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Approximation Methods for Second Kind Weakly Singular Volterra Integral Equations

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Abstract

Projection methods are applied to obtain the convergence rates for Volterra integral equations with weakly singular kernels. We consider Galerkin and multi Galerkin methods and their iterated versions to solve Volterra integral equations with weakly singular kernels, in the space of piecewise polynomials subspaces based on graded mesh. We will show that the iterated multi-Galerkin method improves over iterated Galerkin method. In fact, we show that iterated multi-Galerkin solution converges with the convergence rates $\mathcal{O}(n^{-3m})$ and $\mathcal{O}(n^{-3m}(\log n)^2)$, for algebraic and logarithmic type kernels, respectively. We prove that iterated Galerkin method, for algebraic kernel, converges with the convergence rate $\mathcal{O}(n^{-2m})$ and for logarithmic type kernel converges with the convergence rate $\mathcal{O}(n^{-2m}\log n)$, where n denotes the number of partition points and m is the highest order of the polynomials employed in the approximations. Theoretical results are justified by the Numerical results.

Keywords: Weakly singular Volterra integral equations, Piecewise polynomials, Galerkin method, convergence results, Multi-Galerkin method, Superconvergence results.
2010 MSC: 45D05, 45A05, 65R20

1. Introduction

We consider the approximation methods to find the numerical approximate solutions of the following type of Volterra integral equations of second kind on Banach space $\mathbb{X}$:

$$u(t) = \int_0^t \mathcal{H}(t,s)u(s)ds + g(t), \quad 0 \le t \le 1, \tag{1.1}$$

with the kernel

$$\mathcal{H}(t,s) = \begin{cases} m(t,s)|t-s|^{-\alpha}, & \text{if } 0 < \alpha < 1, \\ m(t,s)\log(|t-s|), & \text{if } \alpha = 1, \end{cases} \tag{1.2}$$

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where $m(\cdot,\cdot)$ and $g(\cdot)$ both are given sufficiently differentiable functions, u is the function to be found in a Banach space $\mathbb{X}$.
In general, it is not easy to obtain the explicit solution of the integral equation of the type (1.1)-(1.2). The derivative u' ($u'(t) \sim t^{-\alpha}$, if $0 < \alpha < 1$ and $u'(t) \sim \log t$, if $\alpha = 1$) of the exact solution of the integral equation of type (1.1)-(1.2) is unbounded at $t = 0$ in the domain of integration $[0, 1]$ [see in Brunner (1983, 1985b, 1985c) and Brunner et al. (2001a, 2001b)]. Various numerical approximation methods are available in literature like Product integration, Galerkin, collocation and Nyström methods etc, for solving Fredholm and Volterra integral equations (see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]). The ordinary collocation method cannot lead to a higher order convergence rates due to the fact that the derivative of the solution u is unbounded at $t = 0$. Several authors have discussed the numerical approximating methods to find the approximate solutions of integral equations of type (1.1)-(1.2). Yanzhao Cao et al. in [13], developed the hybrid collocation method for finding the approximate solution of the Volterra integral equations of type (1.1)-(1.2) and obtained the order of convergence $\mathcal{O}(n^{-m})$, where m is the order of piecewise polynomials and n is the number of partition points. In [2, 3], H. Brunner introduced the notion of the collocation and iterated collocation methods in the space of piecewise polynomial subspaces for the weakly singular Volterra integral equations of type (1.1)-(1.2) and obtained the convergence rates $\mathcal{O}(h^m)$ and $\mathcal{O}(h^{m+1-\alpha})$, respectively, in infinity norm, where h is the norm of the partitions. To our knowledge, there are very few papers related to the Galerkin method for finding the solution of the Volterra integral equations. In [14], Galerkin and iterated Galerkin methods were discussed for Volterra integral equations of the second kind for the smooth kernel by Shuhua Zhang et al. and using an interpolation post-processing technique, they obtained global superconvergence of the order $\mathcal{O}(h^{2m})$, in space of piecewise polynomials of degree not exceed $m - 1$. The main motivation of this paper is to obtain the superconvergence results for weakly singular Volterra integral equation of the type (1.1)-(1.2) by using piecewise polynomial basis functions.

In this article, Galerkin and multi-Galerkin methods with their iterated versions are discussed for solving the Volterra integral equations of type (1.1)-(1.2), in the piecewise polynomials subspaces based on graded mesh. We show that for the algebraic type kernel ($0 < \alpha < 1$), the Galerkin and iterated Galerkin solutions converge with the order $\mathcal{O}(n^{-m})$ and $\mathcal{O}(n^{-2m})$, respectively, where m is the order of the piecewise polynomials. Similarly for the logarithmic type kernel ($\alpha = 1$), we show that the Galerkin and iterated Galerkin solutions converge with the convergence rates $\mathcal{O}(n^{-m})$ and $\mathcal{O}(n^{-2m}\log n)$, respectively. Further, we enhance the above convergence results in iterated multi-Galerkin method. In fact, we prove that the iterated multi Galerkin method converges with the order $\mathcal{O}(n^{-3m})$, for the algebraic type kernel and with the order $\mathcal{O}(n^{-3m}(\log n)^2)$, for logarithmic type kernel.

This article is organized as follows. In section 2, we develop the Galerkin and iterated Galerkin methods for the integral equations of the type (1.1)-(1.2) and analyse the convergence results. In section 3, the multi-Galerkin method with its iterated version are discussed to obtain the improved superconvergence rates. In section 4, we provide numerical examples for verfying our theoretical results. C denotes the generic constant in the article.

2. Projection methods for Volterra integral equation of the second kind with weakly singular kernel

Consider the integral equations of type (1.1) - (1.2) over a Banach space $\mathbb{X} = L^\infty[0,1]$. To obtain the superconvergence results, the domain of integration from $[0,t]$ $(0 < t \leq 1)$, is transformed to the interval $[0,1]$, by the transformation $s(.,.) : ([0,1] \times [0,1]) \to [0,1]$ defined by $s = t\eta, \quad (t,\eta) \in ([0,1] \times [0,1])$. Using this transformation, the integral equation (1.1)-(1.2) can be written as

$$u(t) = \int_0^1 \tilde{\mathcal{H}}(t, s(t,\eta))u(s(t,\eta))d\eta + g(t), \quad t \in [0,1], \tag{2.1}$$

where

$$\tilde{\mathcal{H}}(t, s(t,\eta)) = \begin{cases} \tilde{m}(t, s(t,\eta))|1-\eta|^{-\alpha}, & \text{if } 0 < \alpha < 1, \\ \tilde{m}(t, s(t,\eta)) \log(|t(1-\eta)|), & \text{if } \alpha = 1, \end{cases} \tag{2.2}$$

with

$$\tilde{m}(t, s(t,\eta)) = \begin{cases} m(t, s(t,\eta))|t|^{1-\alpha}, & \text{if } 0 < \alpha < 1, \\ m(t, s(t,\eta))|t|, & \text{if } \alpha = 1. \end{cases} \tag{2.3}$$

Define

$$\mathcal{F}u(t) = \int_0^1 \tilde{\mathcal{H}}(t, s(t,\eta))u(s(t,\eta))d\eta, \quad t \in [0,1]. \tag{2.4}$$

Then the equation (2.1) can be written in the following operator equation form as

$$u - \mathcal{F}u = g. \tag{2.5}$$

Assume that 1 is not an eigen value of $\mathcal{F}$ so that $(\mathcal{I} - \mathcal{F})^{-1}$ exists. Hence the integral equation (2.5) has a unique solution say $u \in \mathbb{X}$.

For any $v \in \mathcal{C}^m([0,1])$, we write

$$\|v\|_{m,\infty} = \max\Big\{\Big\|\frac{\partial^j v}{\partial t^j}\Big\|_\infty : 0 \leq j \leq m\Big\}.$$

Now we discuss Galerkin and iterated Galerkin methods. Since the kerenl $\tilde{\mathcal{H}}(t, s(t,\eta))$ has singularity at $\eta = 1$ for algebraic type kernel and $\tilde{\mathcal{H}}(t, s(t,\eta))$ has singularity at $\eta = 1$ and $t = 0$ for logarithmic type kernel and the exact solution u has singularity at $t = 0$ in [0, 1], we choose the graded mesh on [0, 1] of the type

$$t_l = \frac{1}{2}\Big(\frac{2l}{n}\Big)^q, \quad 0 \le l \le \frac{n}{2},$$
$$t_l = 1 - t_{n-l}, \quad \frac{n}{2} \le l \le n, \tag{2.6}$$

where $q = \frac{m}{1-\alpha}$ in the case of algebraic kernel and $q = m$ in the case of logarithmic kernel, $m \ge 1$. Let h be the norm of partiton and defined as $h = \max_l\{h_l = t_l - t_{l-1} : \ l = 1, \dots, n\} \to 0$, as $n \to \infty$. Let $\sigma_l = [t_{l-1}, t_l]$, $l = 1, 2, \dots, n$ be the subintervals of the interval $[0, 1]$. Assume that the approximating subspace $\mathbb{X}_n = S^{\mu}_{m,n}(\Pi_n)$, denotes the space of piecewise continuous polynomials of degree $\le m-1$ on the interval $[0, 1]$ with the break points at $t_1, t_2, \dots, t_{n-1}$ and with $\mu(-1 \le \mu \le m-2)$ continuous derivatives. Throughout the paper we assume that $\mathbb{P}_m$ denotes the space of polynomials of degree $\le m-1$.

Note that

$$h_1 = t_1 - t_0 = \frac{1}{2}\Big(\frac{2}{n}\Big)^q - 0 = 2^{q-1}\Big(\frac{1}{n}\Big)^q = \mathcal{O}(n^{-q}), \tag{2.7}$$

and

$$h_n = t_n - t_{n-1} = 1 - 1 + \frac{1}{2}\Big(\frac{2}{n}\Big)^q = 2^{q-1}\Big(\frac{1}{n}\Big)^q = \mathcal{O}(n^{-q}). \tag{2.8}$$

Hence

$$h_1 = h_n = \begin{cases} \mathcal{O}(n^{\frac{-m}{1-\alpha}}), & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-m}), & \text{if } \alpha = 1. \end{cases} \tag{2.9}$$

Next for $l = 2, 3, \dots, n-1$, by Mean value theorem, we obtain

$$h_l = t_l - t_{l-1} = \frac{1}{2}\Big(\frac{2l}{n}\Big)^q - \frac{1}{2}\Big(\frac{2(l-1)}{n}\Big)^q = \frac{1}{2}\Big(\frac{2}{n}\Big)^q\{(l)^q - (l-1)^q\} \le \frac{q}{2}\Big(\frac{2}{n}\Big)^q \xi^{q-1}, \tag{2.10}$$

where $l - 1 < \xi < l \le n-1$. Hence

$$h_l \le \frac{q}{2}\Big(\frac{2}{n}\Big)^q l^{q-1} \le \frac{q}{2}\Big(\frac{2}{n}\Big)^q (n-1)^{q-1} \le q2^{q-1}\frac{1}{n}\Big[1 - \frac{1}{n}\Big]^{q-1} \le q2^{q-1}\frac{1}{n} = \mathcal{O}(n^{-1}). \tag{2.11}$$

Orthogonal projection operator: Define $\mathcal{P}_n : L^\infty[0,1] \to \mathbb{X}_n$ be the orthogonal projection operator defined by

$$\langle \mathcal{P}_n v_1, v_2 \rangle = \langle v_1, v_2 \rangle, \quad \forall\ v_1 \in \mathbb{X}, \quad \forall\ v_2 \in \mathbb{X}_n, \tag{2.12}$$

where

$$\langle v_1, v_2 \rangle = \int_0^1 v_1(t) v_2(t) dt.$$

We state the next result from Ivan G. Graham [15], which is helpful in discussing the convergence analysis.
(i) $\exists\ \rho > 0$, such that

$$\|\mathcal{P}_n\|_\infty \leq \rho < \infty. \tag{2.13}$$

(ii) For any $v \in \mathcal{C}^m[0,1]$, $\exists\ C > 0$ such that the following hold:
In the sub-interval $[t_0, t_1]$, we have

$$\|(I - \mathcal{P}_n)v\|_{L^\infty([t_0,t_1])} \leq Ch_1^m \|v\|_{m,L^\infty([t_0,t_1])}, \tag{2.14}$$

in the sub-intervals $[t_{i-1}, t_i],\ i = 2, \cdots, n-1$, we have

$$\|(I - \mathcal{P}_n)v\|_{L^\infty([t_{i-1},t_i])} \leq Ch_i^m \|v\|_{m,L^\infty([t_{i-1},t_i])}, \tag{2.15}$$

and in the sub-interval $[t_{n-1}, t_n]$, we have

$$\|(I - \mathcal{P}_n)v\|_{L^\infty([t_{n-1},t_n])} \leq Ch_n^m \|v\|_{m,L^\infty([t_{n-1},t_n])}, \tag{2.16}$$

where $h_i,\ i = 1, \cdots, n$ are defined in the estimates (2.9)-(2.11).
Now the Galerkin method for finding the approximate solution of integral equation (2.1) is defined as find $u_n \in \mathbb{X}_n$ such that

$$u_n - \mathcal{P}_n \mathcal{F} u_n = \mathcal{P}_n g. \tag{2.17}$$

The iterated approximate solution is defined as

$$\tilde{u}_n = \mathcal{F} u_n + g. \tag{2.18}$$

Using $\mathcal{P}_n \tilde{u}_n = u_n$, the equivalent form of the above equation (2.18) can be written in operator equation form as

$$\tilde{u}_n - \mathcal{F}\mathcal{P}_n \tilde{u}_n = g. \tag{2.19}$$

This is known as the iterated Galerkin method.
We quote a result from (Schumaker [16], P. 92), which is useful in our convergence analysis. For any $v \in L^1[0,1]$, $\exists$ a polynomial ϕ_n of degree $\leq n$ such that

$$\|v - \phi_n\|_{L^1} \leq C_1 \omega_{1,1}\Big(v, \frac{1}{2n}\Big), \tag{2.20}$$

where

$$\omega_{1,1}(v,h) = \sup_{0\leq\delta\leq h} \|\Delta_\delta v\|_{L^1[0,1-\delta]} = \sup_{0\leq\delta\leq h} \Big| \int_0^{1-\delta} [v(t+\delta) - v(t)]dt \Big|, \tag{2.21}$$

denotes the first order modulus of smoothness of the function $v \in L^1[0,1]$ w.r.t. L^1-norm.

Using this result, we prove the following theorem.

Theorem 2.1. *Let the kernel of the integral operator $\mathcal{F}$ is defined by (2.2). Then for any $t \in [0,1]$, $\exists$ a polynomial $v_t \in \mathbb{P}_m$ such that*

$$\|\tilde{\mathcal{H}}(t, s(t,\eta)) - v_t\|_{L^1} = \begin{cases} \mathcal{O}(h^{1-\alpha}), & \text{if } \ 0 < \alpha < 1, \\ \mathcal{O}(h \log h), & \text{if } \ \alpha = 1, \end{cases} \tag{2.22}$$

where h be the norm of the partiton.

Proof. Since $\tilde{\mathcal{H}}(t, s(t,\cdot)) \in L^1[0,1]$, from the estimate (2.20), we have

$$\|\tilde{\mathcal{H}}(t, s(t,\eta)) - v_t\|_{L^1} \leq C_1 \omega_{1,1}(\tilde{\mathcal{H}}(t, s(t,\eta)), h).$$

Let $I_\vartheta = [0, 1-\vartheta]$, by using modulus of smoothness (cf. Schumaker [16], P. 22), we obtain

$$\begin{aligned} \|\tilde{\mathcal{H}}(t, s(t,\eta)) - v_t\|_{L^1} &\leq C_1 \omega_{1,1}(\tilde{\mathcal{H}}(t, s(t,.)), h) = C_1 \sup_{0 \leq \vartheta \leq h} \|\Delta_\vartheta \tilde{\mathcal{H}}(t, s(t,\eta))\|_{L^1(I_\vartheta)} \\ &\leq C_1 \sup_{0 \leq \vartheta \leq h} \int_0^1 |\tilde{\mathcal{H}}(t, s(t,\eta+\vartheta)) - \tilde{\mathcal{H}}(t, s(t,\eta))| d\eta. \end{aligned} \tag{2.23}$$

Case – I : For the algebraic kernel $\tilde{\mathcal{H}}(t, s(t,\eta)) = \tilde{m}(t, s(t,\eta))|1-\eta|^{-\alpha}$:

Consider for any $0 \leq t \leq 1$,

$$\begin{aligned} &\int_0^1 \Big|\tilde{\mathcal{H}}(t, s(t,\eta+\vartheta)) - \tilde{\mathcal{H}}(t, s(t,\eta))\Big| d\eta \\ &= \int_0^1 \Big|\frac{\tilde{m}(t, s(t,\eta+\vartheta))}{|(1-(\eta+\vartheta))|^\alpha} - \frac{\tilde{m}(t, s(t,\eta))}{|(1-\eta)|^\alpha}\Big| d\eta \\ &\leq \int_0^1 \Big|\frac{\tilde{m}(t, s(t,\eta+\vartheta))}{|(1-(\eta+\vartheta))|^\alpha} - \frac{\tilde{m}(t, s(t,\eta+\vartheta))}{|(1-\eta)|^\alpha}\Big| d\eta + \int_0^1 \Big|\frac{\tilde{m}(t, s(t,\eta+\vartheta))}{|(1-\eta)|^\alpha} - \frac{\tilde{m}(t, s(t,\eta))}{|(1-\eta)|^\alpha}\Big| d\eta \\ &\leq \sup_{0 \leq t,\eta \leq 1} |\tilde{m}(t, s(t,\eta+\vartheta))| \int_0^1 \Big|\frac{1}{|(1-(\eta+\vartheta))|^\alpha} - \frac{1}{|(1-\eta)|^\alpha}\Big| d\eta \\ &+ \int_0^1 \Big|\frac{1}{|(1-\eta)|^\alpha}\Big| d\eta \sup_{0 \leq t,\eta \leq 1} \Big|\tilde{m}(t, s(t,\eta+\vartheta)) - \tilde{m}(t, s(t,\eta))\Big| \\ &= I_1 + I_2. \end{aligned} \tag{2.24}$$

For evaluating I_2, note that

$$\begin{aligned} \sup_{0 \leq t,\eta \leq 1} |\tilde{m}(t, s(t,\eta+\vartheta)) - \tilde{m}(t, s(t,\eta))| &= \sup_{0 \leq t,\eta \leq 1} |\tilde{m}^{(0,1)}(t, s(t,\eta_1)) \frac{\partial s(t,\eta)}{\partial \eta}(s(t,\eta+\vartheta) - s(t,\eta))| \\ &= \sup_{0 \leq t,\eta \leq 1} |\tilde{m}^{(0,1)}(t, s(t,\eta_1))||t|^2|\vartheta| < C|\vartheta|, \end{aligned} \tag{2.25}$$

where $\eta < \eta_1 < \eta + \vartheta$, $C = \sup_{0 \leq t,\eta \leq 1} |\tilde{m}^{(0,1)}(t, s(t,\eta))|$ and also

$$\int_0^1 \frac{1}{|(1-\eta)^\alpha|} d\eta = \Big[-\frac{(1-\eta)^{-\alpha+1}}{-\alpha+1}\Big]_0^1 = \frac{1}{1-\alpha} < \infty. \tag{2.26}$$

Hence

$$I_2 \leq \frac{1}{1-\alpha} C|\vartheta|. \tag{2.27}$$

Now for evaluating I_1, consider

$$\begin{aligned}
&\int_0^1 \Big|\frac{1}{|(1-(\eta+\vartheta))^\alpha|} - \frac{1}{|(1-\eta)^\alpha|}\Big| d\eta \\
&= \int_0^{1-\vartheta} \Big|\frac{1}{|1-(\eta+\vartheta)|^\alpha} - \frac{1}{|1-\eta|^\alpha}\Big| d\eta + \int_{1-\vartheta}^{1-\vartheta/2} \Big|\frac{1}{|1-(\eta+\vartheta)|^\alpha} - \frac{1}{|1-\eta|^\alpha}\Big| d\eta \\
&+ \int_{1-\vartheta/2}^{1} \Big|\frac{1}{|1-(\eta+\vartheta)|^\alpha} - \frac{1}{|1-\eta|^\alpha}\Big| d\eta \\
&= \int_0^{1-\vartheta} \Big[\frac{1}{(1-(\eta+\vartheta))^\alpha} - \frac{1}{(1-\eta)^\alpha}\Big] d\eta + \int_{1-\vartheta}^{1-\vartheta/2} \Big[\frac{1}{(\eta-1+\vartheta)^\alpha} - \frac{1}{(1-\eta)^\alpha}\Big] d\eta \\
&+ \int_{1-\vartheta/2}^{1} \Big[\frac{1}{(1-\eta)^\alpha} - \frac{1}{(\eta-1+\vartheta)^\alpha}\Big] d\eta \\
&= \frac{1}{1-\alpha}\Big[4\Big(\frac{\vartheta}{2}\Big)^{1-\alpha} - 1 + (1-\vartheta)^{1-\alpha} - \vartheta^{1-\alpha}\Big] \\
&\leq 4\frac{\vartheta^{1-\alpha}}{2^{1-\alpha}(1-\alpha)},
\end{aligned} \tag{2.28}$$

and using $\sup_{0\leq t,\eta\leq 1} |\tilde{m}(t, s(t, \eta+\vartheta))| < \infty$, we obtain

$$I_1 \leq 4\frac{\vartheta^{1-\alpha}}{2^{1-\alpha}(1-\alpha)}. \tag{2.29}$$

Combining I_1 and I_2 with (2.23), we obtain

$$\begin{aligned}
\|\tilde{\mathcal{H}}(t, s(t,\eta)) - v_t\|_{L^1} &\leq C_1 \sup_{0\leq\vartheta\leq h} \int_0^1 |\tilde{\mathcal{H}}(t, s(t,\eta+\vartheta)) - \tilde{\mathcal{H}}(t, s(t,\eta))| d\eta \\
&\leq C_1 \sup_{0\leq\vartheta\leq h} \Big\{C\vartheta + 4\frac{\vartheta^{1-\alpha}}{2^{1-\alpha}(1-\alpha)}\Big\} \leq C_1 h^{1-\alpha} = \mathcal{O}(h^{1-\alpha}).
\end{aligned} \tag{2.30}$$

Case − II : For the logarithmic type kernel $\tilde{\mathcal{H}}(t, s(t,\eta)) = \tilde{m}(t, s(t,\eta)) \log(|t(1-\eta)|)$:
Consider for any $t \in [0, 1]$:

$$\begin{aligned}
&\int_0^1 |\tilde{\mathcal{H}}(t, s(t,\eta+\vartheta)) - \tilde{\mathcal{H}}(t, s(t,\eta))| \; d\eta \\
&\qquad = \int_0^1 \Big| m(t, s(t,\eta+\vartheta))|t| \log|t - t(\eta+\vartheta)| - m(t, s(t,\eta))|t| \log|t - t\eta|\Big| \; d\eta \\
&\qquad = \int_0^1 \Big| m(t, s(t,\eta+\vartheta))|t| \log|t - t(\eta+\vartheta)| - m(t, s(t,\eta+\vartheta))|t| \log|t - t\eta|
\end{aligned}$$

$$+ m(t,s(t,\eta+\vartheta))|t|\log|t-t\eta| - m(t,s(t,\eta))|t|\log|t-t\eta|\Big|\, d\eta$$
$$\leq \sup_{0\leq t,\eta\leq 1}|m(t,s(t,\eta+\vartheta))||t|\int_0^1 \Big|\log|t-t(\eta+\vartheta)| - \log|t-t\eta|\Big| d\eta$$
$$+ \sup_{0\leq t,\eta\leq 1}\Big|\int_0^1 t\log|t-t\eta| d\eta\Big| \sup_{0\leq t,\eta\leq 1}|m(t,s(t,\eta+\vartheta)) - m(t,s(t,\eta))|$$
$$= I_1' + I_2'. \tag{2.31}$$

For evaluating I_2', note that

$$\int_0^1 t\log(t-t\eta)d\eta = \Big[t\log t - t(1-\eta)\log(1-\eta) + t(1-\eta)\Big]_0^1 \tag{2.32}$$
$$= t\log t - t \leq -1 < \infty. \tag{2.33}$$

Now since $m(t,s(t,\eta)) \in \mathcal{C}^1([0,1]\times[0,1])$, we have

$$\sup_{0\leq t,\eta\leq 1}|m(t,s(t,\eta+\vartheta)) - m(t,s(t,\eta))| = \sup_{0\leq t,\eta\leq 1}\Big|m^{(0,1)}(t,s(t,\eta_1))(s(t,\eta+\vartheta) - s(t,\eta))\Big|$$
$$\leq \sup_{0\leq t,\eta\leq 1}\Big|m^{(0,1)}(t,s(t,\eta_1))|t|\Big|\Big|\frac{\partial s(t,\eta_1)}{\partial\eta}\Big||\vartheta|, \tag{2.34}$$

where $\eta < \eta_1 < \eta+\vartheta$ and $\bar{C} = \sup_{0\leq t,\eta\leq 1}\Big|m^{(0,1)}(t,s(t,\eta_1))||t|\Big|$.
Hence

$$I_2' \leq \bar{C}|\vartheta|. \tag{2.35}$$

Now for evaluating I_1', consider

$$\int_0^1 \Big|\log|t-t(\eta+\vartheta)| - \log|t-t\eta|\Big| d\eta$$
$$= \int_0^1 \Big|\log|t| + \log|1-(\eta+\vartheta)| - \log|t| - \log|1-\eta|\Big| d\eta$$
$$= \int_0^1 \Big|\log|1-(\eta+\vartheta)| - \log|1-\eta|\Big| d\eta$$
$$= \int_0^{1-\vartheta} \Big|\log|1-(\eta+\vartheta)| - \log|1-\eta|\Big| d\eta + \int_{1-\vartheta}^{1-\vartheta/2} \Big|\log|1-(\eta+\vartheta)| - \log|1-\eta|\Big| d\eta$$
$$+ \int_{1-\vartheta/2}^1 \Big|\log|1-(\eta+\vartheta)| - \log|1-\eta|\Big| d\eta$$
$$= \int_0^{1-\vartheta} \Big(\log|1-\eta| - \log|1-(\eta+\vartheta)|\Big) d\eta + \int_{1-\vartheta}^{1-\vartheta/2} \Big(\log|1-\eta| - \log|1-(\eta+\vartheta)|\Big) d\eta$$
$$+ \int_{1-\vartheta/2}^1 \Big(\log|1-(\eta+\vartheta)| - \log|1-\eta|\Big) d\eta$$

$$= \int_0^{1-\vartheta} \Big(\log(1-\eta) - \log(1-(\eta+\vartheta)) \Big) d\eta + \int_{1-\vartheta}^{1-\vartheta/2} \Big(\log(1-\eta) - \log(\eta - 1 + \vartheta) \Big) d\eta$$
$$+ \int_{1-\vartheta/2}^{1} \Big(\log(\eta - 1 + \vartheta) - \log(1-\eta) \Big) d\eta.$$

After evaluating the above integrals, we have

$$\int_0^1 \Big| \log|1-(\eta+\vartheta)| - \log|1-\eta| \Big| d\eta = -4(\vartheta/2)\log(\vartheta/2) + \vartheta \log\vartheta - (1-\vartheta)\log(1-\vartheta), \tag{2.36}$$

and using $\sup_{0\le t,\eta\le 1}|m(t,s(t,\eta+\vartheta))||t| < \infty$, we obtain

$$I_1' \le -4(\vartheta/2)\log(\vartheta/2) + \vartheta\log\vartheta - (1-\vartheta)\log(1-\vartheta). \tag{2.37}$$

Combining I_1' and I_2' with (2.23), we obtain

$$\|\tilde{\mathcal{H}}(t,s(t,\eta)) - v_t\|_{L^1} \le \sup_{0\le\vartheta\le h} \int_0^1 |\tilde{\mathcal{H}}(t,s(t,\eta+\vartheta)) - \tilde{\mathcal{H}}(t,s(t,\eta))| d\eta \le C(h/2)\log(h/2) = \mathcal{O}(h\log h). \tag{2.38}$$

Combining the estimates (2.30) and (2.38), it follows that

$$\|\tilde{\mathcal{H}}(t,s(t,\eta)) - v_t\|_{L^1} = \begin{cases} \mathcal{O}(h^{1-\alpha}), & \text{if } 0<\alpha<1, \\ \mathcal{O}(h\log h), & \text{if } \alpha = 1. \end{cases} \tag{2.39}$$

Hence the proof follows.

We state the following theorem from H. Brunner [2], which is helpful in discussing the convergence results.

Theorem 2.2. *Let $\mathcal{H}(.,.)$ be the kernel of the integral equation (1.1)-(1.2) for $0<\alpha<1$. Then for any $g \in \mathcal{C}^m[0,1]$ and $m(\cdot,\cdot) \in \mathcal{C}^m[0,1]$ with $m(t,t) \neq 0$, the Volterra integral equation (1.1)-(1.2) has a unique solution $u \in \mathcal{C}[0,1]$ and it is uniformly convergent series form*

$$u(t) = g(t) + \sum_{k=1}^{\infty} \psi_k(t) t^{k(1-\alpha)}, \tag{2.40}$$

where $\psi_k(t)$ is smooth function of t.

Let $M_1 = \sup\limits_{0\le t\le 1} |g'(s(t,\eta))|$ and $M_2 = \sup\limits_{0\le t\le 1} \big\{ |\psi_k'(s(t,\eta))| : k = 1,2,.....\big\}$.

Theorem 2.3. *For the exact solution u of the integral equation (2.1)-(2.2) with algebraic type kernel define as in (2.2) for $0<\alpha<1$, then there holds*

$$\omega_\infty(u,h) \le Ch^{1-\alpha}, \tag{2.41}$$

where $\omega_\infty(u,h)$ denotes the modulus of smoothness of u.

Proof. From Theorem 2.2,

$$u(t) = g(t) + \sum_{k=1}^{\infty} \psi_k(t) t^{k(1-\alpha)},$$

where $\psi_k(t)$ be the smooth function of t. By the equation (2.40), the exact solution of transformed equation is

$$u(s(t,\eta)) = g(s(t,\eta)) + \sum_{k=1}^{\infty} \psi_k(s(t,\eta))(s(t,\eta))^{k(1-\alpha)}.$$

Consider for any $t, h, \eta \in [0,1]$ such that $\eta + h \in [0,1]$,

$$\begin{aligned}
&|u(s(t,\eta+h)) - u(s(t,\eta))| \\
&\leq |g(s(t,\eta+h)) - g(s(t,\eta))| + \sum_{k=1}^{\infty} \Big\{ |\psi_k(s(t,\eta+h))(s(t,\eta+h))^{k(1-\alpha)} - \psi_k(s(t,\eta))(s(t,\eta))^{k(1-\alpha)}| \Big\} \\
&\leq \Big\{ \Big| g'(s(t,\eta_1 h))(s(t,\eta+h) - s(t,\eta)) \Big| + \sum_{k=1}^{\infty} |\psi_k(s(t,\eta+h))(s(t,\eta+h))^{k(1-\alpha)} \\
&- \psi_k(s(t,\eta+h))(s(t,\eta))^{k(1-\alpha)} + \psi_k(s(t,\eta+h))(s(t,\eta))^{k(1-\alpha)} - \psi_k(s(t,\eta))(s(t,\eta))^{k(1-\alpha)}| \Big\} \\
&\leq M_1 th + \sum_{k=1}^{\infty} \Big\{ |\psi_k(s(t,\eta+h))(s(t,\eta+h))^{k(1-\alpha)} - \psi_k(s(t,\eta+h))(s(t,\eta))^{k(1-\alpha)}| \\
&+ |\psi_k(s(t,\eta+h))(s(t,\eta))^{k(1-\alpha)} - \psi_k(s(t,\eta))(s(t,\eta))^{k(1-\alpha)}| \Big\} \\
&\leq M_1 h + \sum_{k=1}^{\infty} \Big\{ |\psi_k(s(t,\eta+h))[(s(t,\eta+h))^{k(1-\alpha)} - (s(t,\eta))^{k(1-\alpha)}]| \\
&+ |[\psi_k(s(t,\eta+h)) - \psi_k(s(t,\eta))](s(t,\eta))^{k(1-\alpha)}| \Big\} \\
&\leq M_1 h + \sum_{k=1}^{\infty} \Big\{ |\psi_k(s(t,\eta+h))[(t(\eta+h))^{k(1-\alpha)} - (t(\eta))^{k(1-\alpha)}]| \\
&+ \Big| \psi_k'(s(t,\eta+\eta_2 h))(t(\eta+h-\eta))(s(t,\eta))^{k(1-\alpha)} \Big| \Big\} \\
&\leq M_1 h + \sum_{k=1}^{\infty} |\psi_k(s(t,\eta+h)) t^{k(1-\alpha)}| |(\eta+h)^{k(1-\alpha)} - (\eta)^{k(1-\alpha)}| \\
&+ h^{1-\alpha} \sum_{k=1}^{\infty} \Big| \psi_k'(s(t,\eta+\eta_2 h)) h^{\alpha} (t\eta)^{k(1-\alpha)} \Big|,
\end{aligned} \tag{2.42}$$

in the above, we have used mean value theorem.
Now we show that

$$|(\eta+h)^{k(1-\alpha)} - (\eta)^{k(1-\alpha)}| \leq C h^{1-\alpha}, \quad \forall\, \eta \in [0,1], \quad \forall\, k. \tag{2.43}$$

To prove this, we use the principle of mathematical induction on k. Using the mathematical inequality $(a+b)^p < a^p + b^p$, for $a, b > 0$ and $0 < p < 1$, we have

$$|(\eta+h)^{(1-\alpha)} - (\eta)^{(1-\alpha)}| \le Ch^{1-\alpha}.$$

This proves that for $k = 1$, the result is true. Assume that for $k = n-1$, the result is also true,

$$|(\eta+h)^{(n-1)(1-\alpha)} - (\eta)^{(n-1)(1-\alpha)}| \le Ch^{1-\alpha}.$$

Now we prove for $k = n$. Consider

$$\begin{aligned}
|(\eta+h)^{n(1-\alpha)} &- (\eta)^{n(1-\alpha)}| \\
&= |(\eta+h)^{n(1-\alpha)} - (\eta+h)^{(1-\alpha)}\eta^{(n-1)(1-\alpha)} \\
&+ (\eta+h)^{(1-\alpha)}\eta^{(n-1)(1-\alpha)} - \eta^{n(1-\alpha)}| \\
&\le |(\eta+h)^{(1-\alpha)}||(\eta+h)^{(n-1)(1-\alpha)} - \eta^{(n-1)(1-\alpha)}| \\
&+ |\eta^{(n-1)(1-\alpha)}||(\eta+h)^{(1-\alpha)} - \eta^{(1-\alpha)}| \\
&\le |(\eta+h)^{(1-\alpha)}|Ch^{1-\alpha} + |\eta^{(n-1)(1-\alpha)}|Ch^{1-\alpha} \\
&\le |2^{(1-\alpha)}|Ch^{1-\alpha} + |1^{(n-1)(1-\alpha)}|Ch^{1-\alpha} \\
&\le Ch^{1-\alpha}.
\end{aligned} \tag{2.44}$$

Hence by mathematical induction (2.43) is proved.
Now combining the above estimate (2.44) with the estimate (2.42), we obtain

$$\begin{aligned}
|u(s(t,&\eta+h)) - u(s(t,\eta))| \\
&\le M_1 h + Ch^{1-\alpha}\sum_{k=1}^{\infty}|\psi_k(s(t,\eta+h))t^{k(1-\alpha)}| + h^{1-\alpha}\sum_{k=1}^{\infty}\left|\psi_k'(s(t,\eta+\eta_2 h))h^{\alpha}(t\eta)^{k(1-\alpha)}\right| \\
&\le Ch^{1-\alpha}\Big\{M_1 h^{\alpha} + \sum_{k=1}^{\infty}|\psi_k(s(t,\eta+h))t^{k(1-\alpha)}| + \sum_{k=1}^{\infty}\left|\psi_k'(s(t,\eta+\eta_2 h))h^{\alpha}(t\eta)^{k(1-\alpha)}\right|\Big\}.
\end{aligned} \tag{2.45}$$

Since $\sum_{k=1}^{\infty}|\psi_k(s(t,\eta+h))t^{k(1-\alpha)}|$ and $\sum_{k=1}^{\infty}\left|\psi_k'(s(t,\eta+\eta_2 h))h^{\alpha}(t\eta)^{k(1-\alpha)}\right|$ are uniformly convergent series, hence we obtain

$$\omega_{\infty}(u,h) = \sup_{0\le t,\eta\le 1}|u(s(t,\eta+h)) - u(s(t,\eta))| \le Ch^{1-\alpha}. \tag{2.46}$$

Hence the theorem is proved.

Lemma 1. *For the exact solution u of the integral equation (2.1)-(2.2) with algebraic type kernel defined by (2.2) with $0 < \alpha < 1$. Then there holds*

$$\|(\mathcal{I} - \mathcal{P}_n)u\|_{L^{\infty}} = \mathcal{O}(n^{-m}). \tag{2.47}$$

PROOF. Since the exact solution $u(t)$ has singularity at $t=0$. From estimate (2.9) and Theorem 2.3, it follows

$$\|(\mathcal{I}-\mathcal{P}_n)u\|_{L^\infty(\sigma_1)} = \|(\mathcal{I}-\mathcal{P}_n)(u-\chi_n)\|_{L^\infty(\sigma_1)} \leq c_1 \inf_{\chi_n\in\mathbb{X}_n} \|(u-\chi_n)\|_{L^\infty(\sigma_1)}.$$

Now using Jackson's theorem (see. [17], pages. 144, 147, Theorems III and V), we have

$$\begin{aligned}\|(\mathcal{I}-\mathcal{P}_n)u\|_{L^\infty(\sigma_1)} &\leq c_1 \inf_{\chi_n\in\mathbb{X}_n} \|(u-\chi_n)\|_{L^\infty(\sigma_1)} \\ &\leq c_1\omega_\infty(u,h_1) \leq c_1 h_1^{1-\alpha} = \mathcal{O}\big((n^{\frac{-m}{1-\alpha}})^{1-\alpha}\big) = \mathcal{O}(n^{-m}).\end{aligned} \tag{2.48}$$

Next for $i=2,\dots,n$, using the estimates (2.11), (2.15) and (2.16), we have

$$\|(\mathcal{I}-\mathcal{P}_n)u\|_{L^\infty(\sigma_i)} \leq h_i^m \|u^{(m)}\|_{L^\infty(\sigma_i)} \leq n^{-m}\|u^{(m)}\|_{L^\infty(\sigma_i)} = \mathcal{O}(n^{-m}). \tag{2.49}$$

Combining the estimates (2.48) and (2.49), we obtain

$$\|(\mathcal{I}-\mathcal{P}_n)u\|_{L^\infty} = \mathcal{O}(n^{-m}). \tag{2.50}$$

Hence the Lemma is proved.

Lemma 2. *For the exact solution u of the integral equation (2.1)-(2.2) with logarithmic type kernel defined by (2.2) with $\alpha=1$. Then there holds*

$$\|(\mathcal{I}-\mathcal{P}_n)u\|_{L^\infty} \leq C_3 n^{-m},$$

where $C_3<\infty$ is a constant.

PROOF. Proof follows from the similar analysis of G. Vainikko and P. Uba [18].

Theorem 2.4. *Let the projection operator $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be defined by (2.12) and $\tilde{\mathcal{H}}(t,s(t,\eta))$ be kernel of integral operator $\mathcal{F}$ defined by (2.2), there hold*

$$\|\mathcal{F}(I-\mathcal{P}_n)\|_{L^\infty} = \begin{cases}\mathcal{O}(n^{-m}), & \text{if } 0<\alpha<1,\\ \mathcal{O}(n^{-m}\log(n)), & \text{if } \alpha=1.\end{cases}$$

PROOF. Note that the kernel (2.2) has singularity at $\eta=1$ and in rest of the sub-intervals $[0,t_{n-1}]$, it is sufficiently smooth with respect to the variable η.
Since $\langle v_t, (\mathcal{P}_n - I)u\rangle = 0,\ \forall v_t \in \mathbb{P}_m$, we have

$$\begin{aligned}\|\mathcal{F}(I-\mathcal{P}_n)u\|_\infty &= \sup_{t\in[0,1]} |\mathcal{F}(I-\mathcal{P}_n)u(t)| \\ &= \sup_{t\in[0,1]} \Big\{\Big|\int_0^1 \tilde{\mathcal{H}}(t,s(t,\eta))(I-\mathcal{P}_n)u(s(t,\eta))d\eta\Big|\Big\} \\ &= \sup_{t\in[0,1]} \Big\{\Big|\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \tilde{\mathcal{H}}(t,s(t,\eta))(\mathcal{P}_n-I)u(s(t,\eta))d\eta\Big|\Big\}\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\in[0,1]}\Big\{\Big|\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}\tilde{\mathcal{H}}(t,s(t,\eta))(\mathcal{P}_n-I)u(s(t,\eta))d\eta\Big|+\Big|\int_{t_{n-1}}^{1}\tilde{\mathcal{H}}(t,s(t,\eta))(\mathcal{P}_n-I)u(s(t,\eta))d\eta\Big|\Big\}\\
&=\sup_{t\in[0,1]}\Big\{|\sum_{i=1}^{n-1}\langle\tilde{\mathcal{H}}(t,s(t,.)),(\mathcal{P}_n-I)u\rangle_{\sigma_i}|+|\langle\tilde{\mathcal{H}}(t,s(t,.))-v_t,(\mathcal{P}_n-I)u\rangle_{\sigma_n}|\Big\}\\
&=\sup_{t\in[0,1]}\Big\{|\sum_{i=1}^{n-1}\langle(\mathcal{P}_n-I)\tilde{\mathcal{H}}(t,s(t,.)),(\mathcal{P}_n-I)u\rangle_{\sigma_i}|+|\langle\tilde{\mathcal{H}}(t,s(t,.))-v_t,(\mathcal{P}_n-I)u\rangle_{\sigma_n}|\Big\}\\
&\leq \sup_{t\in[0,1]}\Big\{\sum_{i=1}^{n-1}\|(\mathcal{P}_n-I)\tilde{\mathcal{H}}(t,s(t,.))\|_{L^2(\sigma_i)}\|(\mathcal{P}_n-I)u\|_{L^2(\sigma_i)}\\
&\qquad+\|\tilde{\mathcal{H}}(t,s(t,.))-v_t\|_{L^1(\sigma_n)}\|(\mathcal{P}_n-I)u\|_{L^\infty(\sigma_n)}\Big\}. \qquad (2.51)
\end{aligned}
$$

If $\tilde{\mathcal{H}}(t,s(t,\eta))=\tilde{m}(t,s(t,\eta))|1-\eta|^{-\alpha}$ with $0<\alpha<1$ is of algebraic type kernel, then from Theorem 2.1 and estimate (2.9), (2.11), (2.13), we have

$$
\begin{aligned}
|\mathcal{F}(I-\mathcal{P}_n)u(t)| &\leq \sum_{i=1}^{n-1}Ch_i^m\Big\|\frac{\partial^m\tilde{\mathcal{H}}(t,s(t,\eta))}{\partial\eta^m}\Big\|_{L^2(\sigma_i)}\|(\mathcal{P}_n-I)u\|_{L^2(\sigma_i)}+h_n^{1-\alpha}\|(\mathcal{P}_n-I)u\|_{L^\infty(\sigma_n)}\\
&\leq \sum_{i=1}^{n-1}Ch_i^m h_i^{\frac{1}{2}}\Big\|\frac{\partial^m\tilde{\mathcal{H}}(t,s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty(\sigma_i)}h_i^{\frac{1}{2}}\|(\mathcal{P}_n-I)u\|_{L^\infty(\sigma_i)}+n^{-m}(1+\rho)\|u\|_{L^\infty}\\
&=\sum_{i=1}^{n-1}A_1h_i^{(m+1)}(1+\rho)\|u\|_{L^\infty}+n^{-m}(1+\rho)\|u\|_{L^\infty}\\
&\leq \sum_{i=1}^{n-1}A_1n^{-(m+1)}(1+\rho)\|u\|_{L^\infty}+n^{-m}(1+\rho)\|u\|_{L^\infty}\\
&\leq Cn^{-m}\|u\|_{L^\infty}. \qquad (2.52)
\end{aligned}
$$

If $\tilde{\mathcal{H}}(t,s(t,\eta))=\tilde{m}(t,s(t,\eta))\log(|t(1-\eta)|)$ with $\alpha=1$ is of logarithmic type kernel, then from Theorem 2.1 and estimate (2.9), (2.11), (2.13), we have

$$
\begin{aligned}
|\mathcal{F}(I-\mathcal{P}_n)u(t)| &\leq \sum_{i=1}^{n-1}Ch_i^m\Big\|\frac{\partial^m\tilde{\mathcal{H}}(t,s(t,\eta))}{\partial\eta^m}\Big\|_{L^2(\sigma_i)}\|(\mathcal{P}_n-I)u\|_{L^2(\sigma_i)}+h_n\log h_n\|(\mathcal{P}_n-I)u\|_{L^\infty(\sigma_n)}\\
&\leq \sum_{i=1}^{n-1}Ch_i^m h_i^{\frac{1}{2}}\Big\|\frac{\partial^m\tilde{\mathcal{H}}(t,s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty(\sigma_i)}h_i^{\frac{1}{2}}\|(\mathcal{P}_n-I)u\|_{L^\infty(\sigma_i)}+n^{-m}\log n^{-m}(1+\rho)\|u\|_{L^\infty}\\
&=\sum_{i=1}^{n-1}A_1h_i^{(m+1)}(1+\rho)\|u\|_{L^\infty}+(-m)n^{-m}\log n(1+\rho)\|u\|_{L^\infty}\\
&\leq \sum_{i=1}^{n-1}A_1n^{-(m+1)}(1+\rho)\|u\|_{L^\infty}+(-m)n^{-m}\log n(1+\rho)\|u\|_{L^\infty}\leq Cn^{-m}\log n\|u\|_{L^\infty}. \qquad (2.53)
\end{aligned}
$$

From the estimates (2.52) and (2.53), it folows that

$$\|\mathcal{F}(I-\mathcal{P}_n)\|_{L^\infty} = \begin{cases} \mathcal{O}(n^{-m}), & \text{if } 0<\alpha<1, \\ \mathcal{O}(n^{-m}\log(n)), & \text{if } \alpha = 1. \end{cases} \tag{2.54}$$

This completes the proof.

From the above theorem, we see that $\|\mathcal{F}\mathcal{P}_n - \mathcal{F}\| \to 0$ as $n \to \infty$. Since 1 is not the eigenvalue of the operator $\mathcal{F}$, then $(I-\mathcal{F}\mathcal{P}_n)^{-1}$ exists and uniformly bounded, for sufficiently large n i.e., $\exists$ constant $\mathcal{L} > 0$ such that $\|(I-\mathcal{F}\mathcal{P}_n)^{-1}\|_{L^\infty} \le \mathcal{L} < \infty$.

In the next theorem, the convergence results for the iterated Galerkin method are provided.

Theorem 2.5. *Let $\tilde{\mathcal{H}}(t,s(t,\eta))$ be the kernel of the operator $\mathcal{F}$ given by (2.2) and $\tilde{u}_n$ be the iterated Galerkin approximate solution of u. Then the following results hold*

$$\|u-\tilde{u}_n\|_{L^\infty} = \begin{cases} \mathcal{O}(n^{-2m}), & \text{if } \ 0<\alpha<1, \\ \mathcal{O}(n^{-2m}\log n), & \text{if } \ \alpha = 1. \end{cases}$$

Proof. Consider

$$\begin{aligned} u-\tilde{u}_n &= (\mathcal{I}-\mathcal{F})^{-1}g - (\mathcal{I}-\mathcal{F}\mathcal{P}_n)^{-1}g \\ &= (\mathcal{I}-\mathcal{F}\mathcal{P}_n)^{-1}[\mathcal{I}-\mathcal{F}-\mathcal{I}+\mathcal{F}\mathcal{P}_n](\mathcal{I}-\mathcal{F})^{-1}g \\ &= (\mathcal{I}-\mathcal{F}\mathcal{P}_n)^{-1}\mathcal{F}(\mathcal{I}-\mathcal{P}_n)u. \\ ||u-\tilde{u}_n||_\infty &\le \mathcal{L}||\mathcal{F}(\mathcal{I}-\mathcal{P}_n)u||_\infty. \end{aligned} \tag{2.55}$$

Note that the kernel (2.2) has singularity at $\eta = 1$ and in rest of the sub-intervals $[0, t_{n-1}]$, it is sufficiently smooth with respect to the variable η. Also the exact solution u has singularity at $t = 0$ and in the rest of the sub-intervals $[t_1, 1]$, it is sufficiently smooth.
Now

$$\begin{aligned} \|\mathcal{F}(I-\mathcal{P}_n)u\|_\infty &= \sup_{t\in[0,1]} |\mathcal{F}(I-\mathcal{P}_n)u(t)| \\ &= \sup_{t,s(t,\eta)\in[0,1]} \Big| \int_0^1 \tilde{\mathcal{H}}(t,s(t,\eta))(I-\mathcal{P}_n)u(s(t,\eta))d\eta \Big| \\ &= \sup_{t,\eta\in[0,1]} \Big| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \tilde{\mathcal{H}}(t,s(t,\eta))(\mathcal{P}_n - I)u(s(t,\eta))d\eta \Big| \\ &= \sup_{t,,\eta\in[0,1]} \Big\{ \Big| \int_0^{t_1} \tilde{\mathcal{H}}(t,s(t,\eta))(\mathcal{P}_n - I)u(s(t,\eta))d\eta \Big| \\ &\quad + \Big| \sum_{i=2}^{n-1} \int_{t_{i-1}}^{t_i} \tilde{\mathcal{H}}(t,s(t,\eta))(\mathcal{P}_n - I)u(s(t,\eta))d\eta \Big| \end{aligned}$$

$$
\begin{aligned}
&\quad + \Big| \int_{t_{n-1}}^{1} \tilde{\mathcal{H}}(t,s(t,\eta))(\mathcal{P}_n - I)u(s(t,\eta))d\eta \Big| \Big\} \\
&= \sup_{t,\eta\in[0,1]} \Big\{ |\langle \tilde{\mathcal{H}}(t,s(t,.)), (\mathcal{P}_n - I)u\rangle_{\sigma_1}| + |\sum_{i=2}^{n-1} \langle \tilde{\mathcal{H}}(t,s(t,.)), (\mathcal{P}_n - I)u\rangle_{\sigma_i}| \\
&\quad + |\langle \tilde{\mathcal{H}}(t,s(t,.)) - v_t, (\mathcal{P}_n - I)u\rangle_{\sigma_n}| \Big\} \\
&= \sup_{t,\eta\in[0,1]} \Big\{ |\langle (\mathcal{P}_n - I)\tilde{\mathcal{H}}(t,s(t,.)), (\mathcal{P}_n - I)u\rangle_{\sigma_1}| \\
&\quad + |\sum_{i=2}^{n-1} \langle (\mathcal{P}_n - I)\tilde{\mathcal{H}}(t,s(t,.)), (\mathcal{P}_n - I)u\rangle_{\sigma_i}| \\
&\quad + |\langle \tilde{\mathcal{H}}(t,s(t,.)) - v_t, (\mathcal{P}_n - I)u\rangle_{\sigma_n}| \Big\} \\
&\leq \|(\mathcal{P}_n - I)\tilde{\mathcal{H}}(t,s(t,.))\|_{L^2(\sigma_1)} \|(\mathcal{P}_n - I)u\|_{L^2(\sigma_1)} \\
&\quad + \sum_{i=2}^{n-1} \|(\mathcal{P}_n - I)\tilde{\mathcal{H}}(t,s(t,.))\|_{L^2(\sigma_i)} \|(\mathcal{P}_n - I)u\|_{L^2(\sigma_i)} \\
&\quad + \|\tilde{\mathcal{H}}(t,s(t,.)) - v_t\|_{L^1(\sigma_n)} \|(\mathcal{P}_n - I)u\|_{L^\infty(\sigma_n)}.
\end{aligned} \tag{2.56}
$$

In the above, we have used $\langle v_t, (\mathcal{P}_n - I)u\rangle = 0$, for $v_t \in \mathbb{P}_m$.
Case-I:- If $\tilde{\mathcal{H}}(t,s(t,\eta)) = \tilde{m}(t,s(t,\eta))|1-\eta|^{-\alpha}$ with $0<\alpha<1$ is of algebraic type kernel.
In the subinterval $\sigma_1 = [0,\ t_1]$, for the first term of the above estimate (2.56), using the estimates (2.9), (2.14) and Lemma 1, we obtain

$$
\begin{aligned}
&\|(\mathcal{P}_n - I)\tilde{\mathcal{H}}(t,s(t,\cdot))\|_{L^\infty(\sigma_1)} \|(\mathcal{P}_n - I)u(s(t,\cdot))\|_{L^\infty(\sigma_1)} \\
&\quad \leq C h_1^m \Big\| \frac{\partial^m \tilde{\mathcal{H}}(t,s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_1)} c_1 h_1^{1-\alpha} \leq C n^{-(mq)-q(1-\alpha)} \Big\| \frac{\partial^m \tilde{\mathcal{H}}(t,s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_1)} \\
&\quad \leq C n^{-\left(\frac{m^2}{1-\alpha}+m\right)} \Big\| \frac{\partial^m \tilde{\mathcal{H}}(t,s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_1)}.
\end{aligned} \tag{2.57}
$$

For $\sigma_i = [t_{i-1},\ t_i],\ i = 2,\dots,n-1$, for the middle term of the above estimate (2.56) and using the estimates (2.11) and (2.15), we obtain

$$
\begin{aligned}
&\|(\mathcal{P}_n - I)\tilde{\mathcal{H}}(t,s(t,\cdot))\|_{L^2(\sigma_i)} \|(\mathcal{P}_n - I)u(s(t,\cdot))\|_{L^2(\sigma_i)} \\
&\quad \leq C h_i^m \Big\| \frac{\partial^m \tilde{\mathcal{H}}(t,s(t,\eta))}{\partial \eta^m} \Big\|_{L^2(\sigma_i)} h_i^m \Big\| \frac{\partial^m u(s(t,\eta))}{\partial \eta^m} \Big\|_{L^2(\sigma_i)} \\
&\quad \leq C h_i^m h_i^{\frac{1}{2}} \Big\| \frac{\partial^m \tilde{\mathcal{H}}(t,s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_i)} h_i^m h_i^{\frac{1}{2}} \Big\| \frac{\partial^m u(s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_i)} \\
&\quad \leq C h_i^{2m+1} \Big\| \frac{\partial^m \tilde{\mathcal{H}}(t,s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_i)} \Big\| \frac{\partial^m u(s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_i)} \\
&\quad \leq C n^{-(2m+1)} \Big\| \frac{\partial^m \tilde{\mathcal{H}}(t,s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_i)} \Big\| \frac{\partial^m u(s(t,\eta))}{\partial \eta^m} \Big\|_{L^\infty(\sigma_i)}.
\end{aligned} \tag{2.58}
$$

For $\sigma_n = [t_{n-1},\ 1]$, for the last term of the above estimate (2.56) and using the estimates (2.9), (2.16) and Theorem 2.1, we obtain

$$\begin{aligned}
&\|\tilde{\mathcal{H}}(t,s(t,\cdot)) - v_t\|_{L^1(\sigma_n)}\|(\mathcal{P}_n - I)u(s(t,\cdot))\|_{L^\infty(\sigma_n)} \\
&\qquad \leq \|\tilde{\mathcal{H}}(t,s(t,\cdot)) - v_t\|_{L^1([0,1])}\|(\mathcal{P}_n - I)u(s(t,\cdot))\|_{L^\infty(\sigma_n)} \\
&\qquad \leq Ch_n^{1-\alpha}\Big\|\frac{\partial^m u(s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty} h_n^m \\
&\qquad \leq Cn^{-(mq)-q(1-\alpha)}\Big\|\frac{\partial^m u(s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty} \\
&\qquad \leq Cn^{-\left(\frac{m^2}{1-\alpha}+m\right)}\Big\|\frac{\partial^m u(s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty}. \qquad (2.59)
\end{aligned}$$

Letting $\tilde{M}_{1,i} = \Big\|\frac{\partial^m\tilde{\mathcal{H}}(t,s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty(\sigma_i)}$, $\tilde{M}_{2,i} = \Big\|\frac{\partial^m u(s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty(\sigma_i)}$, $i = 2,..,n-1$, and combining the eatimates (2.57)-(2.59) with the estimate (2.56), we obtain

$$\begin{aligned}
|(\mathcal{F}\mathcal{P}_n - \mathcal{F})u(t)| &\leq Cn^{-\left(\frac{m^2}{1-\alpha}+m\right)}\tilde{M}_{1,1} + \sum_{i=2}^{n-1} Cn^{-(2m+1)}\tilde{M}_{1,i}\tilde{M}_{2,i} + Cn^{-\left(\frac{m^2}{1-\alpha}+m\right)}\tilde{M}_{2,n} \\
&= Cn^{-\left(\frac{m^2}{1-\alpha}+m\right)}\tilde{M}_{1,1} + Cn^{-2m}\sum_{i=2}^{n-1} n^{-1}\tilde{M}_{1,i}\tilde{M}_{2,i} + Cn^{-\left(\frac{m^2}{1-\alpha}+m\right)}\tilde{M}_{2,n} \\
&\leq Cn^{-2m}\max\left\{n^{-\left(\frac{m^2}{1-\alpha}-m\right)},\ 1,\ n^{-\left(\frac{m^2}{1-\alpha}-m\right)}\right\} \\
&\leq C_1 n^{-2m}, \qquad (2.60)
\end{aligned}$$

where $C_1 = C\max\left\{n^{-\left(\frac{m^2}{1-\alpha}-m\right)},\ 1,\ n^{-\left(\frac{m^2}{1-\alpha}-m\right)}\right\}$, which is independent of n.
This implies

$$\|\mathcal{F}(\mathcal{P}_n - \mathcal{I})u\|_\infty = \mathcal{O}(n^{-2m}). \qquad (2.61)$$

Case-II:- If $\tilde{\mathcal{H}}(t,s(t,\eta)) = \tilde{m}(t,s(t,\eta))\log(|t(1-\eta)|)$ with $\alpha = 1$ is of logarithmic type kernel.
In the subinterval $\sigma_1 = [0,\ t_1]$, for the first term of the estimate (2.56) and using the estimates (2.9), (2.14) and Lemma 2, we obtain

$$\begin{aligned}
&\|(\mathcal{P}_n - I)\tilde{\mathcal{H}}(t,s(t,\cdot))\|_{L^\infty(\sigma_1)}\|(\mathcal{P}_n - I)u(s(t,\cdot))\|_{L^\infty(\sigma_1)} \\
&\qquad \leq Ch_1^m\Big\|\frac{\partial^m\tilde{\mathcal{H}}(t,s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty(\sigma_1)} c_2 h_1 \\
&\qquad \leq Cn^{-mq-q}\Big\|\frac{\partial^m\tilde{\mathcal{H}}(t,s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty(\sigma_1)} \\
&\qquad \leq Cn^{-(m^2+m)}\Big\|\frac{\partial^m\tilde{\mathcal{H}}(t,s(t,\eta))}{\partial\eta^m}\Big\|_{L^\infty(\sigma_1)}. \qquad (2.62)
\end{aligned}$$

For $\sigma_i = [t_{i-1},\ t_i]$, $i = 2, \ldots, n-1$, for the middle term of the estimate (2.56) and using the estimates (2.11) and (2.15), we obtain

$$
\begin{aligned}
&\|(\mathcal{P}_n - I)\tilde{\mathcal{H}}(t, s(t,\cdot))\|_{L^2(\sigma_i)}\|(\mathcal{P}_n - I)u(s(t,\cdot))\|_{L^2(\sigma_i)} \\
&\qquad \le Ch_i^m \Big\|\frac{\partial^m \tilde{\mathcal{H}}(t, s(t,\eta))}{\partial \eta^m}\Big\|_{L^2(\sigma_i)} h_i^m \Big\|\frac{\partial^m u(s(t,\eta))}{\partial \eta^m}\Big\|_{L^2(\sigma_i)} \\
&\qquad \le Ch_i^m h_i^{\frac{1}{2}} \Big\|\frac{\partial^m \tilde{\mathcal{H}}(t, s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_i)} h_i^m h_i^{\frac{1}{2}} \Big\|\frac{\partial^m u(s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_i)} \\
&\qquad \le Ch_i^{2m+1} \Big\|\frac{\partial^m \tilde{\mathcal{H}}(t, s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_i)} \Big\|\frac{\partial^m u(s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_i)} \\
&\qquad \le Cn^{-(2m+1)} \Big\|\frac{\partial^m \tilde{\mathcal{H}}(t, s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_i)} \Big\|\frac{\partial^m u(s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_i)}.
\end{aligned}
\tag{2.63}
$$

For $\sigma_n = [t_{n-1},\ 1]$, from the last term of the estimate (2.56) and using the estimates (2.9), (2.16) and Theorem 2.1, we obtain

$$
\begin{aligned}
&\|\tilde{\mathcal{H}}(t, s(t,\cdot)) - v_t\|_{L^1(\sigma_n)}\|(\mathcal{P}_n - I)u(s(t,\cdot))\|_{L^\infty(\sigma_n)} \\
&\qquad \le \|\tilde{\mathcal{H}}(t, s(t,\cdot)) - v_t\|_{L^1([0,1])}\|(\mathcal{P}_n - I)u(s(t,\cdot))\|_{L^\infty(\sigma_n)} \\
&\qquad \le Ch_n \log h_n \Big\|\frac{\partial^m u(s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_n)} h_n^m \\
&\qquad \le Cn^{-(m^2+m)} \log n \Big\|\frac{\partial^m u(s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_n)}.
\end{aligned}
\tag{2.64}
$$

Letting $M_{1,i} = \Big\|\frac{\partial^m \tilde{\mathcal{H}}(t,s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_i)}$, $M_{2,i} = \Big\|\frac{\partial^m u(s(t,\eta))}{\partial \eta^m}\Big\|_{L^\infty(\sigma_i)}$, $i = 2, 3, .., n-1$, and combining the estimates (2.62)-(2.64) with the estimate (2.56), we obtain

$$
\begin{aligned}
|(\mathcal{F}\mathcal{P}_n - \mathcal{F})u(t)| &\le Cn^{-(m^2+m)}M_{1,1} + C\sum_{i=2}^{n-1} n^{-(2m+1)}M_{1,i}M_{2,i} + Cn^{-(m^2+m)}\log n M_{2,1} \\
&= Cn^{-(m^2+m)}M_{1,1} + Cn^{-2m}\sum_{i=2}^{n-1} n^{-1}M_{1,i}M_{2,i} + Cn^{-(m^2+m)}\log n M_{2,1} \\
&\le Cn^{-(m^2+m)}M_{1,1} + Cn^{-2m} + Cn^{-(m^2+m)}\log n M_{2,1} \\
&\le C\max\{n^{-(m^2+m)},\ n^{-2m},\ n^{-(m^2+m)}\log n\} \\
&= Cn^{-2m}\max\{n^{-(m^2-m)},\ 1,\ n^{-(m^2-m)}\log n\} \\
&\le C_2 n^{-2m}\log n,
\end{aligned}
\tag{2.65}
$$

where $C_2 = C\max\{n^{-(m^2-m)},\ 1,\ n^{-(m^2-m)}\}$, which is independent of n.
This implies

$$
\|\mathcal{F}(\mathcal{P}_n - \mathcal{I})u\|_\infty = \mathcal{O}(n^{-2m}\log n). \tag{2.66}
$$

Hence combining (2.61) and (2.66) with (2.55), we have

$$\|u - \tilde{u}_n\|_{L^\infty} = \begin{cases} \mathcal{O}(n^{-2m}), & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-2m}\log n), & \text{if } \alpha = 1. \end{cases} \tag{2.67}$$

This completes the proof.

Theorem 2.6. *Let $\widetilde{\mathcal{H}}(t, s(t,\eta))$ be the kernel of the operator $\mathcal{F}$ given by (2.2) and u_n be the Galerkin approximate solution of integral equation (2.1) then there holds*

$$\|u - u_n\|_{L^\infty} = \mathcal{O}(n^{-m}).$$

PROOF. Using $u_n = \mathcal{P}_n\tilde{u}_n$, we obtain

$$u - u_n = u - \mathcal{P}_n\tilde{u}_n = u - \mathcal{P}_n u + \mathcal{P}_n u - \mathcal{P}_n\tilde{u}_n.$$

From Lemma 1 and Lemma 2, we have

$$\|(I - \mathcal{P}_n)u\|_{L^\infty} \le Cn^{-m}. \tag{2.68}$$

Hence using Theorem 2.5, estimates (2.13) and (2.68), we have

$$\begin{aligned} \|u - u_n\|_{L^\infty} &\le \|(I - \mathcal{P}_n)u\|_{L^\infty} + \|\mathcal{P}_n(u - \tilde{u}_n)\|_{L^\infty}, \\ &\le Cn^{-m} + \rho\ \|u - \tilde{u}_n\|_{L^\infty} = \mathcal{O}(n^{-m}). \end{aligned}$$

Hence the theorem is proved.

In the next section, we improve the convergence results.

3. Superconvergence results for multi-projection method

We analyse the iterated version of multi-Galerkin method to obtain their improved convergence results. The multi Galerkin operator (see [4, 6, 19]) is defined as

$$\mathcal{F}_n^M = \mathcal{P}_n\mathcal{F}\mathcal{P}_n + \mathcal{P}_n\mathcal{F}(I - \mathcal{P}_n) + (I - \mathcal{P}_n)\mathcal{F}\mathcal{P}_n. \tag{3.1}$$

The multi-Galerkin method for integral equation (1.1)-(1.2) is defined as, find $u_n^M \in \mathbb{X}$ such that

$$u_n^M - \mathcal{F}_n^M u_n^M = g, \tag{3.2}$$

and iterated multi-Galerkin solution is defined as

$$\tilde{u}_n^M = \mathcal{F}u_n^M + g. \tag{3.3}$$

From Theorem 2.4, it follows that

$$\|\mathcal{F} - \mathcal{F}_n^M\|_{L^\infty} = \|(I - \mathcal{P}_n)\mathcal{F}(I - \mathcal{P}_n)\|_{L^\infty} \le (1 + \|\mathcal{P}_n\|_{L^\infty})\|\mathcal{F}(I - \mathcal{P}_n)\|_{L^\infty} \to 0 \ \text{ as } \ n \to \infty.$$

Hence $(I - \mathcal{F}_n^M)^{-1}$ exists and uniformly bounded for large enough n. This shows that $u_n^M \in \mathbb{X}$ is the unique solution of the equation (3.2).
Next, we discuss the convergence rates in multi-Galerkin and iterated multi-Galerkin methods.

Theorem 3.1. *Let u_n^M be the multi-Galerkin and $\tilde{u}_n^M$ be the iterated multi-Galerkin approximations of the solution u of (2.1), then there hold*

$$\|u - u_n^M\|_{L^\infty} = \begin{cases} \mathcal{O}(n^{-2m}), & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-2m}\log n), & \text{if } \alpha = 1, \end{cases} \tag{3.4}$$

and

$$\|u - \tilde{u}_n^M\|_{L^\infty} = \begin{cases} \mathcal{O}(n^{-3m}), & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-3m}(\log n)^2), & \text{if } \alpha = 1. \end{cases} \tag{3.5}$$

Proof. From equations (2.5) and (3.2), we obtain

$$\begin{aligned} u - u_n^M = (I - \mathcal{F})^{-1}g - (I - \mathcal{F}_n^M)^{-1}g &= (I - \mathcal{F}_n^M)^{-1}(\mathcal{F} - \mathcal{F}_n^M)(I - \mathcal{F})^{-1}g \\ &= (I - \mathcal{F}_n^M)^{-1}(\mathcal{F} - \mathcal{F}_n^M)u. \end{aligned}$$

Since $\|(I - \mathcal{F}_n^M)^{-1}\|_\infty \leq \mathcal{L}_1 < \infty$ and $\|\mathcal{P}_n\|_\infty \leq \rho < \infty$, we have

$$\begin{aligned} \|u - u_n^M\|_\infty &\leq \|(I - \mathcal{F}_n^M)^{-1}\|_\infty \|(\mathcal{F} - \mathcal{F}_n^M)u\|_\infty \\ &\leq \mathcal{L}_1 \|(\mathcal{F} - \mathcal{F}_n^M)u\|_\infty \\ &\leq \mathcal{L}_1 \|(I - \mathcal{P}_n)\mathcal{F}(I - \mathcal{P}_n)u\|_\infty \\ &\leq \mathcal{L}_1 (1 + \|\mathcal{P}_n\|_\infty)\|\mathcal{F}(I - \mathcal{P}_n)u\|_\infty \\ &\leq \mathcal{L}_1 (1 + \rho)\|\mathcal{F}(I - \mathcal{P}_n)u\|_\infty. \end{aligned} \tag{3.6}$$

Hence by using the estimates (2.61) and (2.66) in the estimate (3.6), we obtain

$$\|u - u_n^M\|_{L^\infty} \leq \begin{cases} \mathcal{L}_1(1 + \rho)Cn^{-2m}, & \text{if } 0 < \alpha < 1, \\ \mathcal{L}_1(1 + \rho)Cn^{-2m}\log n, & \text{if } \alpha = 1. \end{cases}$$

From this the result follows.

$$\|u - u_n^M\|_{L^\infty} = \begin{cases} \mathcal{O}(n^{-2m}), & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-2m}\log n), & \text{if } \alpha = 1. \end{cases} \tag{3.7}$$

Again from equations (2.5) and (3.3),

$$\begin{aligned} u - \tilde{u}_n^M &= \mathcal{F}(u - u_n^M) \\ &= \mathcal{F}(I - \mathcal{F})^{-1}(\mathcal{F} - \mathcal{F}_n^M)(I - \mathcal{F}_n^M)^{-1}g \\ &= (I - \mathcal{F})^{-1}\mathcal{F}(I - \mathcal{P}_n)\mathcal{F}(I - \mathcal{P}_n)(u + u_n^M - u) \\ &= (I - \mathcal{F})^{-1}[\mathcal{F}(I - \mathcal{P}_n)\mathcal{F}(I - \mathcal{P}_n)u - \mathcal{F}(I - \mathcal{P}_n)\mathcal{F}(I - \mathcal{P}_n)(u - u_n^M)]. \end{aligned} \tag{3.8}$$

This implies

$$\begin{aligned} \|u - \tilde{u}_n^M\|_\infty &\leq \|(I - \mathcal{F})^{-1}\|_\infty [\|\mathcal{F}(I - \mathcal{P}_n)\mathcal{F}(I - \mathcal{P}_n)u\|_\infty + \|\mathcal{F}(I - \mathcal{P}_n)\mathcal{F}(I - \mathcal{P}_n)(u - u_n^M)\|_\infty] \\ &\leq \|(I - \mathcal{F})^{-1}\|_\infty [\|\mathcal{F}(I - \mathcal{P}_n)\|_\infty \|\mathcal{F}(I - \mathcal{P}_n)u\|_\infty + \|\mathcal{F}(I - \mathcal{P}_n)\|_\infty^2 \|(u - u_n^M)\|_\infty]. \end{aligned} \tag{3.9}$$

Hence using the estimates (2.54), (2.61), (2.66) and (3.7) in the estimate (3.9), we obtain

$$\|u - \tilde{u}_n^M\|_{L^\infty} = \begin{cases} \mathcal{O}(n^{-3m}), & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-3m}(\log n)^2), & \text{if } \alpha = 1. \end{cases}$$

This completes the proof.

Remark 3.1. *a. From Theorems 2.5 and 2.6, we see that for algebraic type kernel, the Galerkin method and its iterated version converge with the orders $\mathcal{O}(n^{-m})$ and $\mathcal{O}(n^{-2m})$, respectively. For logarithmic type kernel, Galerkin method and its iterated version converge with the orders $\mathcal{O}(n^{-m})$ and $\mathcal{O}(n^{-2m}\log n)$, respectively. From this, we notice that iterated Galerkin solution improves over Galerkin solution.*

b. From Theorem 3.1, we notice that the iterated multi Galerkin approximate solution converges with the rate $\mathcal{O}(n^{-3m})$ and $\mathcal{O}(n^{-3m}(\log n)^2)$ for algebraic and logarithmic type kernels, respectively.

From this, we observe that the iterated multi-Galerkin approximate solution improves over the iterated Galerkin solution.

4. Numerical Illustration

We consider the approximation subspace $\mathbb{X}_n$ as a space of piecewise constant functions with respect to the partition (2.6) that is $(m = 1)$. We present the errors and convergence rates of the approximations and iterated approximations of Galerkin and multi-Galerkin methods.

Denote

$$\|u - u_n\|_{L^\infty} = \mathcal{O}(n^{-\beta}), \qquad \|u - \tilde{u}_n\|_{L^\infty} = \mathcal{O}(n^{-\gamma}),$$
$$\|u - u_n^M\|_{L^\infty} = \mathcal{O}(n^{-\delta}), \qquad \|u - \tilde{u}_n^M\|_{L^\infty} = \mathcal{O}(n^{-\lambda}).$$

The numerical algorithm was executed on a PC Intel(R) CPU @ 3.20 GHz Processor, Core (TM) i5-3470, 4.00GB RAM and 64-bit operating system on Matlab(R2013b).

$\beta,\ \gamma,\ \delta,\ \lambda$ are calculated for the following examples.

Example 4.1. *In this example, we consider the Volterra integral equation with a weakly singular kernel as*

$$u(t) = g(t) + \int_0^t \mathcal{H}(t,s)u(s)ds, \quad 0 < t \leq 1, \quad 0 < \alpha < 1,$$

with $\mathcal{H}(t,s) = \frac{1}{2}(t-s)^{-\alpha}$, $\alpha = \frac{1}{2}$ and the right side function $g(t) = 1 - \frac{1}{4}\pi t$ and the exact solution $u(t) = 1 + t^{\frac{1}{2}}$, which is non-smooth at $t = 0$.

Using transformation $s = t\eta$, we get the following integral equation

$$u(t) = g(t) + \int_0^1 \tilde{\mathcal{H}}(t, s(t,\eta))u(s(t,\eta))d\eta, \quad t \in [0,1], \quad 0 < \alpha < 1,$$

with kernel $\tilde{\mathcal{H}}(t,s(t,\eta)) = \frac{1}{2}t^{\frac{1}{2}}(1-\eta)^{-\frac{1}{2}}$. *Then for* $m=1$ *and* $q=\frac{m}{1-\alpha}$, *the expected order of convergence are* $\beta=1$, $\gamma=2$, $\delta=2$ *and* $\lambda=3$. *For Galerkin and iterated Galerkin methods, the error bounds and convergence rates are presented in Table 1 and for multi-Galerkin and iterated multi-Galerkin methods are presented in Table 2.*

Table 1: Galerkin and iterated Galerkin methods

n	$\Vert u-u_n\Vert_{L^\infty}$	β	$\Vert u-\tilde{u}_n\Vert_{L^\infty}$	γ
2	$3.5937870000\times10^{-1}$	1.04	$1.3595032391\times10^{-1}$	2.03
4	$2.1189406688\times10^{-1}$	1.02	$3.9483026956\times10^{-2}$	2.12
8	$1.1343488703\times10^{-1}$	1.01	$9.9136707350\times10^{-3}$	2.15
16	$4.9835781429\times10^{-2}$	1.06	$3.2185262010\times10^{-3}$	2.04
32	$2.9012135736\times10^{-2}$	1.01	$8.3550341158\times10^{-4}$	2.03
64	$1.3547608069\times10^{-2}$	1.03	$2.4132242067\times10^{-4}$	1.99
128	$8.7037737316\times10^{-3}$	0.97	$5.9745152247\times10^{-5}$	2.00

Table 2: Multi Galerkin and iterated multi Galerkin methods

n	$\Vert u-u_n^M\Vert_{L^\infty}$	δ	$\Vert u-\tilde{u}_n^M\Vert_{L^\infty}$	λ
2	$1.2172765434\times10^{-1}$	2.14	$4.9603664790\times10^{-2}$	3.06
4	$3.9003163004\times10^{-2}$	2.13	$9.1988585949\times10^{-3}$	3.08
8	$1.0260468329\times10^{-2}$	2.13	$1.5097354221\times10^{-3}$	3.02
16	$3.2848147599\times10^{-3}$	2.03	$2.4640931548\times10^{-4}$	2.96
32	$7.5756882417\times10^{-4}$	2.06	$2.3524588289\times10^{-5}$	3.06
64	$1.8794344319\times10^{-4}$	2.05	$3.2159579930\times10^{-6}$	3.03
128	$4.8417180950\times10^{-5}$	2.04	$4.3986038300\times10^{-7}$	3.01

Example 4.2. *Consider the following Volterra integral equation with weakly singular logarithmic kernel*

$$u(t) = g(t) + \int_0^t \mathcal{H}(t,s)u(s)ds, \quad t \in [0,1], \quad \alpha = 1,$$

with $\mathcal{H}(t,s) = \log|(t-s)|$ *and the right side function* $g(t) = t(\log t - 1) + \frac{t^2}{12}(\pi^2 - 21 + 18\log t - 6\log^2 t)$ *and the solution* $u(t) = t(\log t - 1)$*, which is non-smooth at* $t = 0$*.*
Using transformation $s = t\eta$*, we get the following integral equation*

$$u(t) = g(t) + \int_0^1 \tilde{\mathcal{H}}(t, s(t,\eta))u(s(t,\eta))d\eta, \quad t \in [0,1], \quad \alpha = 1,$$

kernel $\tilde{\mathcal{H}}(t, s(t,\eta)) = t\log|(t - t\eta)|$*. Then for* $m = 1$ *and* $q = m$*, the expected order of convergence are* $\beta = 1$*,* $\gamma \approx 2$*,* $\delta \approx 2$ *and* $\lambda \approx 3$*.The error bounds and convergence rates for Galerkin and iterated Galerkin methods are presented in Table 3 and for M-Galerkin and iterated M-Galerkin methods are presented in Table 4.*

Table 3: Galerkin and iterated Galerkin methods

n	$\|u - u_n\|_{L^\infty}$	β	$\|u - \tilde{u}_n\|_{L^\infty}$	γ
2	$3.7826190701 \times 10^{-1}$	0.99	$1.4295496824\times 10^{-1}$	1.98
4	$2.3225864654\times 10^{-1}$	0.96	$4.6269592280\times 10^{-2}$	2.02
8	$1.1478934454\times 10^{-1}$	1.00	$1.8750065292\times 10^{-2}$	1.85
16	$5.9788467281\times 10^{-2}$	1.00	$6.0085808766 \times 10^{-3}$	1.82
32	$3.6197815141\times 10^{-2}$	0.95	$2.0220445497\times 10^{-3}$	1.78
64	$1.3891010776\times 10^{-2}$	1.02	$5.3020659199\times 10^{-4}$	1.81
128	$8.6491699050\times 10^{-3}$	0.97	$1.6540973099\times 10^{-4}$	1.79

Table 4: Multi Galerkin and iterated multi Galerkin methods

n	$\|u - u_n^M\|_{L^\infty}$	δ	$\|u - \tilde{u}_n^M\|_{L^\infty}$	λ
2	$1.3927052015 \times 10^{-1}$	2.00	$5.1914897366 \times 10^{-2}$	3.01
4	$4.9403962519 \times 10^{-2}$	1.97	$9.3664384731 \times 10^{-3}$	3.07
8	$1.2138790300\times 10^{-2}$	2.05	$1.3702678246 \times 10^{-3}$	3.07
16	$3.6296481261 \times 10^{-3}$	2.00	$2.1019322314 \times 10^{-4}$	3.01
32	$1.2007369928 \times 10^{-3}$	1.93	$2.3112874718 \times 10^{-5}$	3.06
64	$3.8346841070\times 10^{-4}$	1.88	$4.4870949940 \times 10^{-6}$	2.95
128	$9.2111510735\times 10^{-5}$	1.91	$6.2344498400\times 10^{-7}$	2.94

From Tables 1, 2, 3 and 4, we see that the computed values of β, γ, δ and λ are well matched with the expected convergence results. From Tables 1 and 2 of Example 4.1 and from Tables 3 and 4 of Example 4.2, we see that the iterated multi Galerkin method improves over iterated Galerkin method. Note that the size of the system of the equations that must be solved, remains the same as in Galekrin method.

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