6Q_5 \parallel Q_5Q_6 \parallel P_6P_7 \parallel P_7Q_6 \parallel Q_6Q_7$ .

3.5. Case 5: points  $P_0$  and  $P_1$  coincide, as well as points  $P_6$  and  $P_7$

In this case, we have  $a_0 = a_2 = 0$ , without loss of generality, we set  $a_1 = 1$ , thus the system (8) becomes

$$\begin{aligned} 7\Delta P_0 &= 0, \\ 42\Delta P_1 &= 2a_1 z_0^2, \\ 105\Delta P_2 &= 8a_1 z_0 z_1, \\ 140\Delta P_3 &= 2a_1 (4z_1^2 + 2z_0 z_2), \\ 105\Delta P_4 &= 8a_1 z_1 z_2, \\ 42\Delta P_5 &= 2a_1 z_2^2, \\ 7\Delta P_6 &= 0. \end{aligned}$$

Therefore, we immediately get

$$\begin{aligned} Q_1 &= Q_2 = P_2, \\ Q_3 &= Q_4 = Q_5 = P_3, \\ Q_7 &= Q_8 = Q_9 = P_5, \\ Q_{10} &= Q_{11} = P_5, \end{aligned}$$

as shown in Fig. 6. Furthermore, let  $Q_6$  be a point such that

$$\begin{aligned} \arg \frac{\Delta P_5}{Q_6 - P_3} &= \arg \frac{Q_6 - P_3}{\Delta P_1}, \\ \arg \frac{P_4 - Q_6}{\Delta P_2} &= \arg \frac{\Delta P_2}{\Delta P_1}. \end{aligned}$$

#### 4. Class III septic PH curves

**Theorem 2** A planar Bézier curve is a class III septic PH curve, if and only if there are points  $Q_i$ ,  $i = 0, \dots, 7$ , such that

$$\begin{aligned} 2\Delta P_0 &: 3(Q_0 - P_1) : 5(Q_1 - P_2) \\ &: 10(P_4 - Q_4) : 30(Q_5 - P_4) \\ &= 72(P_2 - Q_0) : 45\Delta Q_1 : 40\Delta Q_3 \\ &: 45\Delta Q_5 : 72(Q_7 - P_5) \\ &= 30(P_3 - Q_2) : 10(Q_3 - P_3) : 5(P_5 - Q_6) \\ &: 3(P_6 - Q_7) : 2\Delta P_6 \end{aligned} \quad (21)$$

and they further satisfy

$$\begin{aligned} 3(P_2 - Q_0)^2 &= 5\Delta P_0 \cdot (P_3 - Q_2), \\ 15\Delta Q_1^2 &= 32(Q_0 - P_1) \cdot (Q_3 - P_3), \\ 4\Delta Q_3^2 &= 9(Q_1 - P_2) \cdot (P_5 - Q_6), \\ 15\Delta Q_5^2 &= 32(P_4 - Q_4) \cdot (P_6 - Q_7), \\ 3(Q_7 - P_5)^2 &= 5(Q_5 - P_4) \cdot \Delta P_6. \end{aligned} \quad (22)$$

**PROOF.** According to (5), the hodograph of a class III septic PH curve has form

$$P'(t) = \sum_{i=0}^4 a_i B_i^4(t) [z_0(1-t) + z_1 t]^2,$$

which can be expanded as

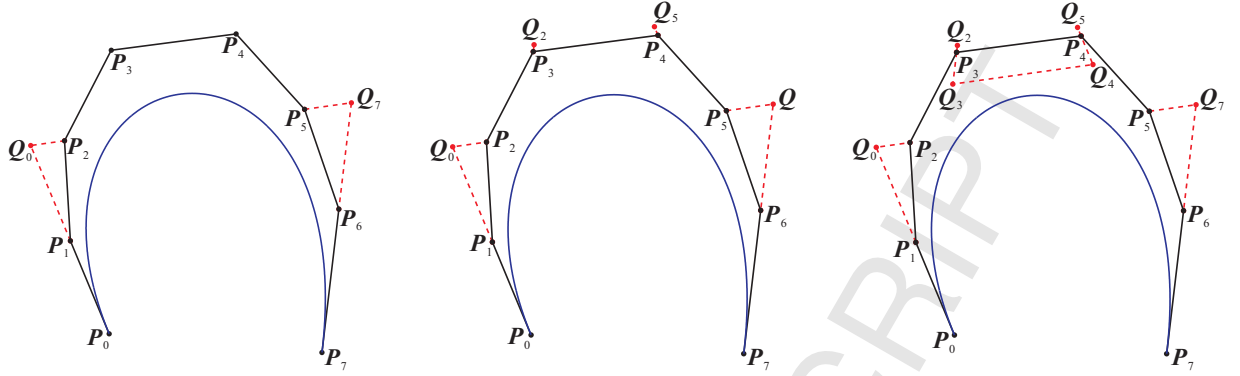
$$\begin{aligned} P'(t) &= a_0 z_0^2 (1-t)^6 \\ &+ (2a_0 z_0 z_1 + 4a_1 z_0^2) (1-t)^5 t \\ &+ (a_0 z_1^2 + 8a_1 z_0 z_1 + 6a_2 z_0^2) (1-t)^4 t^2 \\ &+ (4a_1 z_1^2 + 12a_2 z_0 z_1 + 4a_3 z_0^2) (1-t)^3 t^3 \\ &+ (6a_2 z_1^2 + 8a_3 z_0 z_1 + a_4 z_0^2) (1-t)^2 t^4 \\ &+ (4a_3 z_1^2 + 2a_4 z_0 z_1) (1-t) t^5 \\ &+ a_4 z_1^2 t^6. \end{aligned}$$

By matching the coefficients of Bernstein polynomials with (3), we get decompositions of the control polygon legs as follows:

$$\begin{aligned} 7\Delta P_0 &= a_0 z_0^2, \\ 42\Delta P_1 &= 2a_0 z_0 z_1 + 4a_1 z_0^2, \\ 105\Delta P_2 &= a_0 z_1^2 + 8a_1 z_0 z_1 + 6a_2 z_0^2, \\ 140\Delta P_3 &= 4a_1 z_1^2 + 12a_2 z_0 z_1 + 4a_3 z_0^2, \\ 105\Delta P_4 &= 6a_2 z_1^2 + 8a_3 z_0 z_1 + a_4 z_0^2, \\ 42\Delta P_5 &= 4a_3 z_1^2 + 2a_4 z_0 z_1, \\ 7\Delta P_6 &= a_4 z_1^2. \end{aligned} \quad (23)$$

Thus we define auxiliary points  $Q_i$ ,  $i = 0, \dots, 7$ :

$$Q_0 = P_1 + \frac{2a_1}{21} z_0^2 = P_2 - \frac{a_0}{21} z_0 z_1,$$



(a) Points  $Q_0$  and  $Q_7$  are on lines  $P_0P_1$  and  $P_6P_7$ , respectively, such that  $\arg \frac{P_2 - Q_0}{\Delta P_0} = \arg \frac{\Delta P_6}{P_2 - Q_0}$  and  $Q_0P_2 \parallel P_5Q_7$ .

(b) Points  $Q_2$  and  $Q_5$  satisfy  $P_3Q_2 \parallel P_6P_7$  and  $Q_5P_4 \parallel P_0P_1$ .

(c) Points  $Q_3$  and  $Q_4$  satisfy  $Q_3Q_4 \parallel Q_0P_2$ ,  $P_3Q_3 \parallel P_6P_7$ , and  $P_4Q_4 \parallel P_0P_1$ .

Fig. 8. The procedure of construction of auxiliary points for a class III septic PH curve.

$$\begin{aligned} Q_1 &= P_2 + \frac{2a_2}{35}z_0^2, \\ Q_2 &= Q_1 + \frac{8a_1}{105}z_0z_1 = P_3 - \frac{a_0}{105}z_1^2, \\ Q_3 &= P_3 + \frac{a_1}{35}z_1^2, \\ Q_4 &= Q_3 + \frac{3a_2}{35}z_0z_1 = P_4 - \frac{a_3}{35}z_0^2, \\ Q_5 &= P_4 + \frac{a_4}{105}z_0^2, \\ Q_6 &= Q_5 + \frac{8a_3}{105}z_0z_1 = P_5 - \frac{2a_2}{35}z_1^2, \\ Q_7 &= P_5 + \frac{a_4}{21}z_0z_1 = P_6 - \frac{2a_3}{21}z_1^2. \end{aligned}$$

Therefore, we can derive (21) and (22) immediately.

On the contrary, if (21) and the equations of (22) hold for a given septic Bézier curve, then we may compute the coefficients  $a_i$ ,  $z_j$ ,  $i = 0, \dots, 4$ ,  $j = 0, 1$ , and verify (5). For example, if  $\Delta P_6 \neq 0$ , then we may suppose that  $a_4 = 1$ , and let

$$\begin{aligned} a_0 : a_1 : a_2 : a_3 : a_4 &= 7\Delta P_0 : \frac{21}{2}(Q_0 - P_1) : \frac{35}{2}(Q_1 - P_2) \\ &: 35(P_4 - Q_4) : 105(Q_5 - P_4) \\ &= 21(P_2 - Q_0) : \frac{105}{8}\Delta Q_1 : \frac{35}{3}\Delta Q_3 \\ &: \frac{105}{8}\Delta Q_5 : 21(Q_7 - P_5) \\ &= 105(P_3 - Q_2) : 35(Q_3 - P_3) : \frac{35}{2}(P_5 - Q_6) \\ &: \frac{21}{2}(P_6 - Q_7) : 7\Delta P_6. \end{aligned}$$

Thus, we can further compute  $z_i$ ,  $i = 0, 1$ , that is  $z_1 = \pm \sqrt{7\Delta P_6}$ ,

and

$$z_0 = \frac{21\Delta P_5 - 2a_3z_1^2}{z_1}.$$

Now we propose feasible methods to construct auxiliary points for any given septic Bézier curve. General case and degenerate cases are introduced, respectively. There are six different degenerate cases, regardless of the symmetric property of Bézier control polygon.

4.1. General case: there are eight distinct control points

In general, there are eight distinct control points for any given septic Bézier curve. Thus the auxiliary points can be constructed by following scheme.

Step 1 Let  $Q_0$  and  $Q_7$  be points on lines  $P_0P_1$  and  $P_6P_7$ , respectively, such that  $Q_0P_2 \parallel P_5Q_7$ , and they further satisfy

$$\arg \frac{P_2 - Q_0}{\Delta P_0} = \arg \frac{\Delta P_6}{P_2 - Q_0},$$

as shown in Fig. 8(a).

Step 2 As shown in Fig. 8(b), let  $Q_2$ ,  $Q_5$  be points such that

$$\begin{aligned} Q_2 &= P_3 - \frac{P_2 - Q_0}{Q_7 - P_5} \cdot \frac{\Delta P_6}{15}, \\ Q_5 &= P_4 + \frac{Q_7 - P_5}{P_2 - Q_0} \cdot \frac{\Delta P_0}{15}. \end{aligned}$$

Step 3 Let  $Q_3$ ,  $Q_4$  be points such that

$$Q_3 = P_3 + \frac{Q_0 - P_1}{Q_5 - P_4} \cdot \frac{\Delta P_6}{50},$$

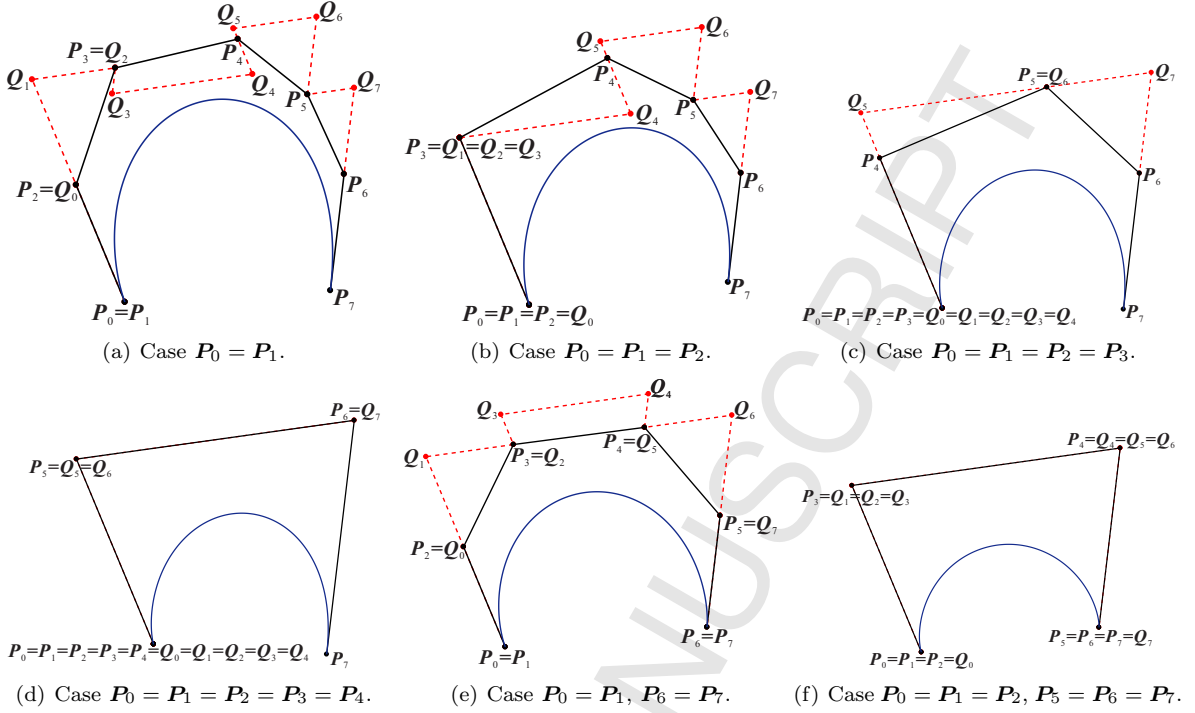


Fig. 9. Degenerate cases of class III septic PH curves.

$$\begin{aligned} Q_3Q_4 &\parallel Q_0P_2, \\ P_4Q_4 &\parallel P_0P_1, \end{aligned}$$

as shown in Fig. 8(c).

Step 4 Finally, let  $Q_1, Q_6$  be points such that  $P_2Q_1 \parallel P_0P_1$ ,  $Q_6P_5 \parallel P_6P_7$ , and  $Q_1Q_2 \parallel Q_5Q_6 \parallel Q_0P_2$ , as shown in Fig. 7.

Notably, the relations of the angles are implied in the construction of the auxiliary points, so identification of a general class III septic PH curve can be performed by verifying the equations in the system (22) using only the norms of the vectors.

#### 4.2. Degenerate cases

In this section, we discuss degenerate cases, where some of the control points coincide.

If the first two control points  $P_0$  and  $P_1$  coincide, then we can derive  $Q_0 = P_2$ ,  $Q_2 = P_3$ , and the other auxiliary points can be constructed by following steps (as shown in Fig. 9(a)):

Step 1 Let  $Q_1$  and  $Q_7$  be points on lines  $P_1P_2$  and  $P_6P_7$ , respectively, such that  $Q_1P_3 \parallel P_5Q_7$ , and they further satisfy

$$\arg \frac{P_3 - Q_1}{\Delta P_0} = \arg \frac{\Delta P_6}{P_3 - Q_1}.$$

Step 2 Let  $Q_3$  and  $Q_5$  be points such that

$$\begin{aligned} Q_3 &= P_3 + \frac{P_3 - Q_1}{Q_7 - P_5} \cdot \frac{\Delta P_6}{8}, \\ Q_5 &= P_4 + \frac{Q_7 - P_5}{P_2 - Q_0} \cdot \frac{4\Delta P_0}{25}. \end{aligned}$$

Step 3 Let point  $Q_4$  be the intersecting point of lines  $Q_3Q_4$  and  $Q_4P_4$ , and point  $Q_6$  be the intersecting point of lines  $Q_5Q_6$  and  $Q_6P_5$  such that

$$\begin{aligned} Q_3Q_4 &\parallel Q_5Q_6 \parallel Q_1P_3, \\ Q_4P_4 &\parallel P_1P_2, \\ Q_6P_5 &\parallel P_6P_7. \end{aligned}$$

If the first three control points  $P_0, P_1, P_2$  coincide, we can derive  $Q_0 = P_2$ ,  $Q_1 = Q_2 = Q_3 = P_3$ , and the other auxiliary points can be constructed by following steps:

Step 1 Let  $Q_7$  be a point on line  $P_6P_7$ , such that  $\arg \frac{Q_7 - P_5}{\Delta P_0} = \arg \frac{\Delta P_6}{Q_7 - P_5}$ .

Step 2 Let  $Q_4$  be a point such that  $P_3Q_4 \parallel P_5Q_7$  and  $Q_4P_4 \parallel P_2P_3$ .

Step 3 Let  $Q_5$  be a point such that  $Q_5 = P_4 + \frac{P_7 - P_5}{Q_4 - P_3} \cdot \frac{3\Delta P_2}{10}$ .

Step 4 Let  $Q_6$  be a point such that  $Q_5Q_6 \parallel P_5Q_7$  and  $Q_6P_5 \parallel P_6P_7$ , as shown in Fig. 9(b).

If the first four control points  $P_0, P_1, P_2$ , and  $P_3$  coincide, then we can derive  $Q_0 = Q_1 = Q_2 = Q_3 = Q_4 = P_3$ ,  $Q_6 = P_5$ . Moreover, it is clear  $Q_5$  and  $Q_7$  are points on  $P_3P_4$  and  $P_6P_7$ , respectively, such that  $Q_5, P_5$ , and  $Q_7$  are co-linear, and they further satisfy

$$\arg \frac{P_5 - Q_5}{\Delta P_3} = \arg \frac{\Delta P_6}{P_5 - Q_5},$$

as shown in Fig. 9(c).

If the control points  $P_0, P_1, P_2, P_3$ , and  $P_4$  coincide, then we have  $Q_0 = Q_1 = Q_2 = Q_3 = Q_4 = P_4$ ,  $Q_5 = Q_6 = P_5$ , and  $Q_7 = P_6$ , as shown in Fig. 9(d).

If the control points  $P_0$  and  $P_1$  coincide, as well as  $P_6$  and  $P_7$ , then we get  $Q_0 = P_2$ ,  $Q_7 = P_5$ ,  $Q_2 = P_3$ ,  $Q_5 = P_4$ , the other auxiliary points are constructed using the scheme as follows:

Step 1 Let  $Q_1$  and  $Q_6$  be points on lines  $P_1P_2$  and  $P_6P_7$ , respectively, such that  $Q_1P_3 \parallel P_4Q_6$  and

$$\arg \frac{Q_6 - P_4}{\Delta P_0} = \arg \frac{\Delta P_6}{Q_6 - P_5}.$$

Step 2 Let  $Q_3$  be a point such that

$$Q_3 = P_3 + \frac{Q_6 - P_4}{P_3 - Q_1} \cdot \frac{3\Delta P_1}{10}.$$

Step 3 Let  $Q_4$  be a point such that  $Q_3Q_4 \parallel Q_1P_3$  and  $Q_4P_4 \parallel P_5P_6$ , as shown in Fig. 9(e).

If the control points  $P_0, P_1, P_2$  coincide, as well as points  $P_5, P_6, P_7$ , then we have

$$\begin{aligned} Q_0 &= P_2, \\ Q_1 &= Q_2 = Q_3 = P_3, \\ Q_4 &= Q_5 = Q_6 = P_4, \\ Q_7 &= P_5, \end{aligned}$$

as shown in Fig. 9(f).

## 5. Construction of rounded corner

In this section, we consider the problem of construction of  $G^2$  rounded corners using septic PH curves [8], that is the initial and the final points, unit tangents, and curvatures are specified by

$$\begin{aligned} P_i &= (0, 0), & T_i &= (1, 0), & k_i &= 0, \\ P_f &= (1, 1), & T_f &= (0, 1), & k_f &= 0. \end{aligned}$$

For class II septic PH curves, the curvatures at two endpoints can be computed by

$$\begin{aligned} k_i &= \frac{4\text{Im}(\bar{z}_0 z_1)}{a_0 |z_0|^4}, \\ k_f &= \frac{4\text{Im}(\bar{z}_1 z_2)}{|z_2|^4}. \end{aligned}$$

Therefore, we immediately have  $z_1 = 0$ . Let  $\|\Delta P_0\| = \|\Delta P_6\| = l$ , we here assume  $a_0 = a_2 = 1$ , then  $z_0 = \pm\sqrt{7}l$ ,  $z_2 = \pm\sqrt{7}li$ . Furthermore, we may compute

$$\begin{aligned} a_1 &= \frac{105(P_6 - P_1) - (z_0^2 + 4z_0z_2 + z_2^2)}{5z_0^2 + 3z_0z_2 + 5z_2^2} \\ &= \frac{30 - (32 \pm 4\sqrt{2})l}{(10 \pm 3\sqrt{2})l}. \end{aligned}$$

Notably, the computation of  $a_1$  depends on the signs of  $z_0$  and  $z_2$ , i.e., plus is used if the signs are the same, otherwise subtraction is used.

Figure 10 shows the corner shape and its curvature profile for class II septic PH curves with  $l = \frac{n}{10}$ ,  $n = 1, \dots, 8$ . In this example, we use the same signs when  $z_0$  and  $z_1$  are computed, because different signs make self-intersecting curves. When  $l$  goes from 0.1 to 0.8, the curve becomes sharper, and the maximum curvature goes from 2.2835 to 5.6036.

For class III septic PH curves, we have

$$\begin{aligned} k_i &= \frac{2\text{Im}(\bar{z}_0 z_1)}{a_0 |z_0|^4}, \\ k_f &= \frac{2\text{Im}(\bar{z}_0 z_1)}{|z_1|^4}. \end{aligned}$$

If  $k_i = k_f = 0$ , then there must be some constant  $c \in \mathbb{R}$ , such that  $z_0 = cz_1 = 7li$ , which means the curve is a line segment. Therefore, we have to relax the curvature condition. Due to the symmetry of the rounded corner, we have  $\|\Delta P_0\| = \|\Delta P_6\| = l$ ,  $a_0 = a_4$ ,  $a_1 = a_3$ , which gives  $z_0^2 \cdot i = z_1^2$ , and  $P_6 - P_1 = (1 - l) + (1 - l)i$ . Let  $l$  and  $a_1$  be specified by user, thus we get

$$\begin{aligned} &\frac{2}{21}z_0z_1 + \frac{1}{105}z_1^2 + \frac{1}{105}z_0^2 \\ &+ (\frac{13}{105}z_0^2 + \frac{16}{105}z_0z_1 + \frac{13}{105}z_1^2)a_1 \\ &+ (\frac{2}{35}z_0^2 + \frac{3}{35}z_0z_1 + \frac{2}{35}z_1^2)a_2 \\ &= P_6 - P_1, \end{aligned}$$

by solving the equation, we get

$$a_2 = \frac{30 - [32 \pm 10\sqrt{2} + (26 \pm 16\sqrt{2})a_1]l}{(12 \pm 9\sqrt{2})l}.$$

Also, the signs in this expression depend on whether the signs of  $z_0$  and  $z_1$  are the same or not.

In Figure 11, we give an example of class III septic PH curves, where we use the same signs when  $z_0$  and  $z_1$  are computed. Here we let  $l = \frac{n}{10}$ ,  $n = 1, \dots, 5$ . When  $l$  is bigger, the curve becomes much closer to the corner, and its curvature profile is shown. When  $l = 0.2$ , the curve is the closest one to a

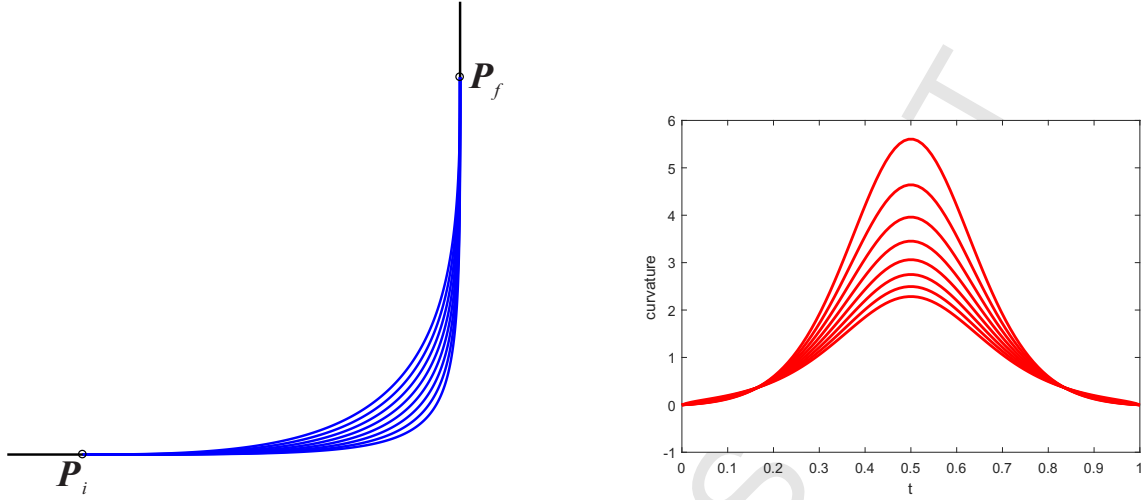


Fig. 10.  $G^2$  rounded corner using class II septic PH curves, with their curvature profiles (right). When  $l = \|\Delta P_0\| = \|\Delta P_6\|$  goes from 0.1 to 0.8, the curve becomes sharper, and the maximum curvature goes from 2.2835 to 5.6036.

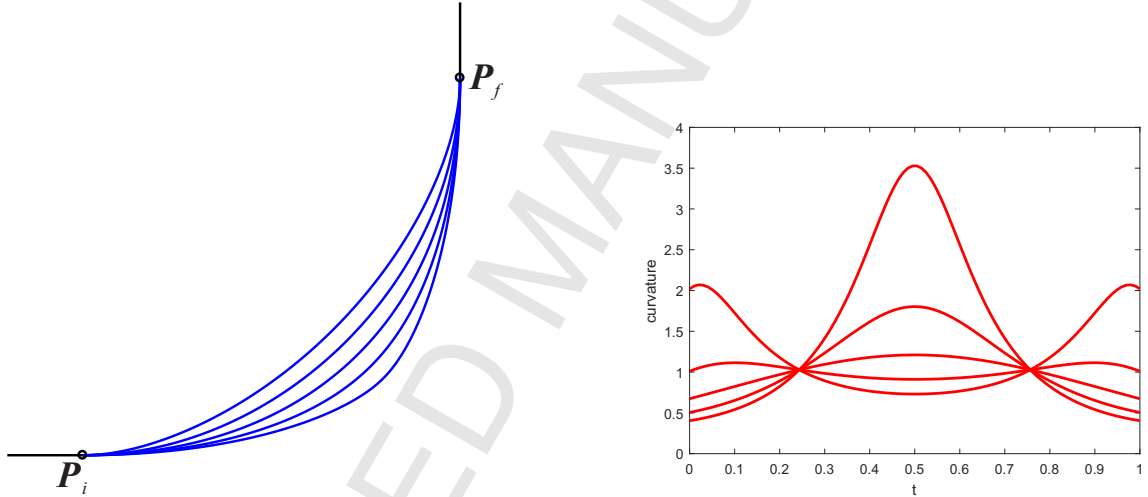


Fig. 11.  $C^1$  rounded corner using class III septic PH curves, with their curvature profiles (right). Here we give  $l = \|\Delta P_0\| = \|\Delta P_6\|$  from 0.1 to 0.5. When  $l$  is bigger, the curve becomes much closer to the corner. Notably, the curve is very close to a circular arc when  $l = 0.2$ , where its curvature is in the interval (0.9111, 1.1145).

quarter of circle, and its curvature is in the interval (0.9111, 1.1145).

## 6. Conclusion

This paper studies geometric characteristics of two classes of septic PH curves. Following our results, identification of septic PH curves can be performed using their Bézier control polygons. So far, the study on geometric characteristics of septic PH curves has been completed. There are some potential applications of our results, for example, construction of various septic PH curves under diverse Hermite condi-

tion, or interactive design of PH splines, etc. We also believe that the proposed approach is adaptable, with some effort, to the discussion of rational PH curves or indirect-PH curves [15]. These directions will be considered in our future work.

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