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An application of Liapunov's method for the analysis of neural networks

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Abstract

The stability is studied of a class of nonlinear dynamical systems which possess many nonlinearities and many equilibrium states. As a special case, the analyzed class of systems includes analog neural networks. Sufficient conditions for the nonoscillatory behaviour of these systems, in the form of frequency domain criteria, are presented. The main result is proved relying on a suitable Liapunov function which is subsequently used for the simultaneous computation of regions of attraction for each stable equilibrium.

Key words: Liapunov's method; Stability; Neural networks

1. Introduction

Analog neural networks are usually implemented as electrical circuits, composed of a number of interconnected nonlinear blocks with identical structures. In general the system possesses several equilibrium states. These nets, proposed as parallel computing devices, offer an approach for solving classification and optimization problems [2,9]. The equilibrium states represent the prototype vectors of the different classes, when the system is used as a classifier, or the optima, when it is used as an optimizer. For its proper operation it is essential that a neural network does not oscillate. Each trajectory, initiated by a state presented to the network, must converge to one of the equilibria. It is important that the regions of attraction of the equilibria can be estimated as precisely as possible because they represent the different classes of a classification network or the basin of attraction of an optimum of an optimization network.

The direct method of Liapunov has been used as a tool for the analysis of analog neural networks by Cohen and Grossberg [1] and by Kosko [5,9]. More recent contributions are due, among others, to Hasler [2] and Keen [4]. Further references can be found in [10]. In this paper

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circuits are considered whose equations belong to a class of nonlinear autonomous dynamical systems of the form

$$\dot{x} = Ax - Bf(\sigma) - h, \quad \sigma \triangleq C'x, \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state and $f(\sigma) \triangleq [f_1(\sigma), \dots, f_m(\sigma)]'$ is a nonlinear function of $\sigma = [\sigma_1, \dots, \sigma_m]'$. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times m}$ are constant matrices, $h \in \mathbb{R}^n$ is a constant vector. A is assumed to be nonsingular and (A, B) is controllable. A stability criterion for system (1) is derived in Section 2. Section 3 presents a systematic method for the simultaneous computation of the regions of attraction of all stable equilibria. The analysis of neural networks using the results of Sections 2 and 3 is given in Section 4. In the discussion of Section 5 the present approach is compared to the classical Cohen and Grossberg [1] and Kosko [5] theorems. The paper concludes with an example.

2. Stability criterion

The Liapunov function $V(x)$ for system (1) has the form

$$V(x) = x'Px + f'(\sigma)Sf(\sigma) + x'Lf(\sigma) + \int_0^\sigma f'(u)\bar{\alpha}\, du + p'x + q'f(\sigma), \tag{2}$$

where $P = P'$, $S = S'$ and $\bar{\alpha} = \mathrm{diag}(\alpha_i)$.

Differentiation of $V(x)$ along the solutions of system (1) and suitable choices for the differential matrices

$$p = -2PA^{-1}h, \qquad q = -L'A^{-1}h, \qquad L = -C\bar{\alpha} - 2PA^{-1}B,$$

$$S = -\tfrac{1}{2}B'A^{-1\prime}L = B'A^{-1\prime}PA^{-1}B + \tfrac{1}{2}B'A^{-1\prime}C\bar{\alpha}$$

yields

$$\dot{V}(x) = \dot{x}'\left[PA^{-1} + A^{-1\prime}P\right]\dot{x} - \dot{x}'A^{-1\prime}\left[C\bar{\alpha} + 2PA^{-1}B\right]f_{\mathrm{d}}(\sigma)C'\dot{x},$$

where $f_{\mathrm{d}}(\sigma) \triangleq \mathrm{diag}[f_{i\mathrm{d}}(\sigma_i)]$ and $f_{i\mathrm{d}}(\sigma_i) = \mathrm{d}f_i(\sigma_i)/\mathrm{d}\sigma_i$.

Since S is symmetric, the matrix $\bar{\alpha}C'A^{-1}B$ must be symmetric as well. Assume the nonlinearities satisfy slope constraints of the form

$$0 \leqslant f_{i\mathrm{d}}(\sigma_i) \leqslant k_i, \quad \text{all } \sigma_i. \tag{3}$$

Then for any $\bar{\gamma} = \mathrm{diag}(\gamma_i) \geqslant 0$ it follows that

$$\lambda \triangleq \dot{x}'C\left[I - K^{-1}f_{\mathrm{d}}(\sigma)\right]\bar{\gamma}f_{\mathrm{d}}(\sigma)C'\dot{x} \geqslant 0,$$

where $K = \mathrm{diag}(k_i)$. Using the transformation

$$W \triangleq A^{-1\prime}PA^{-1}, \tag{4}$$

$\dot{V}(x)$ can be written as

$$\dot{V}(x) = -\lambda - \left[Q'\dot{x} + K^{-1/2}\bar{\gamma}^{1/2}f_{\mathrm{d}}(\sigma)C'\dot{x}\right]'\left[Q'\dot{x} + K^{-1/2}\bar{\gamma}^{1/2}f_{\mathrm{d}}(\sigma)C'\dot{x}\right] - \epsilon\dot{x}'D\dot{x}, \tag{5}$$

for matrices $Q \in \mathbb{R}^{n \times m}$, $D = D' > 0$, $D \in \mathbb{R}^{n \times n}$, and for a scalar $\epsilon > 0$, if the equations

$$A'W + WA = -QQ' - \varepsilon D, \tag{6}$$

$$2WB + A^{-1\prime}C\bar{\alpha} - C\bar{\gamma} = 2QK^{-1/2}\bar{\gamma}^{1/2} \tag{7}$$

have a real solution $W = W'$, Q.

By the Kalman–Yakubovich–Popov main lemma [8,12], the solvability of (6), (7) is guaranteed if $\epsilon > 0$ is sufficiently small and if, after some manipulations,

$$\mathrm{He}\left(\bar{\gamma} - \frac{1}{\mathrm{j}\omega}\bar{\alpha}\right)\left[K^{-1} + C'(\mathrm{j}\omega I - A)^{-1}B\right] > 0, \quad \text{all real } \omega. \tag{8}$$

By (5), $\dot{V}(x) \leqslant 0$ for all x and $\dot{V}(x) = 0$ if and only if $\dot{x} = 0$. Hence, according to [6] all bounded trajectories of the system (1) converge to an equilibrium state. A system which has this property will be called *nonoscillatory*. Eq. (8) is the multivariable version of a simpler "scalar" case which has been reported before [7].

3. Computation of stability regions

In this section it is assumed that all solutions of system (1) are bounded and that the conditions for nonoscillatory behaviour are satisfied. Now, following the method outlined in [7], the Liapunov function (2) is used to construct regions of asymptotic stability for the stable equilibria: as $\dot{V}(x) \leqslant 0$ for all x and there are no trajectories along which $\dot{V}(x) \equiv 0$ except the equilibrium solutions, $V(x)$ reaches a relative minimum at every stable equilibrium. Suppose x_A is such a stable equilibrium. Let C be a constant slightly larger than $V(x_A)$. Then the simply connected subset S_1 of the set $S \triangleq \{x, V(x) < C\}$ which contains x_A, and no other equilibrium state, is a region of attraction of x_A. S_1 grows with increasing C. The largest set S_1, within the scope of the method, which is still an estimate of the region of attraction of x_A is obtained for $C = V(x_e)$. x_e is such that, among all unstable equilibria, $V(x_e)$ assumes the smallest value for which $V(x_e) > V(x_A)$. More generally, any set of the form

$$S \triangleq \{x;\, V(x) < C\},$$

and any simply connected subset of it, is a region of attraction for the union of all equilibria contained in it.

Considering the dependence of $V(x)$ on the matrix P, the Liapunov function can be written as the sum of a term dependent on P and a term independent of P:

$$V(x) = \left[x' - f'(\sigma)B'A^{-1\prime}\right]P\left[x - A^{-1}Bf(\sigma)\right] - 2h'A^{-1\prime}P\left[x - A^{-1}Bf(\sigma)\right] + \Phi(x),$$

where

$$\Phi(x) \triangleq \tfrac{1}{2}f'(\sigma)B'A^{-1\prime}C\bar{\alpha}f(\sigma) - x'C\bar{\alpha}f(\sigma) + \int_0^{\sigma} f'(u)\bar{\alpha}\,\mathrm{d}u + h'A^{-1\prime}C\bar{\alpha}f(\sigma).$$

Since x_e is an equilibrium state of system (1),

$$\dot{x}_e = 0 = Ax_e - Bf(\sigma_e) - h, \quad \sigma_e \triangleq C'x_e,$$

and

$$V(x_e) = -h'A^{-1\prime}PA^{-1}h + \Phi(x_e).$$

Hence, after some manipulations, the set $\{x;\ V(x) < V(x_e)\}$ takes the form

$$\begin{aligned} S &= \{x;\ [x - A^{-1}Bf(\sigma) - A^{-1}h]' P [x - A^{-1}Bf(\sigma) - A^{-1}h] < \Phi(x_e) - \Phi(x)\} \\ &= \{x;\ [Ax - Bf(\sigma) - h]' W [Ax - Bf(\sigma) - h] < \Phi(x_e) - \Phi(x)\}. \end{aligned} \tag{9}$$

W is a solution of (6), (7) or, after elimination of Q, a solution of the algebraic Riccati equation

$$A'W + WA + \left[WB + \tfrac{1}{2}A^{-1\prime}C\bar{\alpha} - \tfrac{1}{2}C\bar{\gamma}\right]\bar{\gamma}^{-1}K\left[WB + \tfrac{1}{2}A^{-1\prime}C\bar{\alpha} - \tfrac{1}{2}C\bar{\gamma}\right]' + \epsilon D = 0. \tag{10}$$

x_e and $\Phi(x)$ are independent of the choice of W. On the other hand, the set S grows monotonically as W decreases. So the best choice of W is the minimal solution $W^-(\epsilon)$ of (10). According to [12] this minimal solution decreases as ϵ and K decrease. So S and the estimate S_1 of the region of attraction of x_A are maximized by letting $\epsilon \downarrow 0$ and by choosing K as small as possible such that the slope constraints (3) are still satisfied.

4. Neural networks

Hopfield's network is an example of a neural classification network [3]. It is an electrical circuit composed of n interconnected nonlinear blocks with identical structures. Each block or artificial neuron consists of a resistor R_i, a capacitor C_i, and a nonlinear amplifier with a bounded S-shaped characteristic. The characteristics satisfy slope constraints of the form (3). The artificial neurons are interconnected by a synaptic resistive network consisting of resistors R_{ij}, $i, j = 1, \ldots, n$. The network is fed with constant currents I_i. The equations for the amplifier input voltages u_i are [2]:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{1}{R_iC_i}u_i + \frac{1}{C_i}\left[\sum_{j=1}^{n}\frac{f_j(u_j) - u_i}{R_{ij}} + I_i\right], \quad i = 1, \ldots, n. \tag{11}$$

These equations have the form of system (1) with corresponding matrices

$$A = \operatorname{diag}(a_i), \quad a_i = -\frac{1}{C_i}\left[\frac{1}{R_i} + \sum_{j=1}^{n}\frac{1}{R_{ij}}\right],$$

$$h = -\left[\frac{I_1}{C_1}, \ldots, \frac{I_n}{C_n}\right]', \qquad f(\sigma) = [f_1(x_1), \ldots, f_n(x_n)]', \qquad C = I, \qquad B = -\Gamma\Lambda,$$

$$\Gamma = \operatorname{diag}\left(\frac{1}{C_i}\right).$$

Λ is the synaptic matrix

$$\Lambda \triangleq \begin{pmatrix} \dfrac{1}{R_{11}} & \cdots & \dfrac{1}{R_{1n}} \\ \vdots & & \\ \dfrac{1}{R_{n1}} & \cdots & \dfrac{1}{R_{nn}} \end{pmatrix}.$$

Using the boundedness of $Bf(\sigma)+h$ and the stability of the matrix A, it is a simple exercise to prove the boundedness of all solutions of the network. Indeed let $U=U'>0$ be the solution of Liapunov's equation [11]

$$UA+A'U=-I.$$

Then it is easily verified that along the solutions of (1), $(\mathrm{d}/\mathrm{d}t)(x'Ux)<0$ for $x'Ux>R^2$ and R^2 sufficiently large. To check the network's nonoscillatory behaviour, choose $\bar{\alpha}=\bar{\gamma}A$, which requires that $\bar{\gamma}B$ is symmetric. The frequency domain condition becomes $\bar{\gamma}K^{-1}>0$, which is satisfied for any $\bar{\gamma}>0$. Choosing $\bar{\gamma}=I$, the remaining condition becomes $B=B'$. Note that if $\Lambda=\Lambda'$, then B can be made symmetric by rescaling the nonlinearities. This means that Hopfield's network cannot oscillate if the synaptic matrix Λ is symmetric. From now on suppose $B=B'$. An estimate of the region of attraction of a stable equilibrium point x_A is the simply connected subset S_1 of the set (9) which contains the equilibrium point. For $C=\bar{\gamma}=I$ and $\bar{\alpha}=A$, Eq. (10) simplifies to

$$A'W+WA+BWKWB+\epsilon D=0,$$

whose minimal solution $W^-(\epsilon)\downarrow 0$ for $\epsilon\downarrow 0$. As a consequence, every point in the set $\hat{S}=\{x;\ \Phi(x)<\Phi(x_e)\}$ is also a point in the set S if ϵ is chosen small enough. The best estimate of the region of attraction of a stable equilibrium point is the simply connected subset $\hat{S}_1$ of $\hat{S}$ which contains that equilibrium point.

5. Discussion

There exists an interesting relationship between the results of this paper and the works of Cohen and Grossberg [1] and of Kosko [5]. The Hopfield network's model (11) is a special case both of the system (1) and of Cohen and Grossberg's neural network model

$$\dot{x}_i=a_i(x_i)\left[b_i(x_i)-\sum_{k=1}^{n}c_{ik}d_k(x_k)\right],\quad i=1,\dots,n. \tag{12}$$

So the symmetry of the synaptic matrix, as a sufficient condition for the nonoscillatory behaviour of (11), is an immediate consequence of the Cohen and Grossberg theorem [1], and of the more general Kosko theorems [5,9]. As a matter of fact, this symmetry condition has also been obtained by other authors [2]. However, model (1) is not a special case of (12) unless the matrix A is diagonal, the matrix $C=I$ and $m=n$. The present analysis relates the symmetry condition on the synaptic matrix of a neural network to the stability conditions of nonlinear feedback systems. In automatic control theory equations of the form (1) have been used to model the dynamics of multivariable nonlinear feedback loops whose forward path component is linear with transfer matrix $H(s)=C'(sI-A)^{-1}B$ (see, e.g., [8]). The structure of Hopfield's network ensures that the frequency stability condition (8) on $H(s)$ holds automatically, while the symmetry condition on $\bar{\alpha}C'A^{-1}B=-\bar{\alpha}H(0)$ translates into the symmetry condition on the synaptic matrix.

The Liapunov function (2) approaches the Liapunov function used in [1,5] as $\epsilon\downarrow 0$ and $\bar{\alpha}=A$. However, there is an advantage in choosing $\epsilon\neq 0$. In the case where the functions $d_i(\xi_i)$

in (12) have thresholds such that $(\mathrm{d}/\mathrm{d}\xi_i)[d_i(\xi_i)]=0$ for $|\xi_i|>\Gamma_i$, Cohen and Grossberg need very complex proofs to show the convergence of the trajectories of (12).

In the present approach the term $\epsilon\dot{x}'D\dot{x}$ in (5) avoids this complexity, even if ϵ is chosen arbitrary small. Furthermore, the nonoscillatory behaviour of system (1) can be proved using any Liapunov function from the class of functions (2) where P is found from (4) and W is any symmetric solution of (10). In the latter equation there may be some freedom of choice in the parameters $\bar{\alpha}$, $\bar{\gamma}$, K and ϵD. This allows an optimization of $V(x)$ for a maximal estimate of the regions of attraction of the stable equilibria.

Finally there may exist neural networks whose equations cannot be cast in the Cohen–Grossberg–Kosko form (12) with a symmetric $\|c_{ik}\|$-matrix, but which could be handled along the lines of the present approach using a model of the class (1). For example, purely additive models of the type

$$\dot{x}_i=-A_ix_i+B_i\left[I_i+\sum_{k=1}^{n}D_{ik}f_k(x_k)\right]-F_i\left[J_i+\sum_{k=1}^{n}G_{ik}g_k(x_k)\right],\quad i=1,\dots,n,$$

which contain both cooperative and competitive interactions (see [1]) can be written in the general form (1). This point remains an open question.

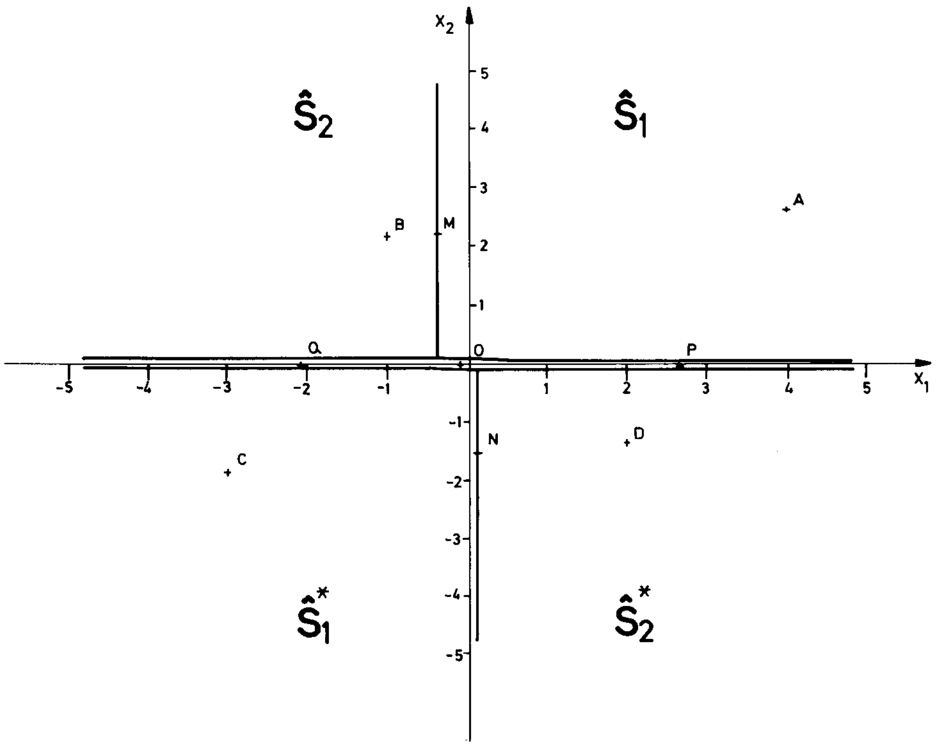

Fig. 1. Example: stability regions of the four stable equilibria of a Hopfield network with two neurons, for the following numerical values: $a_1=2$, $a_2=10$, $b_1=1$, $b_2=5$, $b_3=0.5$, $\rho_1=10$, $\rho_2=40$, $h_1=1$, $h_2=4$, $\alpha_1=5$, $\alpha_2=4$.

6. Example

A Hopfield network with two neurons has the following equations:

$$\begin{pmatrix} \dot{x}_i \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -a_1 & \\ & -a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 & b_3 \\ b_3 & b_2 \end{pmatrix} \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{13}$$

Because of the structure of the Hopfield network, the system parameters must be chosen positive and satisfy the inequalities $a_1 > b_1 + b_3$ and $a_2 > b_2 + b_3$. The nonlinearities $f_i(x_i)$, $i = 1, 2$, have identical structures:

$$f_i(x_i) = \begin{cases} \rho_i x_i, & \text{if } |x_i| \leqslant \dfrac{\alpha_i}{\rho_i},\ \alpha_i > 0,\ \rho_i > 0, \\ \alpha_i \operatorname{sgn}(x_i), & \text{if } |x_i| > \dfrac{\alpha_i}{\rho_i},\ \alpha_i > 0,\ \rho_i > 0. \end{cases}$$

For the numerical values of Fig. 1 the system (13) has nine equilibria:

- four stable nodes: *A, B, C, D*;
- four saddle points: *M, N, P, Q*;
- one unstable node: *O*.

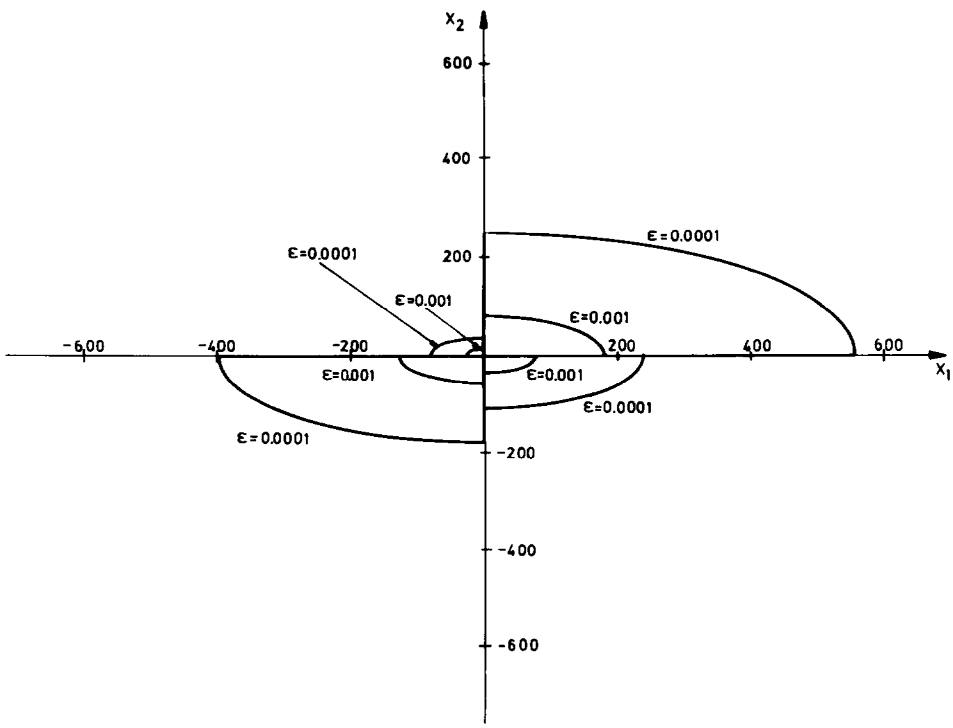

Fig. 2. Example: dependence of the stability regions on ϵ.

The Liapunov function at the equilibria satisfies the inequalities

$$V(A) < V(B) < V(M), \qquad V(C) < V(D) < V(N),$$
$$V(M) < V(N) < V(P) < V(Q) < V(O).$$

According to the previous sections, the disjoint subsets $\hat{S}_1$ and $\hat{S}_2$ of the set $\hat{S} = \{x;\ \Phi(x) < \Phi(M)\}$ are regions of attraction of A respectively B. The disjoint subsets $\hat{S}_1^*$ and $\hat{S}_2^*$ of $\hat{S}^* = \{x;\ \Phi(x) < \Phi(N)\}$ are regions of attraction of C respectively D. These regions of attraction are shown in Fig. 1. Fig. 2 shows the dependence of the obtained stability regions on the choice of ϵ, as $\epsilon \downarrow 0$.

7. Conclusion

A frequency domain criterion for the nonoscillatory behaviour of a class of autonomous nonlinear systems has been presented. As a special case the analyzed class of systems contains Hopfield's neural classification network. The criterion has been derived using a suitable Liapunov function which can be used for the simultaneous calculation of the regions of attraction of all stable equilibria of the system. The obtained results have been compared with the well-known Cohen–Grossberg–Kosko theorems on pattern formation of neural nets. Further research may deal with the application of the method to other types of neural networks and to improve the technique for obtaining better estimates for the exact regions of attraction.

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