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On HSS and AHSS iteration methods for nonsymmetric positive definite Toeplitz systems[☆]

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Two iteration methods are proposed to solve real nonsymmetric positive definite Toeplitz systems of linear equations. These methods are based on Hermitian and skew-Hermitian splitting (HSS) and accelerated Hermitian and skew-Hermitian splitting (AHSS). By constructing an orthogonal matrix and using a similarity transformation, the real Toeplitz linear system is transformed into a generalized saddle point problem. Then the structured HSS and the structured AHSS iteration methods are established by applying the HSS and the AHSS iteration methods to the generalized saddle point problem. We discuss efficient implementations and demonstrate that the structured HSS and the structured AHSS iteration methods have better behavior than the HSS iteration method in terms of both computational complexity and convergence speed. Moreover, the structured AHSS iteration method outperforms the HSS and the structured HSS iteration methods. The structured AHSS iteration method also converges unconditionally to the unique solution of the Toeplitz linear system. In addition, an upper bound for the contraction factor of the structured AHSS iteration method is derived. Numerical experiments are used to illustrate the effectiveness of the structured AHSS iteration method.

1. Introduction

The concept of a Toeplitz matrix appeared in Toeplitz's early work in 1911; see [1]. An $n \times n$ Toeplitz matrix A is defined to be of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & \ddots & \ddots & \ddots & \vdots \\ a_{-2} & \ddots & \ddots & \ddots & a_2 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ a_{-n+1} & \ddots & a_{-2} & a_{-1} & a_0 \end{pmatrix},$$

i.e., the matrix A can be defined by its first row and first column.

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In this paper, we consider the solution of the real, nonsymmetric, and positive definite Toeplitz linear system

$$Ax = b, \tag{1.1}$$

where $A \in R^{n\times n}$, $x, b \in R^n$. Toeplitz linear systems arise in a variety of applications in mathematics, scientific computing and engineering, such as partial differential equations, signal processing, image restoration, integral equations and queuing networks. Toeplitz matrices are often assumed to be generated by a 2π-periodic function, say, $f(x)$, which is called the generating function of the matrix A. Let

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{\imath kx}dx \quad (k = 0, \pm 1, \ldots, \pm(n-1)),\ \imath = \sqrt{-1}.$$

Then we know that $a_k, k = 0, \pm 1, \ldots, \pm(n-1)$, are the Fourier coefficients of the function $f(x)$. The coefficients of the matrix A are given by the Fourier coefficients of the function $f(x)$. We point out that the generating function $f(x)$ is given explicitly in some applications, e.g., kernels of the Wiener–Hopf equations [2] and the spectral density functions in stationary stochastic processes [3]. These applications have motivated both mathematicians and engineers to develop specific algorithms for effectively solving Toeplitz linear systems.

Most of the early works about Toeplitz linear systems were focused on direct methods. Gaussian elimination method results in an algorithm of $O(n^3)$ complexity. However, since $n \times n$ Toeplitz matrices are determined by $(2n-1)$ entries, it is hoped to solve the Toeplitz linear system in $O(n^2)$ operations or less. The fast direct Kumar method of complexity $O(n \log^2 n)$ was developed via the Fast Fourier Transform in 1980; in its implementation a near breakdown can occur, however, when the matrix A is ill-conditioned. Therefore, iteration methods are preferred to solving Toeplitz linear systems.

Recently, the HSS iteration method has attracted many researchers' attention. In [4], Bai, Golub and Ng presented an HSS iteration method for solving the non-Hermitian positive definite linear system

$$\mathcal{A}u = f, \tag{1.2}$$

where $\mathcal{A} \in C^{n\times n}$ is a complex matrix, and $u, f \in C^n$ are complex vectors. They split the matrix as follows:

$$\mathcal{A} = \mathcal{A}_H + \mathcal{A}_S,$$

where $\mathcal{A}_H = \frac{1}{2}(\mathcal{A}+\mathcal{A}^*)$ is the Hermitian part and $\mathcal{A}_S = \frac{1}{2}(\mathcal{A}-\mathcal{A}^*)$ is the skew-Hermitian part of the matrix $\mathcal{A}$, respectively. Here $\mathcal{A}^*$ is used to denote the conjugate transpose of the matrix $\mathcal{A}$.

The HSS iteration method can be described as follows. Given an initial guess $u^{(0)} \in C^n$, compute $u^{(k)}$, for $k = 0, 1, 2, \ldots$, using the following iteration scheme until $\{u^{(k)}\}$ converges:

$$\begin{cases} (\alpha I + \mathcal{A}_H)u^{\left(k+\frac{1}{2}\right)} = (\alpha I - \mathcal{A}_S)u^{(k)} + f, \\ (\alpha I + \mathcal{A}_S)u^{(k+1)} = (\alpha I - \mathcal{A}_H)u^{\left(k+\frac{1}{2}\right)} + f, \end{cases} \tag{1.3}$$

where α is a given positive constant and I is the identity matrix.

The authors of [4] proved that the HSS iteration method converges unconditionally to the unique solution of the linear system (1.2). By further generalizing the HSS iteration method, the NSS iteration method was proposed in [5] and the PSS iteration method was presented in [6]. Actually, the NSS iteration method is a special case of the PSS iteration method. Moreover, according to the special characteristic of Toeplitz linear systems, Ng proposed the CSCS iteration method in [7], which is a special version of the NSS or the PSS iteration method. To further improve the computational efficiency of the HSS iteration method, Bai, Golub and Li applied the HSS iteration method to solve linear systems of block 2×2 structure, and obtained the formula of the optimal parameter that minimizes the spectral radius of the HSS iteration matrix [8]. In 2004, Benzi and Golub applied the HSS iteration method to saddle point problems, which extends the application region of the HSS iteration method from positive definite to positive semidefinite linear systems [9]. Based on this work, Bai, Golub and Pan further studied the PHSS iteration method in [10] for saddle point problems. In 2007, Bai and Golub proposed the AHSS iteration method in [11]; it considerately outperforms the HSS iteration method when solving saddle point problems. In [12], Bai, Golub and Li applied the HSS iteration method with preconditioning to solve non-Hermitian and positive semidefinite linear systems. Hence, we can see that in recent years there have been plenty of research results based on the HSS iteration methods and its convergence theory.

In this paper, the HSS and the AHSS iteration methods are structurally applied to generalized saddle point problems transformed from the real Toeplitz linear system (1.1). The structured HSS and the structured AHSS iteration methods are established, and they are analyzed theoretically and examined numerically.

The paper is organized as follows. In Section 2, we convert a real Toeplitz linear system to a generalized saddle point problem by similarity transformation. In Section 3, we analyze the convergence properties of the HSS and the structured HSS iteration methods, and also explain the application scope of the related method mentioned in [13]. According to the special characteristic of the Toeplitz matrix, we analyze the convergence property of the structured AHSS iteration method in Section 4. Finally, in Section 5, we use several examples to illustrate the numerical effectiveness of our new methods.

2. Some properties of a real Toeplitz matrix

In this section, we transform the Toeplitz linear systems to generalized saddle point problems by technically constructing an orthogonal matrix. For the real $n \times n$ Toeplitz matrix A, its top row and left column can be written as

$$[a_0, a_1, \dots, a_{n-1}]$$

and

$$[a_0, a_{-1}, \dots, a_{-n+1}]^T,$$

respectively.

The real Toeplitz matrix A has the Hermitian and skew-Hermitian (HS) splitting

$$A = H + S,$$

where $H = \frac{1}{2}(A + A^T)$ is a symmetric matrix and $S = \frac{1}{2}(A - A^T)$ is a skew-symmetric matrix. Here, we have used A^T to denote the transpose of the matrix A. Notice that H is a centrosymmetric Toeplitz matrix and S is a skew-centrosymmetric Toeplitz matrix. Denote $J_n = [e_n, e_{n-1}, \dots, e_1]$, where $e_i \in R^n$ is the unit vector of the ith entry being 1, i.e., $J_n \in R^{n\times n}$ is a permutation matrix with ones on the cross diagonal (bottom left to top right) and zeros elsewhere. Then the matrices H and S can be expressed as

$$H = J_n H J_n \quad \text{and} \quad S = -J_n S J_n,$$

respectively.

Now we discuss the structures of H and S in detail according to the cases that n is even and odd.

(1) n is an even number, say, $n = 2m$.

In this case, the matrix H can be written as

$$H = \begin{pmatrix} B & J_m C J_m \\ C & J_m B J_m \end{pmatrix},$$

where $B \in R^{m\times m}$ is a symmetric Toeplitz matrix with its top row being given by

$$\frac{1}{2}[2a_0, (a_1 + a_{-1}), \dots, (a_{m-1} + a_{-m+1})],$$

and C is a Toeplitz matrix with its first row being given by

$$\frac{1}{2}[(a_m + a_{-m}), (a_{m-1} + a_{-m+1}), \dots, (a_1 + a_{-1})]$$

and its first column being given by

$$\frac{1}{2}[(a_m + a_{-m}), (a_{m+1} + a_{-m-1}), \dots, (a_{n-1} + a_{-n+1})]^T.$$

Similarly, the matrix S can be written as

$$S = \begin{pmatrix} D & -J_m E J_m \\ E & -J_m D J_m \end{pmatrix},$$

where D is a skew-symmetric Toeplitz matrix with its top row being given by

$$\frac{1}{2}[0, (a_1 - a_{-1}), \dots, (a_{m-1} - a_{-m+1})],$$

and E is a Toeplitz matrix with its first row being given by

$$-\frac{1}{2}[(a_m - a_{-m}), (a_{m-1} - a_{-m+1}), \dots, (a_1 - a_{-1})]$$

and its first column being given by

$$-\frac{1}{2}[(a_m - a_{-m}), (a_{m+1} - a_{-m-1}), \dots, (a_{n-1} - a_{-n+1})]^T.$$

Define

$$Q = \frac{\sqrt{2}}{2}\begin{pmatrix} I_m & I_m \\ J_m & -J_m \end{pmatrix}. \tag{2.1}$$

Then, Q is an orthogonal matrix. Moreover, it holds that

$$Q^T HQ = \begin{pmatrix} B + J_m C & O \\ O & B - J_m C \end{pmatrix} := \tilde{H}$$

and

$$Q^T SQ = \begin{pmatrix} O & D + J_m E \\ D - J_m E & O \end{pmatrix} := \tilde{S}.$$

Let

$$\tilde{A} = Q^T AQ, \qquad \tilde{x} = Q^T x, \qquad \tilde{b} = Q^T b.$$

Then we can equivalently rewritten the Toeplitz linear system (1.1) as the linear system

$$\tilde{A}\tilde{x} = \tilde{b}. \tag{2.2}$$

Obviously, the coefficient matrix $\tilde{A}$ is still a real nonsymmetric positive definite matrix, and it can be split into symmetric and skew-symmetric parts as

$$\tilde{A} = \tilde{H} + \tilde{S},$$

where

$$\tilde{A} = Q^T AQ = Q^T HQ + Q^T SQ = \tilde{H} + \tilde{S} = \begin{pmatrix} B + J_m C & D + J_m E \\ D - J_m E & B - J_m C \end{pmatrix}.$$

Let

$$G = (B + J_m C), \qquad U = B - J_m C, \qquad W = (D + J_m E).$$

Then we observe that

$$\begin{cases} G^T = (B + J_m C)^T = B^T + C^T J_m = B + J_m C = G, \\ U^T = (B - J_m C)^T = B - J_m C = U, \\ W^T = (D + J_m E)^T = D^T + E^T J_m = -D + J_m E. \end{cases} \tag{2.3}$$

So, $\tilde{A}$ can be expressed as

$$\tilde{A} = \begin{pmatrix} G & W \\ -W^T & U \end{pmatrix}.$$

Because $\tilde{A}$ is real nonsymmetric and positive definite, $\tilde{H}$ is a symmetric positive definite matrix, or equivalently, G and U are symmetric positive definite matrices. Therefore, the linear system (2.2) can be considered as a generalized saddle point problem.

(2) n is an odd number, say, $n = 2m + 1$.

Let

$$H = \begin{pmatrix} B & J_m r & J_m C J_m \\ r^T J_m & a_0 & r^T \\ C & r & J_m B J_m \end{pmatrix},$$

$$S = \begin{pmatrix} D & J_m z & -J_m E J_m \\ -z^T J_m & 0 & z^T \\ E & -z & -J_m D J_m \end{pmatrix}.$$

By defining the orthogonal matrix

$$Q = \frac{\sqrt{2}}{2} \begin{pmatrix} I_m & O & I_m \\ O & \sqrt{2} & O \\ J_m & O & -J_m \end{pmatrix},$$

we have

$$Q^T HQ = \begin{pmatrix} B + J_m C & \sqrt{2} J_m r & O \\ \sqrt{2} r^T J_m & a_0 & O \\ O & O & B - J_m C \end{pmatrix} := \tilde{H}$$

and

$$Q^T S Q = \begin{pmatrix} O & O & D + J_m E \\ O & O & -\sqrt{2} z^T J_m \\ D - J_m E & \sqrt{2} J_m z & O \end{pmatrix} := \tilde{S}.$$

Let

$$G = \begin{pmatrix} B + J_m C & \sqrt{2} J_m r \\ \sqrt{2} r^T J_m & a_0 \end{pmatrix}, \qquad U = (B - J_m C), \qquad W = \begin{pmatrix} D + J_m E \\ -\sqrt{2} z^T J_m \end{pmatrix}.$$

Then we can equivalently rewrite the Toeplitz linear system (1.1) as a generalized saddle point problem of an analogous form to (2.2).

3. The structured HSS iteration method for real Toeplitz linear systems

The structured HSS iteration method was proposed in [13], where the HSS iteration method was used indirectly. A structured HSS iteration method for the Toeplitz linear system (1.1) means that we use the HSS iteration method on the generalized saddle point problem (2.2). In this section, the computational complexity of the HSS and the structured HSS iteration methods are analyzed and compared. Moreover, we point out that the application region of the method in [13] contains only the real Toeplitz linear systems instead of the complex ones. For simplicity, we only consider the case of even n, i.e., $n = 2m$, in the remainder of the paper.

First, we review the convergence criterion for a two-step splitting iteration established in [4].

Lemma 3.1 ([4]). *Let $A \in R^{n\times n}$, $A = M_i - N_i$ $(i = 1, 2)$ be two splittings of the matrix A, and $x^{(0)} \in R^n$ be a given initial vector. If $\{x^{(k)}\}$ is a two-step iteration sequence defined by*

$$\begin{cases} M_1 x^{\left(k+\frac{1}{2}\right)} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{\left(k+\frac{1}{2}\right)} + b, \end{cases}$$

$k = 0, 1, 2, \ldots$, *then*

$$x^{(k+1)} = M_2^{-1} N_2 M_1^{-1} N_1 x^{(k)} + M_2^{-1}(I + N_2 M_1^{-1}) b, \quad k = 0, 1, 2, \ldots.$$

Moreover, if the spectral radius $\rho(M_2^{-1} N_2 M_1^{-1} N_1) < 1$, then the iterative sequence $\{x^{(k)}\}$ converges to the unique solution of the linear system (1.1) for any initial vector $x^{(0)} \in R^n$.

By applying the HSS iteration method to the linear system (2.2), we can obtain a structured HSS iteration method for the real Toeplitz linear system (1.1), which can be described as follows.

Given an initial guess $x^{(0)} \in R^n$, let $\tilde{x}^{(0)} = Q^T x^{(0)}$, and compute $\tilde{x}^{(k)}$, for $k = 0, 1, 2, \ldots$, using the following iteration scheme until $\{\tilde{x}^{(k)}\}$ converges:

$$\begin{cases} (\alpha I + G)\tilde{x}_1^{\left(k+\frac{1}{2}\right)} = \alpha \tilde{x}_1^{(k)} - W \tilde{x}_2^{(k)} + \tilde{b}_1, \\ (\alpha I + U)\tilde{x}_2^{\left(k+\frac{1}{2}\right)} = W^T \tilde{x}_1^{(k)} + \alpha \tilde{x}_2^{(k)} + \tilde{b}_2, \\ (\alpha^2 I + W W^T)\tilde{x}_1^{(k+1)} = \alpha(\alpha I - G)\tilde{x}_1^{\left(k+\frac{1}{2}\right)} - W(\alpha I - U)\tilde{x}_2^{\left(k+\frac{1}{2}\right)} + \alpha \tilde{b}_1 - W \tilde{b}_2, \\ \tilde{x}_2^{(k+1)} = \dfrac{1}{\alpha}\left((\alpha I - U)\tilde{x}_2^{\left(k+\frac{1}{2}\right)} + W^T \tilde{x}_1^{(k+1)} + \tilde{b}_2 \right), \end{cases} \tag{3.1}$$

where α is a given positive constant, $\tilde{x}_i, \tilde{b}_i \in R^m$ $(i = 1, 2)$, $\tilde{x} = (\tilde{x}_1^T, \tilde{x}_2^T)^T$, $\tilde{b} = (\tilde{b}_1^T, \tilde{b}_2^T)^T$. Finally, we can recover the approximate solution $x^{(k+1)}$ through $x^{(k+1)} = Q\tilde{x}^{(k+1)}$.

By (2.3), we find that $\alpha I + G$ and $\alpha I + U$ are Toeplitz-plus-Hankel matrices, and the associated matrix–vector multiplications can be calculated in $O\left(\frac{n}{2}\log\frac{n}{2}\right)$ operations by using the Discrete Fourier Transform. The computational complexities of solving the first and the second equations in (3.1) can be reduced to $O\left(\frac{n}{2}\log\frac{n}{2}\right)$ by the preconditioned conjugate gradient method [14]. However, if we use the HSS iteration method directly on the real Toeplitz linear system (1.1), the computational costs of the corresponding parts are up to $O(n\log n)$ even by the preconditioned conjugate gradient method. The operations for the remaining parts of the two methods are almost the same. Thus, the computational complexity of the structured HSS iteration method (3.1) is, overall, much less than that of the HSS iteration method.

We should notice that the structured HSS iteration method (3.1) is adequate only for real nonsymmetric Toeplitz linear systems instead of complex ones. Therefore, Theorem 3 in [13] is invalid for a complex Toeplitz linear system. This fact is well illustrated by the following example.

Example 3.1. Consider a Toeplitz linear system $Ax = b$ with

$$A = \begin{pmatrix} 10+15\imath & 2-\imath & -1+2\imath & 2-4\imath \\ 3-2\imath & 10+15\imath & 2-\imath & -1+2\imath \\ 2+3\imath & 3-2\imath & 10+15\imath & 2-\imath \\ 1-\imath & 2+3\imath & 3-2\imath & 10+15\imath \end{pmatrix}.$$

Case I: $A = H_1 + S_1$, where $H_1 = \frac{1}{2}(A + A^*)$ and $S_1 = \frac{1}{2}(A - A^*)$. Let

$$U_1 = (I - H_1)(I + H_1)^{-1} \quad \text{and} \quad V_1 = (I - S_1)(I + S_1)^{-1}.$$

Then, by straightforward computation, we get

$$\|U_1\|_2 = 0.8768, \qquad \|V_1\|_2 = 1.$$

So

$$\|U_1\|_2 \cdot \|V_1\|_2 = 0.8768 < 1.$$

This coincides with the result for the HSS iteration method.

Case II: $A = H_2 + S_2$, where $H_2 = \frac{1}{2}(A + A^T)$ and $S_2 = \frac{1}{2}(A - A^T)$. Let

$$U_2 = (I - H_2)(I + H_2)^{-1} \quad \text{and} \quad V_2 = (I - S_2)(I + S_2)^{-1}.$$

Then, by direct calculation, we have

$$\|U_2\|_2 = 0.9753, \qquad \|V_2\|_2 = 4.7236.$$

So

$$\|U_2\|_2 \cdot \|V_2\|_2 = 4.6067 > 1.$$

This contradicts Theorem 3 in [13].

4. The structured AHSS iteration method for real Toeplitz linear systems

In this section, the structured AHSS iteration method for solving the Toeplitz linear system (1.1) is proposed based on the AHSS iteration method in [11]. Because the linear system (2.2) is a generalized saddle point problem, we concentrate on studying the convergence property of the structured AHSS iteration method and deriving an upper bound for its contraction factor.

Let

$$P = \begin{pmatrix} \alpha I & O \\ O & \beta I \end{pmatrix}, \tag{4.1}$$

where α and β are given positive constants. Then the AHSS iteration method for the linear system (2.2) can be described as follows.

Given an initial guess $\tilde{x}^{(0)} \in R^n$, compute $\tilde{x}^{(k)}$, for $k = 0, 1, 2, \ldots$, using the following iteration scheme until $\{\tilde{x}^{(k)}\}$ is convergent:

$$\begin{cases} (P + \tilde{H})\tilde{x}^{\left(k+\frac{1}{2}\right)} = (P - \tilde{S})\tilde{x}^{(k)} + \tilde{b}, \\ (P + \tilde{S})\tilde{x}^{(k+1)} = (P - \tilde{H})\tilde{x}^{\left(k+\frac{1}{2}\right)} + \tilde{b}. \end{cases} \tag{4.2}$$

According to the special structure of the matrix $\tilde{A}$, based on the iteration scheme (4.2), we can obtain the structured AHSS iteration method for the real Toeplitz linear system (1.1) as follows.

First, compute $\tilde{x}^{(k)}$ using the following iteration scheme until $\{\tilde{x}^{(k)}\}$ converges:

$$\begin{cases} (\alpha I + G)\tilde{x}_1^{\left(k+\frac{1}{2}\right)} = \alpha\tilde{x}_1^{(k)} - W\tilde{x}_2^{(k)} + \tilde{b}_1, \\ (\beta I + U)\tilde{x}_2^{\left(k+\frac{1}{2}\right)} = W^T\tilde{x}_1^{(k)} + \beta\tilde{x}_2^{(k)} + \tilde{b}_2, \\ (\alpha\beta I + WW^T)\tilde{x}_1^{(k+1)} = \beta(\alpha I - G)\tilde{x}_1^{\left(k+\frac{1}{2}\right)} - W(\beta I - U)\tilde{x}_2^{\left(k+\frac{1}{2}\right)} + \beta\tilde{b}_1 - W\tilde{b}_2, \\ \tilde{x}_2^{(k+1)} = \dfrac{1}{\beta}\left((\beta I - U)\tilde{x}_2^{\left(k+\frac{1}{2}\right)} + W^T\tilde{x}_1^{(k+1)} + \tilde{b}_2\right). \end{cases} \tag{4.3}$$

Then the approximate solution $x^{(k+1)}$ is recovered from $x^{(k+1)} = Q\tilde{x}^{(k+1)}$.

From (4.3), we can easily see that the computational complexity of the structured AHSS iteration method is the same as that of the structured HSS iteration method.

Applying Lemma 3.1 to the structured AHSS iteration method (4.3), we can obtain the following convergence theorem.

Theorem 4.1. *Let $A \in R^{n\times n}$ be a real, nonsymmetric, and positive definite Toeplitz matrix. Denote*

$$\tilde{A} = \frac{1}{2}Q^T AQ, \qquad \tilde{H} = \frac{1}{2}Q^T (A + A^T)Q, \qquad \tilde{S} = \frac{1}{2}Q^T (A - A^T)Q,$$

where Q is the orthogonal matrix defined by (2.1), and α and β are two positive constants. Then the iteration matrix $M(\alpha, \beta)$ of the structured AHSS iteration method (4.3) is given by

$$M(\alpha, \beta) = Q(P + \tilde{S})^{-1}(P - \tilde{H})(P + \tilde{H})^{-1}(P - \tilde{S})Q^T, \tag{4.4}$$

where P is the parameter matrix defined by (4.1). The spectral radius $\rho(M(\alpha, \beta))$ of $M(\alpha, \beta)$ is bounded by

$$\delta(\alpha, \beta) = \max\left\{\max_{\lambda_i\in\lambda(G)} \frac{|\alpha - \lambda_i|}{|\alpha + \lambda_i|}, \max_{\mu_i\in\lambda(U)} \frac{|\beta - \mu_i|}{|\beta + \mu_i|}\right\}, \tag{4.5}$$

where $\lambda(G)$ denote the spectral set of the matrix G. Therefore, for any positive parameters α and β, it holds that

$$\rho(M(\alpha, \beta)) \le \delta(\alpha, \beta) < 1,$$

i.e., the structured AHSS iteration method (4.3) unconditionally converges to the unique solution of the system of linear equations (1.1).

Proof. Write

$$U = (P - \tilde{H})(P + \tilde{H})^{-1}, \qquad V = (P - \tilde{S})(P + \tilde{S})^{-1}.$$

By similarity transformation, we get

$$\rho(M(\alpha, \beta)) = \rho(UV).$$

Let

$$\tilde{U} = \left(I - P^{-\frac{1}{2}}\tilde{H}P^{-\frac{1}{2}}\right)\left(I + P^{-\frac{1}{2}}\tilde{H}P^{-\frac{1}{2}}\right)^{-1}, \qquad \tilde{V} = \left(I - P^{-\frac{1}{2}}\tilde{S}P^{-\frac{1}{2}}\right)\left(I + P^{-\frac{1}{2}}\tilde{S}P^{-\frac{1}{2}}\right)^{-1}.$$

Then we have

$$\begin{aligned}\rho(M(\alpha, \beta)) &= \rho(UV) = \rho\left(P^{\frac{1}{2}}\tilde{U}\tilde{V}P^{-\frac{1}{2}}\right) = \rho(\tilde{U}\tilde{V}) \\ &\le \|\tilde{U}\tilde{V}\|_2 \le \|\tilde{U}\|_2\|\tilde{V}\|_2 = \|\tilde{U}\|_2 \\ &= \max\left\{\max_{\lambda_i\in\lambda(G)} \frac{|\alpha - \lambda_i|}{|\alpha + \lambda_i|}, \max_{\mu_i\in\lambda(U)} \frac{|\beta - \mu_i|}{|\beta + \mu_i|}\right\}.\end{aligned}$$

Furthermore, it clearly holds that

$$\rho(M(\alpha, \beta)) \le \delta(\alpha, \beta) < 1,$$

since G and U are positive definite matrices, and α and β are positive constants. □

Theorem 4.1 shows that the convergence speed of the structured AHSS iteration method is bounded by $\delta(\alpha, \beta)$, which depends on the spectrum of the symmetric part $\tilde{H}$, but does not depend on the spectrum of $\tilde{A}$ and $\tilde{S}$, nor on the eigenvectors of the matrices involved.

Now, we are going to give the optimal parameters α and β that minimize the quantity $\delta(\alpha, \beta)$ in the following corollary.

Corollary 4.1. *Let $\tilde{A}$, $\tilde{H}$, $\tilde{S}$ and Q be defined as in Theorem 4.1, $\lambda_{\min}$ and $\lambda_{\max}$ be the minimal and the maximal eigenvalues of the matrix G, and $\mu_{\min}$ and $\mu_{\max}$ be the minimal and the maximal eigenvalues of the matrix U, respectively. Then the optimal parameters α^* and β^* that minimize the upper bound of the structured AHSS iteration method are given by*

$$\{\alpha^*, \beta^*\} = \left\{\sqrt{\lambda_{\min}\lambda_{\max}}, \sqrt{\mu_{\min}\mu_{\max}}\right\}, \tag{4.6}$$

and the corresponding optimal upper bound is given by

$$\begin{aligned}\delta(\alpha^*, \beta^*) &= \max\left\{\frac{\sqrt{\lambda_{\max}} - \sqrt{\lambda_{\min}}}{\sqrt{\lambda_{\max}} + \sqrt{\lambda_{\min}}}, \frac{\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}} + \sqrt{\mu_{\min}}}\right\} \\ &= \max\left\{\frac{\sqrt{\kappa(G)} - 1}{\sqrt{\kappa(G)} + 1}, \frac{\sqrt{\kappa(U)} - 1}{\sqrt{\kappa(U)} + 1}\right\} < 1,\end{aligned}$$

where $\kappa(G)$ and $\kappa(U)$ are the spectral condition numbers of the matrices G and U, respectively.

Table 1
The optimal parameters of the structured HSS and AHSS iteration methods for Example 5.1.

n	$\tilde{\alpha}^*$	$\rho(M(\tilde{\alpha}^*))$	α^*	β^*	$\rho(M(\alpha^*, \beta^*))$
16	6.177	0.4748	6.736	6.107	0.4598
32	6.048	0.4872	6.207	6.030	0.4828
64	6.012	0.4906	6.054	6.008	0.4894
128	6.003	0.4914	6.014	6.002	0.4911

Proof. To minimize the upper bound $\delta(\alpha, \beta)$ in (4.5), we only need to solve the following two equations:

$$\frac{|\alpha - \lambda_{\min}|}{|\alpha + \lambda_{\min}|} = \frac{|\alpha - \lambda_{\max}|}{|\alpha + \lambda_{\max}|}, \qquad \frac{|\beta - \mu_{\min}|}{|\beta + \mu_{\min}|} = \frac{|\beta - \mu_{\max}|}{|\beta + \mu_{\max}|}.$$

The result in the corollary is then obtained immediately. □

It should be noted that the optimal parameters α^* and β^* in Corollary 4.1 only minimize the upper bound $\delta(\alpha, \beta)$, but not the spectral radius of the iteration matrix. In particular, when $\alpha = \beta$, we can analogously get the corresponding result for the structured HSS iteration method.

Corollary 4.2. *Let $\tilde{A}$, $\tilde{H}$, $\tilde{S}$ and Q be defined as in Theorem 4.1, and $\tilde{\lambda}_{\min}$ and $\tilde{\lambda}_{\max}$ be the minimal and the maximal eigenvalues of the matrix $\tilde{H}$, respectively. Then the optimal parameter $\tilde{\alpha}^*$ that minimizes the upper bound of the structured HSS iteration method is given by*

$$\tilde{\alpha}^* = \sqrt{\tilde{\lambda}_{\min}\tilde{\lambda}_{\max}}, \tag{4.7}$$

and the corresponding optimal upper bound is given by

$$\tilde{\delta}(\tilde{\alpha}^*) := \delta(\tilde{\alpha}^*, \tilde{\beta}^*) = \frac{\sqrt{\tilde{\lambda}_{\max}} - \sqrt{\tilde{\lambda}_{\min}}}{\sqrt{\tilde{\lambda}_{\max}} + \sqrt{\tilde{\lambda}_{\min}}} = \frac{\sqrt{\kappa(\tilde{H})} - 1}{\sqrt{\kappa(\tilde{H})} + 1} < 1.$$

Corollaries 4.1 and 4.2 show that $\delta(\alpha^*, \beta^*) \le \tilde{\delta}(\tilde{\alpha}^*)$. So we may conclude that the structured AHSS iteration method converges faster than the structured HSS iteration method, at least in theory.

When n is odd, i.e., $n = 2m + 1$, the above results can be demonstrated in an analogous fashion.

5. Numerical examples

In this section, two Toeplitz linear systems are used to illustrate the effectiveness of the structured AHSS iteration method (4.3). All our tests are performed in MATLAB. In actual computations, we assume that the Toeplitz matrices are defined by generating functions, and we choose the right-hand side vectors b such that the exact solutions of the corresponding linear systems (1.1) are $x = (1, 1, \ldots, 1)^T$.

Example 5.1. The generating function is

$$f(x) = 10 + 8\cos(x) + \imath 2\sin(5x).$$

By similarity, we know that the optimal parameters for the upper bounds of the HSS and the structured HSS iteration methods are the same. Moreover, the complexity of the structured HSS iteration method is considerately less than that of the HSS iteration method. Therefore, we only need to compare the optimal convergence factors for the structured HSS and the structured AHSS iteration methods.

In Table 1, we list $\tilde{\alpha}^*$ and $\rho(M(\tilde{\alpha}^*))$ for the structured HSS iteration method, and α^*, β^* and $\rho(M(\alpha^*, \beta^*))$ for the structured AHSS iteration method, with respect to different n. The results in Table 1 show that the structured AHSS iteration method (4.3) has smaller optimal convergence factor than the structured HSS iteration method (1.3). Roughly speaking, $\tilde{\alpha}^*$ is between α^* and β^*.

The experimentally found optimal values $\tilde{\alpha}_t$, α_t, β_t and the corresponding spectral radii for the structured HSS and the structured AHSS iteration methods are listed in Table 2. From this table, we see that $\tilde{\alpha}_t$ is between α_t and β_t, and the spectral radius of the iteration matrix for the structured AHSS iteration method (4.3) is smaller than that of the structured HSS iteration method (3.1) when the experimentally found optimal parameters are employed.

Example 5.2. The generating function is

$$f(x) = 5 + x^2 + 2\cos(3x) + \imath(x + 3\sin x).$$

Table 2
The experimentally found optimal parameters for the structured HSS and AHSS iteration methods for Example 5.1.

n	$\tilde{\alpha}_t$	$\rho(M(\alpha_t))$	α_t	β_t	$\rho(M(\alpha_t, \beta_t))$
16	6.651	0.4461	6.532	6.801	0.4450
32	6.510	0.4587	7.398	5.705	0.4553
64	6.479	0.4617	6.600	6.365	0.4615
128	6.471	0.4625	6.478	6.466	0.4624

Table 3
The optimal parameters for the structured HSS and AHSS iteration methods for Example 5.2.

n	$\tilde{\alpha}^*$	$\rho(M(\alpha^*))$	α^*	β^*	$\rho(M(\alpha^*, \beta^*))$
16	7.204	0.2500	7.085	7.264	0.2482
32	7.128	0.2667	7.031	7.128	0.2636
64	7.126	0.2730	7.058	7.126	0.2709
128	7.138	0.2755	7.099	7.138	0.2748

Table 4
The experimentally optimal parameters for the structured HSS and AHSS iteration methods for Example 5.2.

n	$\tilde{\alpha}_t$	$\rho(M(\alpha_t))$	α_t	β_t	$\rho(M(\alpha_t, \beta_t))$
16	7.110	0.2441	5.449	9.164	0.2095
32	6.979	0.2572	5.361	9.125	0.2216
64	7.032	0.2671	5.333	9.292	0.2316
128	7.106	0.2736	5.331	9.422	0.2377

In Table 3, $\tilde{\alpha}^*$ and $\rho(M(\tilde{\alpha}^*))$ for the structured HSS iteration method, and α^*, β^* and $\rho(M(\alpha^*, \beta^*))$ for the structured AHSS iteration method, with respect to different n, are listed. From this table we see that $\rho(M(\tilde{\alpha}^*))$ are larger than $\rho(M(\alpha^*, \beta^*))$, so the structured AHSS iteration method (4.3) may have a faster convergence factor than the structured HSS iteration method (1.3).

The experimentally found optimal values $\tilde{\alpha}_t$, α_t, β_t and the corresponding spectral radii for the structured HSS and the structured AHSS iteration methods are listed in Table 4. From this table, we observe that the spectral radius of the iteration matrix for the structured AHSS iteration method (4.3) is smaller than that of the structured HSS iteration method (3.1), when the experimentally found optimal parameters are employed.

In summary, from Tables 1–4 we know that for the structured HSS iteration method $\rho(M(\tilde{\alpha}^*))$ is a little larger than $\rho(M(\tilde{\alpha}_t))$. The same phenomenon occurs for the structured AHSS iteration method. Therefore, when the optimal parameters α^* and β^* are used in actual applications, the structured AHSS iteration method outperforms the structured HSS iteration method from both theoretical and experimental viewpoints.

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