Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 213 (2008) 205 – 211
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

The bounds of feasible space on constrained nonconvex quadratic programming☆

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Received 13 November 2006; received in revised form 30 December 2006

Abstract

This paper presents a method to estimate the bounds of the radius of the feasible space for a class of constrained nonconvex quadratic programmings. Results show that one may compute a bound of the radius of the feasible space by a linear programming which is known to be a P-problem [N. Karmarkar, A new polynomial-time algorithm for linear programming, Combinatorica 4 (1984) 373–395]. It is proposed that one applies this method for using the canonical dual transformation [D.Y. Gao, Canonical duality theory and solutions to constrained nonconvex quadratic programming, J. Global Optimization 29 (2004) 377–399] for solving a standard quadratic programming problem.

MSC: 90C; 49N

Keywords: Quadratic programming; Canonical dual theory; Linear programming

1. Introduction

This paper concerns solutions of the following standard quadratic programming (problem (P)):

$$(P): \min_{x \in R^n} P(x) = \frac{1}{2} x^{\mathrm{T}} A x - f^{\mathrm{T}} x \tag{1.1}$$

$$\text{s.t.} \quad Bx \leqslant b, \tag{1.2}$$

where $A = A^{\mathrm{T}} \in R^{n\times n}$ and $B \in R^{m\times n}$ are given two matrices, $f \in R^n$ and $b \in R^m$ are two vectors. The primal feasible space

$$D = \{x \in R^n \mid Bx \leqslant b\} \tag{1.3}$$

is a convex subset of R^n. We assume that $D \neq \emptyset$ and the radius r_0 of D, defined by $\|x\| \leqslant r_0$, $\forall x \in D$, is finite, i.e., D is bounded. It follows that the problem (P) has at least one solution. Also it implies that $b \neq 0$. In fact, if $b = 0$,

☆ This research was partly supported by the National Science Foundation of China under Grant no. 10671145.
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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.01.010

except for the trivial case $D = \{0\}$ (we always assume D is not trivial), there is a nonzero point $x \in D$ such that for every positive real σ, $\sigma x \in D$, which implies that D is unbounded, then it leads to a contradiction to the assumption of D being bounded.

Nonconvex quadratic programming problem has great importance both from the mathematical and application viewpoints. During the last decade, several authors have shown the general quadratic programming problem (P) is an NP-hard problem in global optimization (cf. [8–10,5]). In order to solve this difficult problem, many efforts have been made during the last decade. Duality is a fundamental concept that plays a central role in almost all natural science. For dealing with the general quadratic programming problem (P), Gao (cf. [3]) developed a canonical dual transformation method in general nonconvex systems. The main idea is as follows. In order to use the canonical dual transformation to solve the nonconvex quadratic programming problem (P), an additional normality constraint $\|x\|^2 \leqslant 2\mu$ is introduced, where $\mu > 0$ is a given parameter. By use of this constraint, a parametric optimization problem can be proposed as the following:

$$(P_\mu):\ \min_{x\in R^n} P(x) = \frac{1}{2}x^{\mathrm{T}}Ax - f^{\mathrm{T}}x \tag{1.4}$$

$$\text{s.t.}\quad Bx \leqslant b, \quad \|x\|^2 \leqslant 2\mu. \tag{1.5}$$

It is easy to see that (P_μ) has at least one global minimizer $\bar{x}_\mu$ and if $\mu \geqslant \mu_0 = \frac{1}{2}r_0^2$, with r_0 being the radius of the feasible space D, then $\bar{x}_\mu$ solves also the original problem (P). By the canonical dual transformation method Gao (see [4]) formulated the canonical dual problem (P_μ^{d}) associated with the parametric problem (P_μ) as the following:

$$(P_\mu^{\mathrm{d}}):\ \mathrm{ext}[-\tfrac{1}{2}(f - B^{\mathrm{T}}\varepsilon^*)^{\mathrm{T}}(A+\rho^* I)^{-1}(f - B^{\mathrm{T}}\varepsilon^*) - \mu\rho^* - b^{\mathrm{T}}\varepsilon^*]$$

$$\text{s.t.}\quad \varepsilon^* \geqslant 0, \quad \rho^* \geqslant 0, \quad \det(A+\rho^* I) \neq 0, \tag{1.6}$$

where $\mathrm{ext}\, P(x)$ stands for finding all the extremum values of $P(x)$. It is shown in [4] that the primal problem (P_μ) is equivalent to the canonical dual problem (P_μ^{d}) in the sense that they have the same set of KKT points.

In this paper we study to estimate the bounds of the radius of the feasible space D by solving a linear programming which can be solved by a polynomial-time algorithm according to the classical linear programming theory (see [6]).

The rest of the paper is organized as follows. In Section 2, we present a way to find a bound of the radius of the feasible space D by constructing a linear programming. We give an example and a remark in Section 3.

2. Estimating the bounds of the radius of the feasible space

To estimate the bounds of the radius of the feasible space $D = \{Bx \leqslant b\}$, we consider the following negative definite programming problem:

$$\Phi(b) = \min_{x\in R^n} Q(x) = -\frac{1}{2}x^{\mathrm{T}}x \tag{2.1}$$

$$\text{s.t.}\quad Bx \leqslant b. \tag{2.2}$$

Assumption 2.1. $\{Bx \geqslant b\} \neq \emptyset$.

Remark 2.1. It is easy to see that Assumption 2.1 means that there is a vector $\eta \in R^n$ such that $B\eta \geqslant b$.

Let $c(x)$ denote $-\frac{1}{2}x$. For a given nonzero feasible point $\hat{x}$ of the problem (2.1)–(2.2), consider the following linear programming:

$$\Psi(b, \hat{x}) = \min_{x\in R^n} [c(\hat{x})]^{\mathrm{T}}x \tag{2.3}$$

$$\text{s.t.}\quad Bx \leqslant b. \tag{2.4}$$

It follows from $\hat{x} \neq 0$ and D being bounded that $\Psi(b, \hat{x})$ is well defined. It follows from D being not trivial that $b \neq 0$. The dual problem of (2.2)–(2.3) is

$$L(b, \hat{x}) = \max y^{\mathrm{T}} b \tag{2.5}$$

$$\text{s.t.} \quad y^{\mathrm{T}} B = [c(\hat{x})]^{\mathrm{T}}, \quad y \leqslant 0. \tag{2.6}$$

For a given nonzero point $\hat{x} \in D$, let $D(\hat{x}) = \{(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} \mid y^{\mathrm{T}} B = [c(\hat{x})]^{\mathrm{T}},\ y \leqslant 0\}$. By the classical theory of the linear programming [7], we see that $D(\hat{x}) \neq \emptyset$ and

$$L(b, \hat{x}) = \Psi(b, \hat{x}). \tag{2.7}$$

Since D is bounded, there exists a global minimizer for the problem (2.1)–(2.2). Let x^* be an global minimizer of (2.1)–(2.2). Since D is not trivial, noting that D has a nonzero point $\bar{x}$ such that $-\frac{1}{2}\bar{x}^{\mathrm{T}}\bar{x} < 0$, we have $x^* \neq 0$. It follows from (2.7) that

$$L(b, x^*) = \Psi(b, x^*). \tag{2.8}$$

We need the following lemma.

Lemma 2.1. *If x^* is a global optimum of* (2.1)–(2.2), *then we have*

$$\Psi(b, x^*) = \Phi(b), \tag{2.9}$$

where, *by the notation* (2.3)–(2.4),

$$\Psi(b, x^*) = \min_{x \in R^n} [c(x^*)]^{\mathrm{T}} x$$

$$\text{s.t.} \quad Bx \leqslant b.$$

Proof. Since $D = \{x \mid Bx \leqslant b\}$ is convex and $x^* \in D$, when $x \in D$, $x - x^*$ is a feasible direction at x^* with respect to D. By the classical optimization theory [7], it follows from x^* being a global optimum that, for each $x \in D$,

$$[\bigtriangledown Q(x^*)]^{\mathrm{T}}(x - x^*) \geqslant 0. \tag{2.10}$$

Noting that

$$c(x^*) = -\tfrac{1}{2}x^* = \tfrac{1}{2}\bigtriangledown Q(x^*), \tag{2.11}$$

and x^* is a global minimizer of (2.1)–(2.2), we deduce by (2.10) that, for each $x \in D$,

$$\Phi(b) = -\tfrac{1}{2}(x^*)^{\mathrm{T}}x^* = [c(x^*)]^{\mathrm{T}}x^* = \tfrac{1}{2}[\bigtriangledown Q(x^*)]^{\mathrm{T}}x^* \leqslant \tfrac{1}{2}[\bigtriangledown Q(x^*)]^{\mathrm{T}}x = [c(x^*)]^{\mathrm{T}}x. \tag{2.12}$$

It follows from the definition of $\Psi(b, x^*)$ that

$$\Psi(b, x^*) = \Phi(b). \qquad \square \tag{2.13}$$

It is well known that a point $x \in D$ is a K–K–T point [1] of the problem (2.1)–(2.2) if the following relationships hold:

$$Bx \leqslant b;$$

$$-x + B^{\mathrm{T}}\lambda = 0, \quad \lambda \geqslant 0;$$

$$\lambda^{\mathrm{T}}(Bx - b) = 0.$$

Define $L(b)$ for a given $b(\neq 0)$ as follows:

$$L(b) = \min y^{\mathrm{T}} b \tag{2.14}$$

$$\text{s.t.} \quad y^{\mathrm{T}} B = -\tfrac{1}{2} x^{\mathrm{T}}, \quad y \leqslant 0; \tag{2.15}$$

$$Bx \leqslant b; \tag{2.16}$$

$$-x + B^{\mathrm{T}} \lambda = 0, \quad \lambda \geqslant 0; \tag{2.17}$$

$$\lambda^{\mathrm{T}}(Bx - b) = 0. \tag{2.18}$$

Let D_1 denote the feasible set of the above mathematical programming (2.14)–(2.18):

$$\{(x^{\mathrm{T}}, y^{\mathrm{T}}, \lambda^{\mathrm{T}})^{\mathrm{T}} \mid y^{\mathrm{T}} B = -\tfrac{1}{2} x^{\mathrm{T}}, \quad y \leqslant 0;\ Bx \leqslant b;\ -x + B^{\mathrm{T}} \lambda = 0, \quad \lambda \geqslant 0;\ \lambda^{\mathrm{T}}(Bx - b) = 0\}.$$

Since $D \neq \{0\}$, there is a nonzero global minimizer x^* for the problem (2.1)–(2.2). Thus $D(x^*) \neq \emptyset$. x^* is also a K–K–T point of the problem (2.1)–(2.2). It implies D_1 being not empty, noting that $D(x^*) \subset D_1$. If $y^{\mathrm{T}} b$ is bounded below on D_1, then $L(b)$ is well defined. It is clear that

$$L(b) \leqslant L(b, x^*). \tag{2.19}$$

But we need to show that $y^{\mathrm{T}} b$ is bounded below on D_1. We will do it after defining $l(b)$ as follows:

$$l(b) = \min y^{\mathrm{T}} b \tag{2.20}$$

$$\text{s.t.} \quad y^{\mathrm{T}} B = -\tfrac{1}{2} x^{\mathrm{T}}, \quad y \leqslant 0; \tag{2.21}$$

$$Bx \leqslant b. \tag{2.22}$$

Let D_2 denote the feasible set of the above linear programming (2.20)–(2.22)

$$\{(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} \mid y^{\mathrm{T}} B = -\tfrac{1}{2} x^{\mathrm{T}}, \quad y \leqslant 0;\ Bx \leqslant b\}. \tag{2.23}$$

It is clear that $D_1 \subset D_2$. If we have shown that $y^{\mathrm{T}} b$ is bounded below on D_2, then we can deduce that $y^{\mathrm{T}} b$ is bounded below on D_1, i.e., $l(b)$ is finite, and, at the same time,

$$l(b) \leqslant L(b). \tag{2.24}$$

Lemma 2.2. *If D is bounded and $\{Bx \geqslant b\} \neq \emptyset$, then $y^{\mathrm{T}} b$ is bounded below on D_2.*

Proof. Since $\{Bx \geqslant b\} \neq \emptyset$, there are vectors $\eta \in R^n$ and $\zeta \in R^m$ such that $\zeta = B\eta \geqslant b$. Noting that $y \leqslant 0$ on D_2, we have, for $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} \in D_2$,

$$y^{\mathrm{T}} b \geqslant y^{\mathrm{T}} \zeta = y^{\mathrm{T}} B \eta = -\tfrac{1}{2} x^{\mathrm{T}} \zeta. \tag{2.25}$$

It follows from D being bounded and ζ being fixed that $y^{\mathrm{T}} b$ is bounded below on D_2. □

As a conclusion of all of the above we reach the following theorem.

Theorem 2.3. *If D is bounded and $\{Bx \geqslant b\} \neq \emptyset$, then $\sqrt{-2l(b)}$ is a bound of the radius of D.*

Proof. By Lemma 2.2, $l(b)$ is finite. Further by (2.24), (2.19), (2.8), (2.9) we have, for each $x \in D$,

$$l(b) \leqslant L(b) \leqslant L(b, x^*) = \Psi(b, x^*) = \Phi(b) \leqslant -\tfrac{1}{2} x^{\mathrm{T}} x \leqslant 0. \tag{2.26}$$

Thus

$$-\Phi(b) \leqslant -l(b).$$

By (2.1)–(2.2), $-\Phi(b) = \max_D \{\tfrac{1}{2} x^{\mathrm{T}} x\}$. It follows that $\sqrt{-2l(b)}$ is a bound of the radius of D. □

3. An example and a remark

Example 3.1. Estimate the bounds of the radius of the following feasible space:

$$D=\{x_1+x_2\leqslant 1;\quad x_1-x_2\leqslant 1;\quad -x_2\leqslant 0\}. \tag{3.1}$$

Solution: Let $b=(1,1,0)^{\mathrm{T}}$ and

$$B=\begin{pmatrix}1, & 1\\ 1, & -1\\ 0, & -1\end{pmatrix}.$$

Thus, for $x=(x_1,x_2)^{\mathrm{T}}\in R^2$,

$$D=\{x\mid Bx\leqslant b\}.$$

For seeing the fact that $\{x\mid Bx\geqslant b\}\neq\emptyset$, one may choose $\eta=(1,5,2)^{\mathrm{T}}$ and $\zeta=(3,-2)$ to deduce easily the following relationship:

$$B\zeta=\eta\geqslant b.$$

Then we can estimate the bounds of the radius of D by Theorem 2.3 as follows.

Consider the following negative programming problem:

$$\Phi(b)=\min -\tfrac{1}{2}x^{\mathrm{T}}x \tag{3.2}$$

$$\text{s.t.}\quad x\in D. \tag{3.3}$$

To find a bound of $-\Phi(b)$, by Theorem 2.3, we solve the following linear programming to get $l(b)$:

$$l(b)=\min(y_1+y_2)$$

$$\text{s.t.}\quad y_1+y_2+\tfrac{1}{2}x_1=0;$$

$$y_1-y_2-y_3+\tfrac{1}{2}x_2=0;$$

$$y_1\leqslant 0,\quad y_2\leqslant 0,\quad y_3\leqslant 0;$$

$$x_1+x_2\leqslant 1,\quad x_1-x_2\leqslant 1,\quad -x_2\leqslant 0.$$

By Matlab, we got

$$l(b)=-2.6361. \tag{3.4}$$

By Theorem 2.3 we obtain a bound of the radius of D (noting that the exact radius of D is equal to 1):

$$\sqrt{-2l(b)}=\sqrt{4.2722}.$$

Remark 3.2. In the following, the meanings of notations like D, D_1, D_2 are the same as that appearing in the last section. Let $b=(1,1,1,1)^{\mathrm{T}}$ and

$$B=\begin{pmatrix}1, & 1\\ 1, & -1\\ -1, & -1\\ -1, & 1\end{pmatrix}.$$

It is easy to see that, for $x=(x_1,x_2)^{\mathrm{T}}\in R^2$,

$$\{x\mid Bx\geqslant b\}=\emptyset.$$

Noting that $D=\{x \mid Bx \leqslant b\}$ is bounded, let us follow the way as in the last section to estimate a bound of the radius of D by the following programming:

$$\min y^{\mathrm{T}} b \tag{3.5}$$

$$\text{s.t.} \quad y^{\mathrm{T}} B = -\tfrac{1}{2} x^{\mathrm{T}}, \quad y \leqslant 0; \tag{3.6}$$

$$Bx \leqslant b. \tag{3.7}$$

But the dual problem of the linear programming (3.4)–(3.7) is the following:

$$\max \lambda^{\mathrm{T}} b \tag{3.8}$$

$$\text{s.t.} \quad B\mu \geqslant b; \tag{3.9}$$

$$\tfrac{1}{2} \mu^{\mathrm{T}} - \lambda^{\mathrm{T}} B \leqslant 0; \tag{3.10}$$

$$-\tfrac{1}{2} \mu^{\mathrm{T}} + \lambda^{\mathrm{T}} B \leqslant 0; \tag{3.11}$$

$$\lambda \geqslant 0. \tag{3.12}$$

Since $\{B\mu \geqslant b\} = \emptyset$, the feasible space of the problem (3.8)–(3.12) is empty. We note that $D_1 \neq \emptyset$. Since $D_1 \subset D_2$, $D_2 \neq \emptyset$. It follows from the classical theory [7,2] of the linear programming that $y^{\mathrm{T}} b \longrightarrow -\infty$ on $D_2 = \{(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}} \mid y^{\mathrm{T}} B = -\frac{1}{2} x^{\mathrm{T}},\ y \leqslant 0;\ Bx \leqslant b\}$. We conclude that the way using the linear programming (3.5)–(3.7) is not valid in the case that $\{x \mid Bx \geqslant b\} = \emptyset$.

4. More details on using the presented estimate in Gao's parametric optimization problem

Although the relationship between the presented estimate and Gao's parametric optimization problem is obvious, it still needs to note that it usually takes a key role in Gao's dual problem (Eq. (1.6)) to provide numerical solutions. For clarifying this issue, we consider to restate Example 2 in [4] without the extra ball constraint $x_1^2 + x_2^2 \leqslant 4$ as follows:

$$\min P(x_1, x_2) = \tfrac{1}{2}(-0.5x_1^2 - 0.3x_2^2) - 0.3x_1 - 0.3x_2 \tag{4.1}$$

$$\text{s.t.} \quad \tfrac{1}{2} x_1 + x_2 \leqslant 2, \quad x_1 \geqslant 0, x_2 \geqslant 0. \tag{4.2}$$

We have $b = (1, 0, 0)^{\mathrm{T}}$ and

$$B = \begin{pmatrix} 0.5, & 1 \\ -1, & 0 \\ 0, & -1 \end{pmatrix}.$$

Thus, for $x = (x_1, x_2)^{\mathrm{T}} \in R^2$,

$$D = \{x \mid Bx \leqslant b\}.$$

By Theorem 2.3, we solve the following linear programming to get $l(b)$:

$$l(b) = \min(y_1)$$

$$\text{s.t.} \quad \tfrac{1}{2} y_1 - y_2 + \tfrac{1}{2} x_1 = 0;$$

$$y_1 - y_3 + \tfrac{1}{2} x_2 = 0;$$

$$y_1 \leqslant 0; \quad y_2 \leqslant 0; \quad y_3 \leqslant 0;$$

$$\tfrac{1}{2} x_1 + x_2 \leqslant 1; \quad -x_1 \leqslant 0; \quad -x_2 \leqslant 0.$$

By Matlab, we got

$$l(b) = -2.0. \tag{4.3}$$

By Theorem 2.3 we obtain the bounds of the radius of D (noting that the exact radius of D is exactly equal to 2):

$$\sqrt{-2l(b)} = \sqrt{4.0} = 2.0, \tag{4.4}$$

which is just used in Example 2 in [4] as an extra ball constraint.

This presentation shows that Theorem 2.3 is very useful to obtain a good bound of the radius of the feasible space D which is used in Gao's parametric optimization problem.

5. Conclusion

It follows from the classical theory [6] that a linear programming is a P-problem, i.e., it can be solved by an algorithm of polynomial time. A way of estimating a bound of the radius of the feasible space D presented in this paper will be helpful in using the canonical dual transformation method developed in [4] to solve a standard quadratic programming.

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