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Oscillation properties for even order neutral equations with distributed deviating arguments[☆]

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Abstract

In this paper, we establish several oscillatory criteria for even order neutral differential equations with distributed deviating arguments, which generalize and improve some known results.

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1. Introduction

The oscillatory behavior of solutions of higher order neutral differential equations are of both theoretical and practical interest. Some applicable example can be found in the monograph of Hale [8]. There have been some results on the oscillatory and asymptotic behavior of even order neutral equations. We mention here the monographs of Bainov and Mishev [2], Erbe et al. [4], and literatures of Grace and Lalli [7], Ladas and Sficas [10], Zhahriev and Bainov [15], Grace [5,6], Ruan [13], Das and Nayak [3], Wang et al. [14], Ou and Wong [11], Agarwal et al. [1] and references cited therein. To the best of our knowledge, very little has been done with distributed deviating arguments. However, we note that in many areas of

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their actual application, models describing these problems are often effected by such factors as seasonal changes. Therefore it is necessary, either theoretically or practically, to study a type of equation in a more general sense—differential equations with distributed deviating arguments.

In this paper, we consider the oscillatory behavior of solutions of the even order neutral differential equations of the form

$$[x(t) + c(t)x(t - \tau)]^{(n)} + \int_a^b p(t, \xi)x[g(t, \xi)]d\sigma(\xi) = 0, \quad t \geq t_0, \quad (1)$$

in which $\tau > 0$ is a constant, n is an even positive integer; $c(t) \in C(I, \mathbb{R})$, $0 \leq c(t) \leq 1$, and $p(t, \xi) \in C(I \times J, \mathbb{R}_+)$ is not eventually zero on any $I_\mu \times J$, $I = [t_0, \infty)$, $J = [a, b]$, $I_\mu = [t_\mu, \infty)$, $t_\mu \geq t_0$, $\mathbb{R}_+ = [0, \infty)$; $g(t, \xi) \in C(I \times J, \mathbb{R})$ is nondecreasing with respect to t and ξ , respectively, $(d/dt)g(t, a)$ exists, $g(t, \xi) \leq t$ for $\xi \in J$, and $\liminf_{t \rightarrow \infty, \xi \in J} \{g(t, \xi)\} = \infty$; $\sigma(\xi) \in (J, \mathbb{R})$ is nondecreasing, integral of Eq. (1) is a Stieltjes one.

The objective of this paper is to derive some general oscillatory criteria of solutions of Eq. (1). In the oscillatory criteria obtained, there is a general class of function $H(t, s)$ as the parameter function. By choosing different function $H(t, s)$, we are able to derive some useful corollaries. The corollaries generalize and improve some known results.

We restrict our attention to proper solutions of Eq. (1), that is, to nonconstant solutions existing on $[T, \infty)$ for some $T \geq t_0$ and satisfying $\sup_{t \geq T} |x(t)| > 0$. A proper solution $x(t)$ of Eq. (1) is called oscillatory if it does not have the largest zero, otherwise, it is called nonoscillatory. Eq. (1) is called oscillatory if all its proper solutions are oscillatory.

To obtain the oscillatory criteria of Eq. (1), we first give the following lemmas.

Lemma 1 (Kiguradze [9]). *Let $u(t)$ be a positive and n times differentiable function on \mathbb{R}_+ . If $u^{(n)}(t)$ is of constant sign and not identically zero on any ray $[t_1, +\infty)$ for $(t_1 > 0)$, then there exists a $t_u \geq t_1$ and an integer l ($0 \leq l \leq n$), with $n + l$ even for $u(t)u^{(n)}(t) \geq 0$ or $n + l$ odd for $u(t)u^{(n)}(t) \leq 0$; and for $t \geq t_u$,*

$$u(t)u^{(k)}(t) > 0, \quad 0 \leq k \leq l; \quad (-1)^{k-l}u(t)u^{(k)}(t) > 0, \quad l \leq k \leq n.$$

Lemma 2 (Philos [12]). *Suppose that the conditions of Lemma 1 is satisfied, and*

$$u^{(n-1)}(t)u^{(n)}(t) \leq 0, \quad t \geq t_u,$$

then there exists a constant $\theta \in (0, 1)$ such that for sufficiently large t , there exists a constant $M_\theta > 0$ satisfying

$$|u'(t/2)| \geq M_\theta t^{n-2} |u^{(n-1)}(t)|.$$

2. Main results

Now we give the main results of this paper.

Let $D_0 = \{(t, s) | t > s \geq t_0\}$, $D = \{(t, s) | t \geq s \geq t_0\}$.

Theorem 1. Assume that there exist function $H(t, s) \in C'(D; \mathbb{R})$, $h(t, s) \in C(D_0; \mathbb{R})$ and $\rho(t) \in C'(I, (0, \infty))$, such that

$$(A_1) \quad H(t, t) = 0, H(t, s) > 0;$$

$$(A_2) \quad H'_s(t, s) \leq 0, \text{ and } -H'_s(t, s) - H(t, s)(\rho'(s)/\rho(s)) = h(t, s)\sqrt{H(t, s)}.$$

If

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s)h^2(t, s)}{2M_0g^{n-2}(s, a)g'(s, a)} \right] ds = \infty, \quad (2)$$

then Eq. (1) is oscillatory.

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of Eq. (1) on I , such that $x(t) \neq 0$ on I . Without loss of generality, assume that $x(t) > 0$, $t \geq t_0$. Then, from $\liminf_{t \rightarrow \infty, \xi \in J} \{g(t, \xi)\} = \infty$, there exists a $t_1 \geq t_0$ such that

$$x(t) > 0, \quad x(t - \tau) > 0 \quad \text{and} \quad x[g(t, \xi)] > 0, \quad t \geq t_1, \quad \xi \in [a, b].$$

Let

$$y(t) = x(t) + c(t)x(t - \tau), \quad (3)$$

Thus, from (1) and (3), we have

$$\begin{aligned} 0 &= y^{(n)}(t) + \int_a^b p(t, \xi)x[g(t, \xi)]d\sigma(\xi) \\ &= y^{(n)}(t) + \int_a^b p(t, \xi)\{y[g(t, \xi)] - c[g(t, \xi)]x[g(t, \xi) - \tau]\}d\sigma(\xi). \end{aligned} \quad (4)$$

From the assumptions of $c(t)$ and $p(t, \xi)$, we have $y(t) \geq x(t) > 0$, $y^{(n)}(t) \leq 0$ for $t \geq t_1$, and $y^{(n)}(t)$ is not eventually zero. Thus, from Lemma 1, there exists a $t_2 \geq t_1$ and an odd number l ($0 < l < n$), such that for $t \geq t_2$, we have

$$y^{(k)}(t) > 0, \quad 0 \leq k \leq l, \quad (-1)^{k-l}y^{(k)}(t) > 0, \quad l \leq k \leq n.$$

By choosing $k = 1$ and $n - 1$, we have

$$y'(t) > 0, \quad y^{(n-1)}(t) > 0, \quad t \geq t_2. \quad (5)$$

From (5) and $y(t) \geq x(t)$, we have $y[g(t, \xi)] \geq y[g(t, \xi) - \tau] \geq x[g(t, \xi) - \tau]$, thus

$$y^{(n)}(t) + \int_a^b p(t, \xi)\{1 - c[g(t, \xi)]\}y[g(t, \xi)]d\sigma(\xi) \leq 0, \quad t \geq t_2, \quad (6)$$

furthermore, noting that $g(t, \xi)$ is nondecreasing with respect to ξ , we have

$$y^{(n)}(t) + y[g(t, a)] \int_a^b p(t, \xi)\{1 - c[g(t, \xi)]\}d\sigma(\xi) \leq 0, \quad t \geq t_2. \quad (7)$$

Let

$$z(t) = \rho(t) \frac{y^{(n-1)}(t)}{y[g(t, a)/2]}, \quad (8)$$

then $z(t) \geq 0$, and noting that $(d/dt)g(t, a)$ exists, we have $y'[g(t, a)] = (dy/dg)(d/dt)g(t, a)$, and in view of $g(t, \xi)$ is nondecreasing with respect to ξ , $g(t, \xi) \leq t$, $\xi \in [a, b]$, and $y^{(n)}(t) \leq 0$, we obtain $y^{(n-1)}(t) \leq y^{(n-1)}[g(t, a)] \leq y^{(n-1)}[g(t, a)/2]$, thus, from Lemma 2, for $t \geq t_2$, we have

$$\begin{aligned} z'(t) &= \frac{\rho'(t)y^{(n-1)}(t) + \rho(t)y^{(n)}(t)}{y[g(t, a)/2]} - \frac{\rho(t)y^{(n-1)}(t)y'[g(t, a)/2]g'(t, a)}{2y^2[g(t, a)]} \\ &\leq \frac{\rho'(t)y^{(n-1)}(t) + \rho(t)y^{(n)}(t)}{y[g(t, a)/2]} - \frac{M_\theta g^{n-2}(t, a)g'(t, a)}{2\rho(t)} z^2(t), \end{aligned}$$

furthermore, from $y'(t) > 0$ and (7), we have

$$z'(t) \leq \frac{\rho'(t)}{\rho(t)} z(t) - \rho(t) \int_a^b p(t, \xi) \{1 - c[g(t, \xi)]\} d\sigma(\xi) - \frac{M_\theta g^{n-2}(t, a)g'(t, a)}{2\rho(t)} z^2(t)$$

that is

$$\rho(t) \int_a^b p(t, \xi) \{1 - c[g(t, \xi)]\} d\sigma(\xi) \leq -z'(t) + \frac{\rho'(t)}{\rho(t)} z(t) - \frac{M_\theta g^{n-2}(t, a)g'(t, a)}{2\rho(t)} z^2(t). \quad (9)$$

Integrating by parts for any $t > T \geq t_1$, and using properties (A₁) and (A₂), we have

$$\begin{aligned} &\int_T^t H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\ &\leq - \int_T^t H(t, s) z'(s) ds + \int_T^t H(t, s) \frac{\rho'(s)}{\rho(s)} z(s) ds \\ &\quad - \frac{M_\theta}{2} \int_T^t H(t, s) \frac{g^{n-2}(s, a)g'(s, a)}{\rho(s)} z^2(s) ds \\ &= H(t, T)z(T) + \int_T^t H(t, s) \left[\frac{\partial H}{\partial s} + H(t, s) \frac{\rho'(s)}{\rho(s)} \right] z(s) ds \\ &\quad - \frac{M_\theta}{2} \int_T^t H(t, s) \frac{g^{n-2}(s, a)g'(s, a)}{\rho(s)} z^2(s) ds \\ &= H(t, T)z(T) - \int_T^t h(t, s) \sqrt{H(t, s)} z(s) ds - \frac{M_\theta}{2} \int_T^t H(t, s) \frac{g^{n-2}(s, a)g'(s, a)}{\rho(s)} z^2(s) ds \\ &= H(t, T)z(T) - \frac{1}{2} \int_T^t \left[\sqrt{\frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)} h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right]^2 ds \\ &\quad + \int_T^t \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} ds. \end{aligned} \quad (10)$$

Furthermore, we have

$$\begin{aligned} & \int_T^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \\ & \leq H(t, T) z(T) - \frac{1}{2} \int_T^t \left[\sqrt{\frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)} h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right]^2 ds. \end{aligned} \quad (11)$$

From (A₂), $H'_s(t, s) \leq 0$, for $t_1 \geq t_0$, we have $H(t, t_1) \leq H(t, t_0)$, thus

$$\begin{aligned} & \int_{t_1}^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \\ & \leq H(t, t_1) z(t_1) - \frac{1}{2} \int_{t_1}^t \left[\sqrt{\frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)} h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right]^2 ds \\ & \leq H(t, t_1) z(t_1) \leq H(t, t_0) z(t_1) \end{aligned}$$

which implies that

$$\begin{aligned} & \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \\ & = \frac{1}{H(t, t_0)} \left[\int_{t_0}^{t_1} + \int_{t_1}^t \right] \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) \right. \\ & \quad \left. - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \\ & \leq z(t_1) + \int_{t_0}^{t_1} \frac{H(t, s)}{H(t, t_0)} \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\ & \leq z(t_1) + \int_{t_0}^{t_1} \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds. \end{aligned} \quad (12)$$

Let $t \rightarrow \infty$, and taking upper limits, we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \\ & \leq z(t_1) + \int_{t_0}^{t_1} \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds = M < \infty, \end{aligned} \quad (13)$$

where M is a constant, which contradicts (2). This completes the proof of Theorem 1. \square

From Theorem 1, we can obtain the following corollary.

Corollary 1. *If condition (2) of Theorem 1 is replaced by*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds = \infty, \quad (14)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{\rho(s) h^2(t, s)}{g^{n-2}(s, a) g'(s, a)} ds < \infty, \quad (15)$$

then Eq. (1) is oscillatory.

Remark 1. By introducing various $H(t, s)$ from Theorem 1 or Corollary 1, we can obtain some oscillatory criteria of Eq. (1). For example, let $H(t, s) = (t - s)^{m-1}$, $t \geq s \geq t_0$, in which $m > 2$ is a integer. By choosing

$$h(t, s) = (t - s)^{(m-3)/2} \left[m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right],$$

it is clear that the conditions of (A₁) and (A₂) hold, then, from Theorem 1 and Corollary 1, we have

Corollary 2. *Assume that there exists a function $\rho(t) \in C'(I, (0, \infty))$ such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t \left[(t - s)^{m-1} \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s)}{2M_0 g^{n-2}(s, a) g'(s, a)} \left[m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right]^2 (t - s)^{m-3} \right] ds = \infty, \quad (16)$$

then Eq. (1) is oscillatory.

Corollary 3. *Assume that there exists a function $\rho(t) \in C'(I, (0, \infty))$ such that*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t (t - s)^{m-1} \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds = \infty, \quad (17)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t \frac{\rho(s)}{g^{n-2}(s, a) g'(s, a)} \left[m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right]^2 (t - s)^{m-3} ds < \infty, \quad (18)$$

then Eq. (1) is oscillatory.

If taking $\rho(t) \equiv 1$, then we have

Corollary 4. *If*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t (t - s)^{m-1} \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds = \infty, \quad (19)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t \frac{(m - 1)^2 (t - s)^{m-3}}{g^{n-2}(s, a) g'(s, a)} ds < \infty, \quad (20)$$

then Eq. (1) is oscillatory.

If $H(t, s) = \varphi(t - s)$, taking $\rho(t) \equiv 1$, $t \geq s \geq 0$, in which $\varphi(t)$ is an any function satisfying $\varphi(t) \in C'(I, (0, \infty))$, $\varphi(0) = 0$, $\varphi'(t) \geq 0$, $t \geq 0$. Choosing

$$h(t, s) = \begin{cases} -\varphi'(t - s)\sqrt{\varphi(t - s)}, & t > s, \\ 0, & t = s, \end{cases}$$

then the conditions of (A₁) and (A₂) are satisfied, thus we have

Corollary 5. *If*

$$\limsup_{t \rightarrow \infty} \frac{1}{\varphi(t)} \int_{t_0}^t \left[\varphi(t - s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{(\varphi'(t - s))^2}{2M_\theta \varphi(t - s) g^{n-2}(s, a) g'(s, a)} \right] ds = \infty, \quad (21)$$

then Eq. (1) is oscillatory.

Theorem 2. *Assume that the conditions of Theorem 1 hold, and*

$$0 < \inf_{s \geq t_0} \left[\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right] \leq \infty, \quad (22)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{\rho(s) h^2(t, s)}{g^{n-2}(s, a) g'(s, a)} ds < \infty. \quad (23)$$

If there exists a function $\varphi(t) \in C(I, \mathbb{R})$ satisfying

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, u)} \int_u^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \geq \varphi(u), \quad u \geq t_0, \quad (24)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{g^{n-2}(u, a) g'(u, a)}{\rho(u)} \varphi_+^2(u) du = \infty, \quad \varphi_+(u) = \max_{u \geq t_0} \{\varphi(u), 0\}, \quad (25)$$

then Eq. (1) is oscillatory.

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of Eq. (1) on I , such that $x(t) \neq 0$ on I . Without loss of generality, assume that $x(t) > 0$, $t \geq t_0$. Then, proceeding as in the proof of Theorem 1, for $t > u \geq t_1 \geq t_0$, we have

$$\begin{aligned} & \frac{1}{H(t, u)} \int_u^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \\ & \leq z(u) - \frac{1}{2H(t, u)} \int_u^t \left[\sqrt{\frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)} h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right]^2 ds. \end{aligned}$$

Let $t \rightarrow \infty$, and taking upper limits, we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, u)} \int_u^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \\ & \leq z(u) - \liminf_{t \rightarrow \infty} \frac{1}{2H(t, u)} \int_u^t \left[\sqrt{\frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)}} z(s) \right. \\ & \quad \left. + \frac{\sqrt{\rho(s)} h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right]^2 ds, \end{aligned}$$

thus, from (24), we have

$$\begin{aligned} z(u) & \geq \varphi(u) + \liminf_{t \rightarrow \infty} \frac{1}{2H(t, u)} \int_u^t \left[\sqrt{\frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)}} z(s) \right. \\ & \quad \left. + \frac{\sqrt{\rho(s)} h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right]^2 ds, \end{aligned}$$

then $z(u) \geq \varphi(u)$, and

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{H(t, u)} \int_u^t \left[\sqrt{\frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)} h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right]^2 ds \\ & \leq 2(z(u) - \varphi(u)) = M < \infty, \end{aligned} \quad (26)$$

where M is a constant. On the other hand, for $t > t_1$, we have

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{H(t, t_1)} \int_{t_1}^t \left[\sqrt{\frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)} h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a) g'(s, a)}} \right]^2 ds \\ & \geq \liminf_{t \rightarrow \infty} \left[\frac{1}{H(t, t_1)} \int_{t_1}^t \frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)} z^2(s) ds \right. \\ & \quad \left. + \frac{1}{H(t, t_1)} \int_{t_1}^t \sqrt{H(t, s)} h(t, s) z(s) ds \right]. \end{aligned} \quad (27)$$

Let

$$\begin{aligned} v(t) &= \frac{1}{H(t, t_1)} \int_{t_1}^t \frac{M_\theta H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)} z^2(s) ds, \\ w(t) &= \frac{1}{H(t, t_1)} \int_{t_1}^t \sqrt{H(t, s)} h(t, s) z(s) ds. \end{aligned} \quad (28)$$

From (26) and (27), we have

$$\liminf_{t \rightarrow \infty} [v(t) + w(t)] < \infty. \quad (29)$$

Now we can claim that

$$\int_{t_1}^{\infty} \frac{M_{\theta} g^{n-2}(s, a) g'(s, a)}{\rho(s)} z^2(s) \, ds < \infty, \quad t > t_1. \quad (30)$$

In fact, assume the contrary, that

$$\int_{t_1}^{\infty} \frac{M_{\theta} g^{n-2}(s, a) g'(s, a)}{\rho(s)} z^2(s) \, ds = \infty, \quad t > t_1. \quad (31)$$

From (22), there exists a constant $L > 0$ such that

$$\inf_{s \geq t_0} \left[\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} \right] > L > 0, \quad (32)$$

then it follows from (31), for any positive number $\mu > 0$, that there exists a $T > t_1$ such that

$$\int_{t_1}^t \frac{M_{\theta} g^{n-2}(s, a) g'(s, a)}{\rho(s)} z^2(s) \, ds \geq \frac{\mu}{L}, \quad t > T, \quad (33)$$

then, for $t \geq T > t_1$, integrating by parts, we have

$$\begin{aligned} v(t) &= \frac{1}{H(t, t_1)} \int_{t_1}^t \frac{M_{\theta} H(t, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)} z^2(s) \, ds \\ &= \frac{1}{H(t, t_1)} \int_{t_1}^t H(t, s) \, d \left[\int_{t_1}^s \frac{M_{\theta} g^{n-2}(u, a) g'(u, a)}{\rho(u)} z^2(u) \right] \\ &= \frac{1}{H(t, t_1)} \int_{t_1}^t \left[\int_{t_1}^s \frac{M_{\theta} g^{n-2}(u, a) g'(u, a)}{\rho(u)} z^2(u) \right] \left[-\frac{\partial H(t, s)}{\partial s} \right] \, ds \\ &\geq \frac{1}{H(t, t_1)} \int_T^t \left[\int_{t_1}^s \frac{M_{\theta} g^{n-2}(u, a) g'(u, a)}{\rho(u)} z^2(u) \right] \left[-\frac{\partial H(t, s)}{\partial s} \right] \, ds \\ &\geq \frac{\mu}{L} \frac{1}{H(t, t_1)} \int_T^t \left[-\frac{\partial H(t, s)}{\partial s} \right] \, ds \\ &= \frac{\mu}{L} \frac{H(t, T)}{H(t, t_1)} \geq \frac{\mu}{L} \frac{H(t, T)}{H(t, t_0)}. \end{aligned} \quad (34)$$

It follows from (32) that

$$\liminf_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} > L > 0, \quad s \geq t_0,$$

therefore, there exists a $t_2 > T$ such that

$$\frac{H(t, T)}{H(t, t_0)} \geq L, \quad t \geq t_2. \quad (35)$$

From (34) and (35), we have $v(t) \geq \mu$, $t \geq t_2$. Furthermore, since μ is arbitrary, thus we have

$$\lim_{t \rightarrow \infty} v(t) = \infty. \quad (36)$$

From (29), there exists a sequence $\{t_n\}_1^\infty$ on $[t_1, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \infty$ and satisfying

$$\lim_{n \rightarrow \infty} [v(t_n) + w(t_n)] = \liminf_{t \rightarrow \infty} [v(t) + w(t)] < \infty.$$

Thus, there exists a positive integer N_1 and constant M such that

$$v(t_n) + w(t_n) < M, \quad n > N_1, \quad (37)$$

and from (36), we have

$$\lim_{n \rightarrow \infty} v(t_n) = \infty. \quad (38)$$

Furthermore, from (37) and (38), we obtain

$$\lim_{n \rightarrow \infty} w(t_n) = -\infty, \quad (39)$$

and for any $\varepsilon \in (0, 1)$, there exists a positive integer N_2 such that $w(t_n)/v(t_n) + 1 < \varepsilon$, $n > N_2$, then

$$\frac{w(t_n)}{v(t_n)} < \varepsilon - 1 < 0. \quad (40)$$

From (39) and (40), we have

$$\lim_{t \rightarrow \infty} \frac{w(t_n)}{v(t_n)} w(t_n) = \infty. \quad (41)$$

On the other hand, by using the Schwartz inequality, for $t \geq t_1$, we obtain

$$\begin{aligned} 0 \leq w^2(t_n) &= \frac{1}{H^2(t_n, t_1)} \left\{ \int_{t_1}^{t_n} \sqrt{H(t_n, s)} h(t_n, s) z(s) \, ds \right\}^2 \\ &\leq \left\{ \frac{1}{H(t_n, t_1)} \int_{t_1}^{t_n} \frac{M_\theta H(t_n, s) g^{n-2}(s, a) g'(s, a)}{\rho(s)} z^2(s) \, ds \right\} \\ &\quad \times \left\{ \frac{1}{H(t_n, t_1)} \int_{t_1}^{t_n} \frac{\rho(s)}{M_\theta g^{n-2}(s, a) g'(s, a)} h(t_n, s) \, ds \right\} \\ &= v(t_n) \frac{1}{H(t_n, t_1)} \int_{t_1}^{t_n} \frac{\rho(s)}{M_\theta g^{n-2}(s, a) g'(s, a)} h(t_n, s) \, ds, \end{aligned}$$

thus

$$\frac{w^2(t_n)}{v(t_n)} \leq \frac{1}{H(t_n, t_1)} \int_{t_1}^{t_n} \frac{\rho(s)}{M_\theta g^{n-2}(s, a) g'(s, a)} h(t_n, s) \, ds. \quad (42)$$

From (35), we have

$$\frac{1}{H(t_n, t_1)} \leq \frac{1}{H(t_n, T)} = \frac{H(t_n, t_0)}{H(t_n, T)} \frac{1}{H(t_n, t_0)} \leq \frac{1}{LH(t_n, t_0)}, \quad (43)$$

therefore, from (42) and (43), we have

$$\frac{w^2(t_n)}{v(t_n)} \leq \frac{1}{LH(t_n, t_0)} \int_{t_1}^{t_n} \frac{\rho(s)}{M_\theta g^{n-2}(s, a) g'(s, a)} h(t_n, s) \, ds, \quad (44)$$

from (41) and (44), we have

$$\lim_{n \rightarrow \infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} \frac{\rho(s)}{M_\theta g^{n-2}(s, a) g'(s, a)} h(t_n, s) \, ds = \infty, \quad (45)$$

thus,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{\rho(s)}{M_\theta g^{n-2}(s, a) g'(s, a)} h(t, s) \, ds = \infty, \quad (46)$$

which contradicts (23). Furthermore, from (30) and $z(s) \geq \varphi(s)$, we have

$$\int_{t_1}^\infty \frac{M_\theta g^{n-2}(s, a) g'(s, a)}{\rho(s)} \varphi_+^2(s) \, ds \leq \int_{t_1}^\infty \frac{M_\theta g^{n-2}(s, a) g'(s, a)}{\rho(s)} z^2(s) \, ds < \infty, \quad (47)$$

which contradicts (25). This completes the proof of Theorem 2. \square

Remark 2. By introducing various $H(t, s)$ from Theorem 2, we can obtain some oscillatory criteria of solution of Eq. (1). For example, let $H(t, s) = (t - s)^{m-1}$, $t \geq s \geq t_0$, in which $m > 2$ is a integer, then

$$h(t, s) = (t - s)^{(m-3)/2} \left[m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right],$$

$$\lim_{t \rightarrow \infty} \frac{H(t, s)}{H(t, t_0)} = \lim_{t \rightarrow \infty} \frac{(t - s)^{m-1}}{(t - t_0)^{m-1}} = 1 > 0,$$

thus, the conditions of Theorem 2 are satisfied, we have

Corollary 6. Assume that there exist function $\rho(t) \in C(I, (0, \infty))$ and $\varphi(t) \in C(I, \mathbb{R})$ satisfying

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t \frac{\rho(s)}{g^{n-2}(s, a) g'(s, a)} (t - s)^{m-3} \left[m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right]^2 \, ds < \infty. \quad (48)$$

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_u^t \left\{ (t - s)^{m-1} \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} \, d\sigma(\xi) \right. \\ \left. - \frac{\rho(s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \left[m - 1 - (t - s) \frac{\rho'(s)}{\rho(s)} \right]^2 \right\} \, ds \geq \varphi(u), \quad u \geq t_0, \end{aligned} \quad (49)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{m-1}} \int_{t_0}^t \frac{g^{n-2}(u, a) g'(u, a)}{\rho(u)} \varphi_+^2(u) \, du = \infty, \quad \varphi_+(u) = \max_{u \geq t_0} \{\varphi(u), 0\}, \quad (50)$$

then Eq. (1) is oscillatory.

Theorem 3. Assume that the conditions of Theorem 1 and (22) hold, and

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} \, d\sigma(\xi) \, ds < \infty. \quad (51)$$

If there exists a function $\varphi(t) \in C(I, \mathbb{R})$ satisfying

$$\liminf_{t \rightarrow \infty} \frac{1}{H(t, u)} \int_u^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s)h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \geq \varphi(u), \quad u \geq t_0, \quad (52)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{g^{n-2}(u, a)g'(u, a)}{\rho(u)} \varphi_+^2(u) du = \infty, \quad \varphi_+(u) = \max_{u \geq t_0} \{\varphi(u), 0\}, \quad (53)$$

then Eq. (1) is oscillatory.

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of Eq. (1) on I , such that $x(t) \neq 0$ on I . Without loss of generality, assume that $x(t) > 0$, $t \geq t_0$. Then, proceeding as in the proof of Theorem 1, for $t > u \geq t_1 \geq t_0$, we have

$$\begin{aligned} & \frac{1}{H(t, u)} \int_u^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s)h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \\ & \leq z(u) - \frac{1}{2H(t, u)} \int_u^t \left[\sqrt{\frac{M_\theta H(t, s)g^{n-2}(s, a)g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)}h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a)g'(s, a)}} \right]^2 ds, \end{aligned}$$

Let $t \rightarrow \infty$, and taking lower limits, we have

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{H(t, u)} \int_u^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) - \frac{\rho(s)h^2(t, s)}{2M_\theta g^{n-2}(s, a)g'(s, a)} \right] ds \\ & \leq z(u) - \limsup_{t \rightarrow \infty} \frac{1}{2H(t, u)} \int_u^t \left[\sqrt{\frac{M_\theta H(t, s)g^{n-2}(s, a)g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)}h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a)g'(s, a)}} \right]^2 ds, \end{aligned}$$

which implies that $\varphi(u) \leq z(u)$, $u \geq t_0$, and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, u)} \int_u^t \left[\sqrt{\frac{M_\theta H(t, s)g^{n-2}(s, a)g'(s, a)}{\rho(s)}} z(s) + \frac{\sqrt{\rho(s)}h(t, s)}{\sqrt{M_\theta g^{n-2}(s, a)g'(s, a)}} \right]^2 ds \\ & \leq 2(z(u) - \varphi(u)) = M_1 < \infty. \end{aligned} \quad (54)$$

Let $v(t)$ and $w(t)$ as same as Theorem 2, then, from (54), we have

$$\limsup_{t \rightarrow \infty} [v(t) + w(t)] \leq M_1 < \infty. \quad (55)$$

From (52), we have

$$\begin{aligned} \varphi(t_0) &\leq \liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) \right. \\ &\quad \left. - \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} \right] ds \\ &\leq \liminf_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \rho(s) \int_a^b p(s, \xi) \{1 - c[g(s, \xi)]\} d\sigma(\xi) ds \\ &\quad - \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} ds, \end{aligned} \quad (56)$$

from (51) and (56), we have

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} ds < \infty. \quad (57)$$

Thus, there exists a sequence $\{t_n\}_1^\infty$ on $[t_1, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \infty$ and satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} \frac{\rho(s) h^2(t_n, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} ds \\ = \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{\rho(s) h^2(t, s)}{2M_\theta g^{n-2}(s, a) g'(s, a)} ds < \infty, \end{aligned} \quad (58)$$

then, (23) holds in Theorem 2. The following proof is similar to Theorem 2, we omit the details. This completes the proof of Theorem 3. \square

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