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An improvement on the positivity results for 2-stage explicit Runge–Kutta methods

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ABSTRACT

In this paper, we investigate the positivity property for a class of 2-stage explicit Runge–Kutta (RK2) methods of order two when applied to the numerical solution of special nonlinear initial value problems (IVPs) for ordinary differential equations (ODEs). We also pay particular attention to monotonicity property. We obtain new results for positivity which are important in practical applications. We provide some numerical examples to illustrate our results.

1. Introduction

Consider an initial value problem for a positive system of ordinary differential equations (ODEs) of type

$$U'(t) = F(t, U(t)), \quad (t \geq 0), \qquad U(0) = U_0. \tag{1}$$

With positivity, we mean, the component-wise non-negativity of the initial vector, is preserved in time for the exact solution ($U(t) \geq 0, t > 0$ if $U_0 \geq 0$). There are many problems of practical interest that can be modelled by positive ODEs. For example positive ODEs arise from modelling chemical reactions or semidiscretizing partial differential equations of advection-diffusion type (see e.g. [1]). In both cases, the components of the unknown can denote concentrations or densities which are physical quantities and they need to remain positive.

Solving a positive ODE numerically with a non-negative initial vector, it is a natural demand that the resulting numerical approximations $U_n \approx U(t_n),\ t_n = n\Delta t,\ \Delta t$ being the time step, should be non-negative. Furthermore, a negative value may cause undershoots or overshoots near a steep gradient. Therefore, we need to analyse numerical methods from the point of view of positivity (preservation of non-negativity).

As our numerical scheme, we consider the following 2-stage explicit Runge–Kutta scheme

$$\begin{aligned} U_{n_1} &= U_n \\ U_{n_2} &= U_n + \kappa \Delta t F(t_n, U_{n_1}) \\ U_{n+1} &= U_n + \Delta t \left(\left(1 - \frac{1}{2\kappa}\right) F(t_n, U_{n_1}) + \frac{1}{2\kappa} F(t_n + \kappa\Delta t, U_{n_2}) \right). \end{aligned} \tag{2}$$

If $\kappa = 1$, (2) gives the explicit trapezoidal method. This method is also known as the improved Euler method or Heun's method. With $\kappa = \frac{1}{2}$, we have the one-step explicit midpoint rule which is also known as the modified Euler method.

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In the literature, we can find several papers devoted to discussing positivity property (e.g., [2–7]). In [1], positivity results have been presented for some Runge–Kutta methods. Based on these results, for example, with explicit midpoint rule, positivity is not ensured when applied to the inhomogeneous linear systems and the same result is regained on nonlinear positivity for this method. In [8], a step size condition has been obtained for monotonicity ($\|U_n\| \le \|U_0\|$ for all $n \ge 1$, $U_0 \in \mathbb{R}^m$) with arbitrary convex function $\|.\|$, for general linear methods. $\|.\|$ is a convex function on $\mathbb{V}$ (the vector space on which the differential equation is defined) if $\|\lambda v + (1-\lambda)w\| \le \lambda\|v\| + (1-\lambda)\|w\|$ for $0 \le \lambda \le 1$ and $v, w \in \mathbb{V}$. Usually, *step size coefficients* γ are determined such that monotonicity, in the sense of mentioned above, is present for all Δt with $0 < \Delta t \le \gamma\tau_0$ ($\tau_0 > 0$ is a maximal step size such that $\|v + \Delta t F(t, v)\| \le \|v\|$ for all t, $0 < \Delta t \le \tau_0$ and $v \in \mathbb{R}^m$), see e.g. [3,9,8,6]. General monotonicity of Runge–Kutta methods presented in [8] shows that the maximal step size coefficient γ for explicit midpoint rule, is equal to 0. Monotonicity-preserving methods, can prevent the occurrence of negative values where even very small negative values are unacceptable, as for example, in the advective transport of chemical species see e.g. [10]. On the other hand, positivity preservation may be obtained from monotonicity-preserving methods (such a relation will be discussed in Lemma 1).

An experimental study shows that the necessity of the step size restriction on positivity in general theory (see e.g. [1, p. 190]) for explicit midpoint method is somewhat too strict. Applying the explicit midpoint rule to special nonlinear ODEs (positive semi-discrete systems arising 1D and 2D advection with limited third-order upwind-biased spatial discretization), it is observed that the step size restriction here, is comparable to the step size restriction for the explicit trapezoidal rule with respect to positivity. In fact, in the special cases of ODEs, a similar qualitative behavior and temporal accuracy are observed with both methods. From this practical point of view, the question arises whether it is theoretically possible to have positivity preservation for the explicit midpoint method. To answer this question, the class of RK2 in (2) which includes the explicit midpoint method and the explicit trapezoidal rule, is applied to a special ODE and some results are achieved theoretically that coincide with numerical experiments. Here, we focus on positivity for this RK2 methods.

In Section 2, general positivity results are presented for RK2 methods. In Section 3, the special positivity results are obtained for RK2 methods and a comparison is made with existing general positivity results. The numerical results obtained are then compared in Section 4 with respect to positivity. Both one and two-dimensional linear scalar advection equations are used as test cases.

2. General results on positivity for 2-stage explicit Runge–Kutta methods

In this section, we study the general positivity for RK2 methods. In many papers, one starts from an assumption about F which, $\tau_0 \ge 0$, to be the maximal step size such that positivity holds for the forward Euler method i.e.

$$U + \Delta t F(t, U) \ge 0 \quad (\text{for all } t \text{ and } U \ge 0), \tag{3}$$

whenever $0 < \Delta t \le \tau_0$ and $U \in \mathbb{R}^m$. We shall determine step size coefficients $\gamma(\kappa)$, $\kappa > 0$ such that the positivity is valid for (2) under the step size restriction $\Delta t \le \gamma(\kappa)\tau_0$. Based on the positivity results for Runge–Kutta methods (see e.g. [1]), the explicit trapezoidal method is nonlinearly positive under assumption (3) for $\Delta t \le \gamma(\kappa)\tau_0$ whenever $\gamma(\kappa) = 1$. With the explicit midpoint rule, positivity is not ensured when applied to the inhomogeneous linear systems ($\gamma(\kappa) = 0$).

The standard positivity theory of Runge–Kutta methods is based on the Shu–Osher [5,6] form of the methods, whereby it is written as convex combinations of Euler steps, under the assumption (3) of the ODE. Following this idea the last stage of (2) is written as

$$U_{n+1} = (1-\theta)U_n + \left(1 - \frac{1}{2\kappa} - \kappa\theta\right)\Delta t F(t_n, U_n) + \theta U_{n_2} + \frac{1}{2\kappa}\Delta t F(t_n + \kappa\Delta t, U_{n_2}), \tag{4}$$

with $0 \le \theta \le 1$ and $U_{n_2} = U_n + \kappa\Delta t F(U_{n_1})$. Then, $U_{n+1} \ge 0$ under the step size restriction $\Delta t \le \gamma(\kappa)\tau_0$, where

$$\gamma(\kappa) = \min\left(\frac{1-\theta}{1 - \frac{1}{2\kappa} - \kappa\theta}, 2\kappa\theta\right).$$

Obviously, we are interested in largest $\gamma(\kappa)$ so that we will have better positivity properties of the scheme. With $\frac{1}{2} < \kappa < 1$ and $\kappa > 1$, the largest $\gamma(\kappa)$ obtained when $\theta = \frac{2\kappa^2 - 4\kappa + 1}{4\kappa^3 - 4\kappa^2}$ and $\theta = \frac{1}{2\kappa^2}$, respectively, and for $0 < \kappa < \frac{1}{2}$, since $1 - \frac{1}{2\kappa} < 0$, the largest $\gamma(\kappa)$ is 0. Therefore we have

$$\gamma(\kappa) = \begin{cases} 0, & 0 < \kappa < \dfrac{1}{2} \\ 2 - \dfrac{1}{\kappa}, & \dfrac{1}{2} \le \kappa \le 1 \\ \dfrac{1}{\kappa}, & \kappa > 1. \end{cases} \tag{5}$$

The values of $\gamma(\kappa)$ are plotted in Fig. 1.

General monotonicity results have been obtained in [8]. In that paper it has been shown that the step size coefficient $\gamma(\kappa)$ in (5) is necessary for monotonicity in the maximum norm. It follows that the Shu–Osher form (4) is optimal.

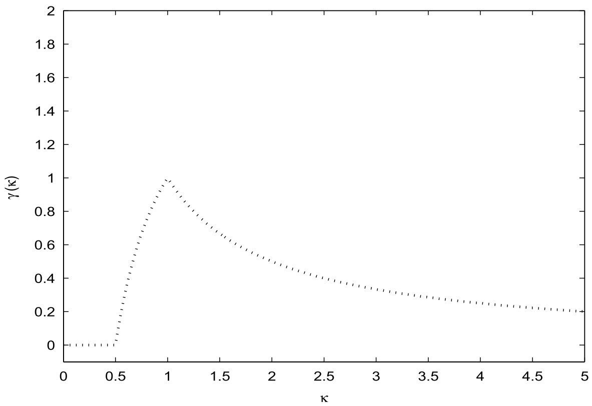

Fig. 1. Step size coefficients for general positivity of (2).

Lemma 1. *Monotonicity with step size coefficient γ implies positivity with the same step size coefficient.*

Proof. As in [9], we consider the following convex function

$$\|V\|_0 = -\min\{0, V_1, \ldots, V_m\}, \quad \text{for } V \in \mathbb{R}^m.$$

For arbitrary V with convex function as defined above, if $\|V\|_0 = 0$, then $V \geq 0$ and conversely. Let $U_0 \geq 0$, then $\|U_0\|_0 = 0$ and under the monotonicity assumption (here, $\|U_n\|_0 \leq \|U_0\|_0$), we have $\|U_n\|_0 = 0$ and then $U_n \geq 0$. □

3. Special results on positivity for 2-stage explicit Runge–Kutta methods

In this section, we obtain the largest step size for explicit RK2 methods for which the corresponding numerical approximations are non-negative (component-wise non-negative) with arbitrary non-negative initial vector. The new results are determined, whenever the underlying ODE possesses the related positivity preserving property. Let us consider

$$U_i' = \frac{q_i\,(U(t))}{\Delta x}\,(U_{i-1}(t) - U_i(t))\,, \quad i = 1, 2 \ldots, m, \tag{6}$$

with the nonlinear function $q_i(U)$ satisfying

$$q_i(U) \geq 0 \quad \text{for any vector } U, \tag{7}$$

and $\Delta x = \frac{1}{m}$, $U = [U_1, U_2, \ldots, U_m]^T$, $U_0 = U_m$. This special semi-discrete system arises from a linear advection problem after discretization using a flux limiter.

Lemma 2. *Assuming (7) and Lipschitz continuity for q_i in (6) with respect to U, this nonlinear system is positive.*

Proof. To demonstrate that (6) is positive, we use the forward Euler method

$$U_i^{n+1} = U_i^n + \Delta t\frac{q_i(U_n)}{\Delta x}(U_{i-1}^n - U_i^n) = \left(1 - \Delta t\frac{q_i(U_n)}{\Delta x}\right)U_i^n + \Delta t\frac{q_i(U_n)}{\Delta x}U_{i-1}^n, \tag{8}$$

where $U_i^n \approx U(x_i, t_n)$ is the *fully discrete* approximation with step size Δt for the time levels $t_n = n\Delta t,\ n = 1, 2, \ldots$ and $U_n = [U_1^n, U_2^n, \ldots, U_m^n]^T \in \mathbb{R}^m$. By spatial periodicity we have $U_{i\pm m}^n = U_i^n$ and $q_{i\pm m} = q_i$. In view of (7) we find that the forward Euler method (8) is positive if

$$\Delta t\frac{q_i(U)}{\Delta x} \leq 1 \quad \text{for all } U \in \mathbb{R}^m \text{ and } i = 1, 2, \ldots, m. \tag{9}$$

By assuming Lipschitz continuity and letting $\Delta t \to 0$, we can conclude that the semi-discrete system (6) is positive. □

In the following we assume that there is a maximal step size $\tau_0 > 0$ under which positivity holds for the forward Euler method,

$$U + \Delta t\frac{q_i(U)}{\Delta x}\,(U_{i-1} - U_i) \geq 0 \quad \text{for all } 0 < \Delta t \leq \tau_0,\ U \geq 0, \tag{10}$$

and we shall determine $\gamma(\kappa)$ such that the positivity is valid for (2) under the step size restriction $\Delta t \le \gamma(\kappa)\tau_0$. Application of (2) to (6) with $\nu_i^l = \Delta t \frac{q_i(U_l)}{\Delta x}$ and $l = n_1, n_2$, gives

$$(U_{n_2})_i = U_i^n + \kappa\Delta t \frac{q_i(U_{n_1})}{\Delta x}(U_{i-1}^n - U_i^n) = U_i^n + \kappa\nu_i^{n_1}(U_{i-1}^n - U_i^n) \quad i = 1, 2, \dots, m,$$

where $U_i^n \approx U(x_i, t_n)$ as mentioned above is the fully discrete approximation. Therefore, we have

$$U_i^{n+1} = U_i^n + \nu_i^{n_1}\left(1 - \frac{1}{2\kappa}\right)(U_{i-1}^n - U_i^n) + \frac{1}{2\kappa}\nu_i^{n_2}\left(U_{i-1}^n + \kappa\nu_{i-1}^{n_1}(U_{i-2}^n - U_{i-1}^n) - U_i^n - \kappa\nu_i^{n_1}(U_{i-1}^n - U_i^n)\right),$$

and by rearranging

$$\begin{aligned} U_i^{n+1} = {} & \frac{1}{2}\nu_{i-1}^{n_1}\nu_i^{n_2}U_{i-2}^n + \left(\left(1 - \frac{1}{2\kappa}\right)\nu_i^{n_1} + \frac{1}{2\kappa}\nu_i^{n_2} - \frac{1}{2}\nu_i^{n_2}\nu_{i-1}^{n_1} - \frac{1}{2}\nu_i^{n_2}\nu_i^{n_1}\right)U_{i-1}^n \\ & + \left(1 - \left(1 - \frac{1}{2\kappa}\right)\nu_i^{n_1} - \frac{1}{2\kappa}\nu_i^{n_2} + \frac{1}{2}\nu_i^{n_2}\nu_i^{n_1}\right)U_i^n. \end{aligned} \tag{11}$$

Theorem 1. *Sufficient for scheme (2) applied to (6), to be positive is*

$$0 \le \Delta t \frac{q_i(U)}{\Delta x} \le \gamma(\kappa), \quad \gamma(\kappa) = \begin{cases} 0, & 0 < \kappa < \dfrac{1}{2} \\ 1, & \dfrac{1}{2} \le \kappa \le 1 \\ \dfrac{1}{\kappa}, & \kappa > 1, \end{cases} \tag{12}$$

for all $U \in \mathbb{R}^m$ and $i = 1, 2, \dots, m$.

Proof. From (11) it is enough to show that

$$\begin{aligned} P &= \left(1 - \frac{1}{2\kappa}\right)x_1 + \frac{1}{2\kappa}x_2 - \frac{1}{2}x_2x_3 - \frac{1}{2}x_2x_1 \ge 0, \\ Q &= 1 - \left(1 - \frac{1}{2\kappa}\right)x_1 - \frac{1}{2\kappa}x_2 + \frac{1}{2}x_2x_1 \ge 0, \end{aligned} \tag{13}$$

where $x_1 = \nu_i^{n_1}$, $x_2 = \nu_i^{n_2}$ and $x_3 = \nu_{i-1}^{n_1}$.

(i) First, let us suppose that $\frac{1}{2} \le \kappa \le 1$. Considering P and Q as functions of 3 and 2 variables, respectively, and parameter κ, our goal is to find the global minimum of these two functions in the cube $0 \le x_1 \le 1$, $0 \le x_2 \le 1$ and $0 \le x_3 \le 1$ for P, and the square $0 \le x_1 \le 1$, $0 \le x_2 \le 1$ for Q. Since the functions P and Q in (13) are algebraic, to find critical points, we set the partial derivatives equal to 0 and solved for variables. It can be shown that there is no interior critical point and the global minimum occurs only at corner points ($(0, 0, 1)$ and $(1, 1, 1)$ for P and $(0, 1)$ for Q) with respect to parameter κ. After evaluation functions P and Q, one can easily find that the global minimum is 0 and, therefore this concludes the sufficiency of $\gamma(\kappa) = 1$ for P and Q to be non-negative.

(ii) Next, let $\kappa > 1$. Under this assumption, we have $1 - \frac{1}{2\kappa} > \frac{1}{2}$. In order to show sufficiency of $\gamma(\kappa) = \frac{1}{\kappa}$ for (13), we write

$$P = \left(1 - \frac{1}{2\kappa} - \frac{1}{2}x_2\right)x_1 + \frac{1}{2}\left(\frac{1}{\kappa} - x_3\right)x_2.$$

Now, if $\gamma(\kappa) = \frac{1}{\kappa}$, since $0 \le x_1, x_2, x_3 \le \gamma(\kappa) = \frac{1}{\kappa} < 1$, we derive

$$\frac{1}{\kappa} - x_3 \ge 0 \quad \text{and} \quad 1 - \frac{1}{2\kappa} - \frac{1}{2}x_2 > \frac{1}{2} - \frac{1}{2}x_2 > 0,$$

therefore, it can be easily seen that $P \ge 0$. In order to showing $Q \ge 0$, we write

$$\begin{aligned} &0 \le \left(1 - \frac{1}{2\kappa}\right)x_1 < 1 - \frac{1}{2k}, \\ &0 \le \frac{1}{2\kappa}x_2 < \frac{1}{2\kappa}, \end{aligned}$$

on the other hand, we have

$$0 < 1 - \left(1 - \frac{1}{2\kappa}\right)x_1 - \frac{1}{2\kappa}x_2 \le 1,$$

and, since $\frac{1}{2}x_2x_1 \ge 0$, we have $Q \ge 0$. This completes the proof. □

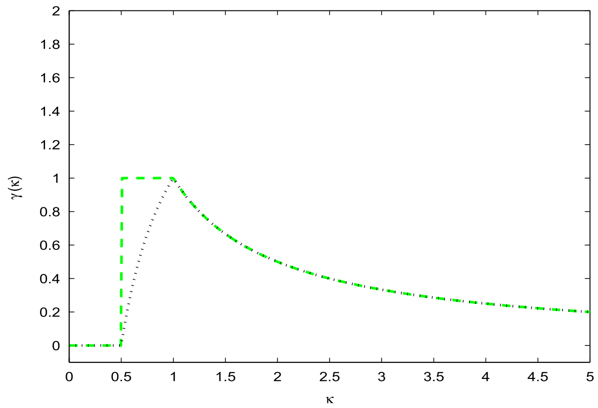

Fig. 2. Step size coefficients for positivity of RK2 (green) when applied to (6) compared with the classical results (black-dotted). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

With step size coefficients $\gamma(\kappa)$ obtained in (12), it follows that the method (2) is positive for (6) satisfying (10) under the step size restriction

$$\Delta t \leq \gamma(\kappa)\tau_0. \tag{14}$$

The result is plotted in Fig. 2. We see that the explicit midpoint rule ($\kappa = \frac{1}{2}$) and explicit trapezoidal method ($\kappa = 1$) allow the same time step with respect to positivity. Also, comparing with the classical results, the RK2 methods allow larger time steps whenever $\frac{1}{2} < \kappa < 1$.

4. Numerical experiments

In this section, first we have considered the scalar linear advection equation in one dimension $U_t + U_x = 0$ with $t > 0$, $0 < x < 1$ and a periodic boundary condition. We have discretized in space on uniformly distributed grid points $x_i = i\Delta x$, and $\Delta x = \frac{1}{500}$ by means of the flux form

$$U_i'(t) = \frac{1}{\Delta x}\left(F_{i-\frac{1}{2}}(t, U(t)) - F_{i+\frac{1}{2}}(t, U(t))\right), \qquad F_{i\pm\frac{1}{2}}(t, U) = U_{i\pm\frac{1}{2}} \quad i = 1, 2, \dots, 500, \tag{15}$$

where the values $U_{i\pm 1/2}$ are defined at the cell boundaries $x_{i\pm 1/2}$. With the third-order upwind-biased flux we have

$$F_{i+\frac{1}{2}}(t, U) = \frac{1}{6}\left(-U_{i-1} + 5U_i + 2U_{i+1}\right) = \left(U_i + \left(\frac{1}{3} + \frac{1}{6}\theta_i\right)(U_{i+1} - U_i)\right),$$

where θ_i is the ratio

$$\theta_i = \frac{U_i - U_{i-1}}{U_{i+1} - U_i} \quad i = 1, 2, \dots, 500.$$

The general discretization (14) written out in full gives

$$U_i' = \frac{1}{\Delta x}\left(1 - \psi(\theta_{i-1}) + \frac{1}{\theta_i}\psi(\theta_i)\right)(U_{i-1} - U_i) \quad i = 1, 2, \dots, 500,$$

with the *limiter function* ψ, here

$$\psi(\theta) = \max\left(0, \min\left(1, \frac{1}{3} + \frac{1}{6}\theta, \theta\right)\right).$$

This limiter function was introduced by Koren [11]. For details see [1, p. 216].

Table 1 gives some numerical solutions for the four schemes ($\kappa = 0.5, 0.75, 1, 2$) with fixed time step sizes Δt and two initial profiles, viz. the peaked function $U_0(x, t) = \sin^{100}(\pi x)$ and the block function $U_0(x, t) = 1$ for $0.3 \leq x \leq 0.7$ and 0 otherwise. Our final time is $t_f = 1$. For each of these schemes, it has used the number steps $N = 400, 450, 500, 550, 600$ and this leads to values of $\Delta t \approx 0.0025, 0.0022, 0.0020, 0.0018, 0.0016$ and the Courant (CFL) numbers $\nu = \frac{\Delta t}{\Delta x} = 500/N \approx 1.25, 1.11, 1.00, 0.90, 0.83$. Furthermore, in order to characterize positivity, the value of the smallest component of the solutions is given. The corresponding biggest component of the solutions shows that the positivity may also imply a maximum principle ($\min_i U_i^0 \leq U_i^n \leq \max_i U_i^0$ for all $n \geq 1$). Practical experience indicates that the smallest number N is needed to achieve positivity with the peaked function, for these four schemes, is equal to 500. In the case of block function, we see little difference for these methods with respect to positivity. With he block function these schemes, are

Table 1
Results for the scalar linear advection. N denotes the number of time steps.

κ	N	1D advection with smooth profile		1D advection with non-smooth profile	
		$\min_{i,n}(U_i^n)$	$\max_{i,n}(U_i^n)$	$\min_{i,n}(U_i^n)$	$\max_{i,n}(U_i^n)$
	400	−1.35e+081	1.35e+081	−1.79e+082	1.79e+082
	450	−2.06e+040	2.06e+040	−4.94e+040	4.94e+040
0.5	500	1.33e−203	1.00	−8.39e−037	1+3e−013
	550	1.10e−203	1.00	0.00	1+2e−014
	600	9.26e−204	9.995e−001	0.00	1.00
	400	−9.18e+080	9.18e+080	−2.02e+082	2.02e+082
	450	−1.79e+040	1.79e+040	−1.30e+041	1.30e+041
0.75	500	1.30e−203	1.00	−3.06e−030	1+3e−012
	550	1.10e−203	1.00	−1.35e−037	1+1e−012
	600	9.26e−204	9.993e−001	0.00	1.00
	400	−6.80e+080	6.80e+080	−2.47e+082	2.47e+082
	450	−1.28e+040	1.28e+040	−1.33e+041	1.33e+041
1	500	1.33e−203	1.00	−9.46e−037	1+3e−014
	550	1.10e−203	1.00	0.00	1+5e−014
	600	9.26e−204	9.991e−001	0.00	1.00
	400	−1.44e+080	1.44e+080	−2.47e+082	2.47e+082
	450	−6.23e+038	6.23e+038	−1.33e+041	1.33e+041
2	500	1.33e−203	1.00	−2.69e−030	1+2e−014
	550	1.10e−203	1.00	0.00	1.00
	600	9.26e−204	9.990e−001	0.00	1.00

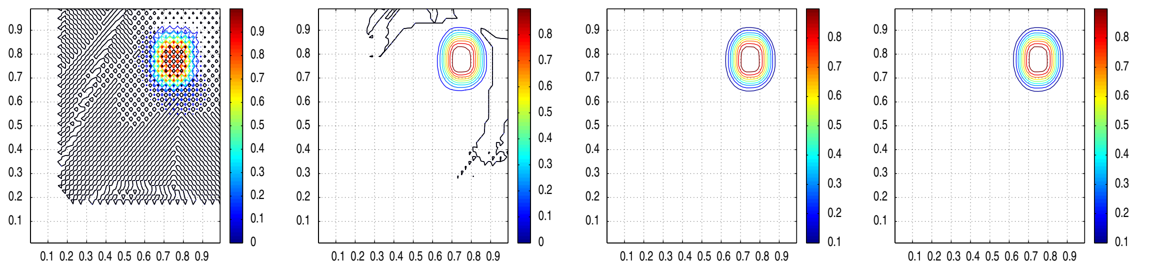

Fig. 3. Advection for the cylinder profile on a 50 × 50 grid. From left, solutions for the RK2 ($\kappa = 0.5$), time stepping with 25, 35, 50, 80 time steps, respectively. Corresponding Courant numbers are 1, 0.7, 0.5, 0.3. Contour lines at levels 0.1, 0.2, . . . , 0.9.

free from negative values for $N > 550$. Considering, an approximate solution positive if the smallest component is greater than -10^{-25}, the trapezoidal method ($\kappa = 1$) and the explicit midpoint rule ($\kappa = 0.5$) perform well up to CFL numbers = 1 but their results quickly deteriorates when applied with larger and larger CFL numbers. Therefore, in practical sense these two methods are equally efficient with regard to positivity. With other RK2 methods ($\frac{1}{2} < \kappa < 1$) a similar behavior was observed. The necessity of the step size restriction (14) was experimentally studied for several RK2 methods for $\kappa > 1$ and $\kappa < \frac{1}{2}$. In general it was found that (14) is somewhat strict. For more details, see Table 1.

Next, we apply the RK2 methods to the 2D problem for $\kappa = 0.5$ and $\kappa = 1$. Consider the model problem $U_t + a_1 U_x + a_2 U_y = 0$ on the unit square with constant $a_1, a_2 = 1$. The initial profile is a cylinder with height 1, centered at (0.25, 0.25) with radius 0.1. Our final time is $t_f = 0.5$, and at the inflow boundaries, homogeneous Dirichlet conditions are imposed. Also here, in two spatial directions, we use the third-order upwind-biased discretization with mentioned above limiter. Then the semi-discrete system can be written as

$$U'_{ij}(t) = \alpha_{ij}(U(t))\left(U_{i-1,j}(t) - U_{ij}(t)\right) + \beta_{ij}(U(t))\left(U_{i,j-1}(t) - U_{ij}(t)\right),$$

with nonlinear functions α_{ij}, β_{ij} satisfying

$$0 \le \alpha_{ij}(U) \le \frac{2}{\Delta x}, \qquad 0 \le \beta_{ij}(U) \le \frac{2}{\Delta y},$$

where Δx and Δy being the mesh width in the x-direction and y-direction, respectively. For more details see [1, p. 307]. In Figs. 3 and 4, some numerical results have been shown on a 50×50 grid for the RK2 methods with $\kappa = 0.5$ and 1, respectively. The pictures for both methods are self-evident. For the solution the qualitative behavior and temporal accuracy is good with both methods for the CFL numbers ≤ 0.5. Furthermore, we found that there are no global undershoots or overshoots for these RK2 methods. Our final time is taken as $t_f = 0.5$.

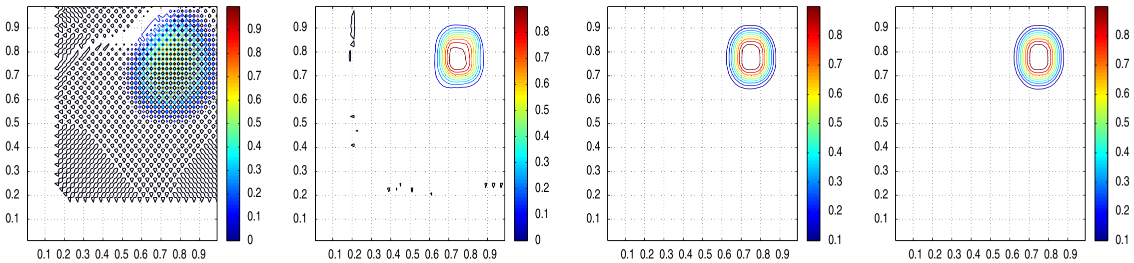

Fig. 4. Advection for the cylinder profile on a 50×50 grid. From left, solutions for the RK2 ($\kappa = 1$), time stepping with 25, 35, 50, 80 time steps, respectively. Corresponding Courant numbers are 1, 0.7, 0.5, 0.3. Contour lines at levels 0.1, 0.2, . . . , 0.9.

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