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# A posteriori error estimation of residual type for anisotropic diffusion–convection–reaction problems

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ABSTRACT

This paper presents a robust a posteriori residual error estimator for diffusion–convection–reaction problems with anisotropic diffusion, approximated by a SUPG finite element method on isotropic or anisotropic meshes in $\mathbb{R}^d$, $d = 2$ or 3. The equivalence between the energy norm of the error and the residual error estimator is proved. Numerical tests confirm the theoretical results.



## 1. Introduction

This paper is devoted to the singularly perturbed diffusion–convection–reaction problem with a special focus on anisotropic diffusion: for $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, let $u$ be the solution of

$$\begin{cases} -\operatorname{div}(A\nabla u) + \mathbf{b}\cdot\nabla u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ A\nabla u \cdot n = g & \text{on } \Gamma_N, \end{cases} \tag{1}$$

where the matrix $A$ and the functions $\mathbf{b}$ and $c$ satisfy assumptions (A1)–(A6) below, and $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is a bounded domain with a polygonal ($d = 2$) or polyhedral ($d = 3$) boundary $\Gamma$. This boundary is divided into two parts $\Gamma_D$ and $\Gamma_N$, where Dirichlet and Neumann boundary conditions are imposed, respectively.

We are particularly interested in the case when $A$ becomes small in some direction, for instance the cases

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \quad (d = 2), \quad \text{or} \quad A = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (d = 3),$$

$\varepsilon > 0$. In the case when $\varepsilon$ is small with respect to $\mathbf{b}$ and $c$, the problem is singularly perturbed and the solution may generate sharp boundary or interior layers, where the solution of the limit problem (corresponding to $\varepsilon = 0$) is not smooth or does

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not satisfy the boundary condition. Let us quote [1–3] for the a priori error analysis in two dimensions. It is shown that anisotropic finite elements must be used in order to achieve convergence uniformly in the perturbation parameter $\varepsilon$.

There is a vast amount of literature on a posteriori error estimation. For singularly perturbed problems with convection we cite [4–12], where anisotropic finite element meshes were considered in [5,7,8] only. An anisotropic diffusion tensor is considered only in [13].

In this paper we combine all those ingredients and derive a residual type error estimator. We prove the reliability and efficiency of this error estimator where the involved constants are independent of the coefficients of the operator, namely $A$, $\mathbf{b}$ and $c$. The lower bound is, as usual, mainly based on inverse inequalities and integration by parts, but the efficiency is achieved independently of the coefficients of the operator. The reliability is based on the introduction of an *alignment measure* as it was done in [14,15]. This quantity is of the order one if the mesh is well-adapted to the problem, see the discussion in Section 3.3.

Let us mention that, to our knowledge, no approach is known that leads to two-sided estimates on anisotropic meshes without any assumption on the mesh. The classical results as summarized in [16,17] are obtained for isotropic meshes only. The dual weighted residual method, see [18] for an overview, is applied in [5,19] on anisotropic meshes, but there is no estimate from below. The more recent approach in [20] is not yet analyzed for anisotropic meshes and two different error estimators are used for the upper and lower bounds. Let us finally mention the approach by Picasso [8] who considers anisotropic meshes and proves reliability for an estimator that depends on $\nabla(u-u_h)$ where $\nabla u$ is replaced in practice by a recovered gradient $\nabla^R u$. We note that we can control the alignment measure in the same way.

In this paper we develop an estimator of residual type for problems with convection, reaction and anisotropic diffusion. For the discretization we use the $h$-version of the streamline upwind Petrov–Galerkin method (SUPG). Without the stabilization term, the method reduces to a standard Galerkin method and produces non-physical oscillations. We note that our error estimator works as well in this case.

In comparison with the paper [13], where a posteriori error estimation is investigated for an isotropic discretization of a problem with anisotropic diffusion but without convection, our residual error estimator allows one to prove an optimal lower bound. The factor $\varepsilon^{-1/2}$ in the upper bound in [13] is retained in our analysis, since the alignment measure is of the same order in the isotropic case. Our experiments show, however, that the effectivity index is bounded uniformly in $\varepsilon$ on adequately refined anisotropic meshes. In this sense, our analysis is sharper.

The outline of the paper is as follows. In Sections 2 and 3 we introduce the discretization, notation, and estimates for bubble functions and the interpolation. The a posteriori error estimator is introduced in Section 4 where also the the upper and lower bounds are proved. The paper is completed with a numerical test in Section 5 and with conclusions.

As usual, we denote by $L^2(.)$ the Lebesgue spaces and by $H^s(.)$, $s \geq 0$, the standard Sobolev spaces. The usual norm and seminorm of $H^s(D)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$. For the sake of brevity the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$ and in the case $D = \Omega$, we will drop the index $\Omega$. The space $H^1_{\Gamma_D}(\Omega)$ is defined, as usual, by $H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) : v = 0$ on $\Gamma\}$. In the sequel the symbol $|\cdot|$ will denote either the Euclidean norm in $\mathbb{R}^d$, $d = 2$ or 3, or the length of a line segment, or the measure of a domain of $\mathbb{R}^d$. Finally the notation $a \lesssim b$ means here and below that there exists a positive constant $C$ independent of $a$ and $b$ (of the mesh size of the triangulation, as well as the diffusion matrix $A$, the convection function $\mathbf{b}$ and the reaction term $c$) such that $a \leq C\, b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

## 2. Discretization of the diffusion–convection–reaction equation

We consider the standard elliptic problem: for $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, let $u$ be the solution of (1) where $A$, $\mathbf{b}$ and $c$ satisfy the following assumptions:

(A1) $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$,
(A2) $\exists c_0 \geq 0 : c - \frac{1}{2}\operatorname{div}\mathbf{b} \geq c_0$ and if $c_0 = 0$ then $c \equiv 0$,
(A3) $\mathbf{b} \cdot n \geq 0$ on $\Gamma_N$,
(A4) $A \in \mathbb{R}^{d\times d}$ is symmetric,
(A5) $\exists \alpha_0 > 0 : A\xi \cdot \xi \geq \alpha_0,\ \forall \xi \in \mathbb{R}^d$.
(A6) $\Gamma_D \neq \emptyset$ if $c_0 = 0$.

Note that the assumption "if $c_0 = 0$ then $c \equiv 0$" is not necessary for our proofs but simplifies the presentation.

Now we define the weighted $H^1$ (semi-)norm

$$|||u|||^2_\omega := \int_\omega (|A^{1/2}\nabla u|^2 + c_0|u|^2) \tag{2}$$

on a subdomain $\omega$ of $\Omega$. Let us further introduce the space

$$H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$$

and the forms

$$B(u, v) = \int_\Omega (A\nabla u \cdot \nabla v + \mathbf{b} \cdot \nabla u v + cuv)\mathrm{d}x,$$

$$F(v) = \int_\Omega f v \,\mathrm{d}x + \int_{\Gamma_N} g v \mathrm{d}\Gamma(x).$$

For further purposes, we denote by $B_\omega$ the restriction of $B$ on a subset $\omega$ of $\Omega$, namely

$$B_\omega(u, v) = \int_\omega (A\nabla u \cdot \nabla v + \mathbf{b} \cdot \nabla uv + cuv)\mathrm{d}x.$$

With this notation, the variational formulation of problem (1) reads: Find $u \in H^1_{\Gamma_D}(\Omega)$ such that

$$B(u, v) = F(v) \quad \forall v \in H^1_{\Gamma_D}(\Omega). \tag{3}$$

In order to obtain a robust lower bound, as in [12], we need to use the dual norm of the convective derivative. Hence for $\phi \in L^2(\Omega)$, let us denote by $|||\phi|||_*$ its norm as a element of the dual of $H^1_{\Gamma_D}(\Omega)$, namely

$$|||\phi|||_* = \sup_{v \in H^1_{\Gamma_D}(\Omega)\setminus\{0\}} \frac{\int_\Omega \phi v}{|||v|||}. \tag{4}$$

Note that the assumption $\phi \in L^2(\Omega)$ guarantees that there exists at least one $v_1 \in H^1_{\Gamma_D}(\Omega)$ such that

$$|||v_1||| \le 1 \quad \text{and} \quad |||\phi|||_* = \int_\Omega \phi v_1. \tag{5}$$

The assumptions (A1)–(A6) guarantee that $B$ is continuous and coercive, i.e., $B$ satisfies

$$B(v, v) \ge |||v|||^2 \quad \forall v \in H^1_{\Gamma_D}(\Omega), \tag{6}$$

$$|B(u, v)| \le (\kappa |||u||| + |||\mathbf{b} \cdot \nabla u|||_*)|||v||| \quad \forall u, v \in H^1_{\Gamma_D}(\Omega), \tag{7}$$

where

$$\kappa = \max\{1, c_0^{-1}\|c\|_{\infty,\Omega}\}. \tag{8}$$

If $c_0 = 0$, then the term $c_0^{-1}\|c\|_{\infty,\Omega}$ disappears and $\kappa$ is equal to one. By the Lax–Milgram lemma, problem (3) has a unique solution $u \in H^1_{\Gamma_D}(\Omega)$.

Note that the case $\operatorname{div} \mathbf{b} = 0$, $c = 0$, i.e. $c_0 = 0$, is admitted. It is excluded in several other publications.

To approximate problem (3) by a finite element scheme we fix a family $\{T_h\}_{h>0}$ of meshes of $\Omega$ that satisfies the usual conformity conditions, cf. [21, Chapter 2]. In 2D we assume that all elements of $T_h$ are triangles, while in 3D the mesh is made up of tetrahedra. For $T \in T_h$ we denote by $h_T$ the diameter of $T$, and $h = \max_{T \in T_h} h_T$.

Let $V_h$ be the subspace of $H^1_{\Gamma_D}(\Omega)$ defined by

$$V_h = \{v_h \in H^1_{\Gamma_D}(\Omega) : v_{h|T} \in \mathbb{P}^k(T)\ \forall T \in T_h\},$$

where $k$ is a positive integer.

Problem (3) is now approximated by a Streamline Upwind Petrov Galerkin scheme (SUPG): Find $u_h \in V_h$ such that

$$B_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h, \tag{9}$$

where

$$B_h(u_h, v_h) = B(u_h, v_h) + \sum_{T \in T_h} \delta_T(-\operatorname{div}(A\nabla u_h) + \mathbf{b} \cdot \nabla u_h + cu_h, \mathbf{b} \cdot \nabla v_h)_T,$$

$$F_h(v_h) = F(v_h) + \sum_{T \in T_h} \delta_T(f, \mathbf{b} \cdot \nabla v_h)_T.$$

The parameters $\delta_T \ge 0$ should satisfy similar assumptions as in [7] where the case of isotropic diffusion was investigated:

$$\delta_T \le \frac{h_{\min,A,T}}{\|A^{-1/2}\mathbf{b}\|_{\infty,T}}, \tag{10}$$

$$\delta_T \le 2(1-\alpha)\mu^{-2}h^2_{\min,T}\|A\|^{-1}_{2\to 2}, \tag{11}$$

$$\delta_T \le 2(1-\alpha)c_0 \left(\max_{x \in T} c(x)^2\right)^{-1} \quad \text{if } c \not\equiv 0, \tag{12}$$

for all $T \in T_h$ and some $\alpha \in (0, 1)$ (where $\|A\|_{2\to 2}$ means the spectral matrix norm of $A$, induced by the Euclidean vector norm). The element quantities $h_{\min,T}$ and $h_{\min,A,T}$ are introduced below, and $\mu$ is the constant in the inverse inequality $\|\nabla \cdot \nabla v_h\|_T \le \mu h^{-1}_{\min,T}\|\nabla v_h\|_T$. Note further that (11) and (12) guarantee the coercivity $B_h(v_h, v_h) \ge \alpha|||v_h|||^2$ with the above $\alpha \in (0, 1)$, compare [22, Lemma 3.25] for the case of isotropic meshes. The optimal choice of $\delta_T$ was discussed for the slightly different Galerkin-Least-Squares method and for the case of isotropic diffusion in [23]. This choice satisfies the conditions (10)–(12). We note also that the choice $\delta_T = 0$ (pure Galerkin method) satisfies these conditions. Meanwhile it is well-known that this choice is suited within boundary layers if adequately refined anisotropic meshes are used there [22, p. 391 ff.]. Outside the layers, the choice $\delta_T = 0$ leads in general to non-physical oscillations. Therefore this choice is not advisable, but the error estimator still works and estimates the large error.

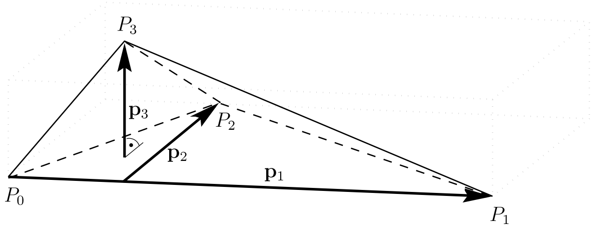


**Fig. 1.** Notation of tetrahedron $T$.

## 3. Analytical tools

Let us define $E_h$ as the set of edges ($d = 2$) or faces ($d = 3$) of the triangulation and let $E_h^{\text{int}} = \{E \in E_h / E \subset \Omega\}$ be the set of interior edges/faces of $T_h$, while $E_h^{\text{ext}} = E_h \setminus E_h^{\text{int}}$ is the set of boundary edges/faces of $T_h$.

For an edge/face $E$ of a 2D/3D element $T$ we denote by $n_{T,E}$ the unit outward normal vector to $T$ along $E$. Furthermore we fix one of the two normal vectors of $E$ and denote it by $n_E$. The jump of some function $v$ across an edge/face $E$ at a point $y \in E$ is defined as

$$[\![ v(y) ]\!]_E := \begin{cases} \lim\limits_{\alpha \to +0} v(y + \alpha n_E) - v(y - \alpha n_E) & \forall E \in E_h^{\text{int}}, \\ v(y) & \forall E \in E_h^{\text{ext}}. \end{cases}$$

Finally we will need local subdomains, also called patches. For any $T \in T_h$, let, as usual, $\omega_T$ be the union of all elements having a common vertex with $T$. Similarly let $\omega_E$ be the union of the elements having $E$ as edge/face.

### 3.1. Some anisotropic quantities

As explained in the introduction, anisotropic discretizations can be very advantageous or, in certain situations, even mandatory. More information and arguments concerning anisotropy can be found in [24,14].

Let us shortly recall some useful anisotropic quantities from Kunert [14], see also [25,7]. We start with an arbitrary (anisotropic) tetrahedron $T$ and enumerate its vertices so that $P_0P_1$ is the longest edge, $\text{meas}_2(\triangle P_0P_1P_2) \ge \text{meas}_2(\triangle P_0P_1P_3)$, and $\text{meas}_1(P_1P_2) \ge \text{meas}_1(P_0P_2)$. Further, we introduce three orthogonal vectors $\mathbf{p}_{i,T}$ of length $h_{i,T} := |\mathbf{p}_{i,T}|$, as described in Fig. 1.

The minimal element size is particularly important; thus we define

$$h_{\min,T} := h_{3,T}.$$

The three main anisotropic directions $\mathbf{p}_{i,T}$ play an important role in several proofs. They span the matrix

$$C_T := (\mathbf{p}_{1,T}, \mathbf{p}_{2,T}, \mathbf{p}_{3,T}) \in \mathbb{R}^{3\times 3}.$$

This matrix may be considered as a transformation matrix which defines implicitly a *reference element* $\hat{T}_T$ via

$$\hat{T}_T := C_T^{-1}(T - \vec{P}_0),$$

cf. Fig. 2. Note in particular that the reference element $\hat{T}_T$ is of size $\mathcal{O}(1)$.

In 2D the notation is similar. For a triangle $T$ the enumeration is as in the bottom triangle $P_0P_1P_2$ of Fig. 1. We set $h_{\min,T} := h_{2,T}$, and $C_T$ becomes a $2 \times 2$ matrix.

The new idea is now to transform any $T \in T_h$ by the matrix $A^{-1/2}$. More precisely, we transform $T$ into $T_A$ by the affine transformation

$$F_{A,T} : T \to T_A : x \to A^{-1/2}(x - g_T) + g_T, \tag{13}$$

where $g_T$ is the center of gravity of $T$. This element $T_A$ is a triangle in 2D or a tetrahedron in 3D that can be isotropic or not. Therefore we use its anisotropic quantities $h_{i,T_A}$, $h_{\min,T_A}$, $C_{T_A}$ as introduced before.

**Remark 3.1.** In 2D, take

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix},$$

$c = 1, \mathbf{b} = 0$ and $\Omega = (0, 1)^2$, $\Gamma_N = \emptyset$. If $f$ is smooth then $u$ has boundary layers near $x_1 = 0$ and $x_1 = 1$. The transformation

$$\Omega \to \tilde{\Omega} : x \to A^{-1/2}x,$$

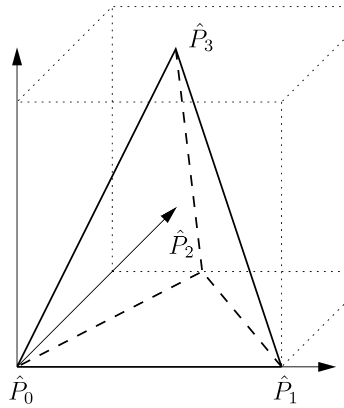


**Fig. 2.** Reference tetrahedron $\hat{T}_T$.

replaces the problem (1) into

$$-\Delta\tilde{u} + \tilde{u} = \tilde{f} \quad \text{in } \tilde{\Omega} = \left(0, \frac{1}{\sqrt{\varepsilon}}\right) \times (0, 1),$$
$$\tilde{u} = 0 \quad \text{on } \partial\tilde{\Omega}.$$

Take for simplicity a uniform triangular triangulation of $\tilde{\Omega}$. Then the triangle $\tilde{T} \in \tilde{\Omega}$ with nodes $(0, 0)$, $(h, 0)$, and $(0, h)$ becomes the triangle $T \in \Omega$ with vertices $(0, 0)$, $(\sqrt{\varepsilon}h, 0)$, and $(0, h)$ by the inverse transformation. This element $T$ is a good one to capture adequately the boundary layer near $x_1 = 0$. Moreover by using $F_{A,T}$, the triangle $T$ is transformed into an isotropic element $T_A$ which is a translation of $\tilde{T}$, and therefore $C_{\tilde{T}} = h^{-1}Id$. $\square$

For further use, we denote

$$h_{\min,A,T} = \min_{T' \subset \omega_T} h_{\min,T'_A}. \tag{14}$$

Note that the composition of the transformation

$$\hat{T}_{T_A} \to T_A : \hat{x} \to C_{T_A}\hat{x} + P_0,$$

with $F_{A,T}^{-1}$, see (13), yields the following transformation from $\hat{T}_{T_A}$ to $T$:

$$\hat{T}_{T_A} \to T : \hat{x} \to C_{A,T}\hat{x} + b_T \tag{15}$$

with

$$C_{A,T} = A^{1/2}C_{T_A}. \tag{16}$$

Note that $\hat{T}_{T_A}$ depends on $T$ and $A$ but is of unit size in the sense of Fig. 2.

Finally we introduce a scaling factor $\alpha_T$ that will be used quite often:

$$\alpha_T = \min\{c_0^{-1/2}, h_{\min,A,T}\}. \tag{17}$$

Here and below, we use the convention that $c_0^{-1/2} = +\infty$ if $c_0 = 0$.

For an edge/face $E$ of an element $T$ introduce the height $h_{E,T} = \frac{|T|}{|E|}$.

We finally require, as usual, that

$$|T| \sim |T'| \quad \text{if } T \cap T' \neq \emptyset, \tag{18}$$

the number of elements containing a vertex $x$ is bounded uniformly. (19)

**Remark 3.2.** If $A = \varepsilon Id$ (the case treated by Kunert in [25,7]), then $T_{A,T}$ is simply a homothetic transformation of $T$ with a factor $\varepsilon^{-1/2}$. Therefore the matrix $C_{\varepsilon Id,T}$ defined by (16) is equal to $C_T$. Moreover we have

$$h_{\min,\varepsilon Id,T} = \varepsilon^{-1/2}h_{\min,T}.$$

This last property implies that

$$\alpha_T = \min\{c_0^{-1/2}, \varepsilon^{-1/2}h_{\min,T}\},$$

which is exactly the scaling factor introduced in [7].

### 3.2. Bubble functions, extension operator, and inverse inequalities

For our further analysis we require standard bubble functions and extension operators that satisfy certain properties recalled here for the sake of completeness.

We need two types of bubble functions, namely $b_T$ and $b_E$ associated with an element $T$ and an edge $E$, respectively. For a triangle or a tetrahedron $T$, denoting by $\lambda_{a_i^T}$, $i = 1, \dots, d+1$, the barycentric coordinates of $T$ and by $a_i^E$, $i = 1, \dots, d$, the vertices of the edge/face $E \subset \partial T$ we recall that

$$b_T = \prod_{i=1}^{d+1} \lambda_{a_i^T} \quad \text{and} \quad b_E = \prod_{i=1}^{d} \lambda_{a_i^E}.$$

We note that

$$b_T = 0 \quad \text{on } \partial T, \qquad b_E = 0 \quad \text{on } \partial\omega_E, \qquad \|b_T\|_{\infty,T} = \|b_E\|_{\infty,\omega_E} \sim 1.$$

In 2D, denote by $\bar{T}$ the standard reference element with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. For an edge $\bar{E}$ of $\bar{T}$ included into the $\bar{x}_1$ axis, the extension $\mathrm{F}_{\mathrm{ext}}(v_{\bar{E}})$ of $v_{\bar{E}} \in C(\bar{E})$ to $\bar{T}$ is defined by $\mathrm{F}_{\mathrm{ext}}(v_{\bar{E}})(\bar{x}_1, \bar{x}_2) = v_{\bar{E}}(\bar{x}_1)$. The extension operator $\mathrm{F}_{\mathrm{ext}}(v_E)$ of $v_E \in C(E)$ to $T$ for an edge $E \subset \partial T$ is obtained using the affine transformation mapping $T$ to $\bar{T}$ and $E$ to $\bar{E}$ and the extension operator defined above. We proceed similarly in 3D.

Now we may state the so-called inverse inequalities that are proved using classical scaling techniques, cf. [17] for the isotropic case and [14] for the anisotropic case.

**Lemma 3.3** (*Inverse Inequalities*)**.** *Let $v_T \in \mathbb{P}_{k_0}(T)$ and $v_E \in \mathbb{P}_{k_1}(E)$, for some nonnegative integers $k_0$ and $k_1$. Then the following inequalities hold, with the constants in the inequality depending on the polynomial degrees $k_0$ or $k_1$ but not on $T$, $E$ or $v_T$, $v_E$.*

$$\|v_T b_T^{1/2}\|_T \sim \|v_T\|_T, \tag{20}$$

$$\|v_E b_E^{1/2}\|_E \sim \|v_E\|_E, \tag{21}$$

$$|||v_T b_T|||_T \lesssim \alpha_T^{-1} \|v_T\|_T. \tag{22}$$

**Proof.** The equivalences (20) and (21) are proved in [14], see also Lemma 1 of [7]. For the last estimate, we write

$$\begin{aligned} |||v_T b_T|||_T^2 &= c_0 \|v_T b_T\|_T^2 + \|A^{1/2} \nabla(v_T b_T)\|_T^2 \\ &\le c_0 \|v_T\|_T^2 + \|C_{T_A}^{-T}\|^2 \|C_{T_A}^T A^{1/2} \nabla(v_T b_T)\|_T^2. \end{aligned}$$

Now recalling that $\|C_{T_A}^{-T}\| \sim h_{\min,T_A}^{-1}$ and using the affine transformation (15), we obtain

$$|||v_T b_T|||_T^2 \lesssim c_0 \|v_T\|_T^2 + h_{\min,A,T}^{-2} |T| \; \|\hat{\nabla}(\hat{v}_T \hat{b}_T)\|_{\hat{T}}^2.$$

Since $\hat{T}$ is regular in Ciarlet's sense we can use the inverse inequality with $h_{\hat{T}} \sim 1$ to deduce that

$$|||v_T b_T|||_T^2 \lesssim c_0 \|v_T\|_T^2 + h_{\min,A,T}^{-2} |T| \; \|\hat{v}_T\|_{\hat{T}}^2.$$

Going back to $T$ again using the affine transformation (15), we obtain (22). □

As usual for singularly perturbed problems, we need to use squeezed edge/face bubble functions $b_{E,\gamma}$. Here according to our previous point of view, they are defined through the transformation $F_{A,T}$ from (13). Namely for a fixed edge/face $E$ of $T$, the mapping (13) transforms $E$ into an edge/face $E_A$ of $T_A$. Now for a parameter $\gamma \in (0, 1]$, we define the squeezed element $T_{E_A,\gamma}$ of $T_A$ as in [7]. The squeezed element $T_{E,\gamma}$ of $T$ is simply the element obtained by the inverse transformation

$$T_{E,\gamma} = F_{A,T}^{-1} T_{E_A,\gamma}.$$

Note that $T_{E,\gamma}$ is the usual squeezed element on $T$ with the parameter $\gamma$ depending on $A$. For the sake of simplicity we do not write this dependence.

The squeezed edge/face bubble function $b_{E,\gamma}$ is defined on the two elements $T_{1,E,\gamma}$ and $T_{2,E,\gamma}$ sharing $\gamma$, as the usual edge/face bubble function on these elements and extended by zero outside $T_{1,E,\gamma} \cup T_{2,E,\gamma}$.

**Lemma 3.4** (*Further Inverse Inequalities*)**.** *Under the assumptions of Lemma 3.3, we have*

$$\|b_{E,\gamma} \mathrm{F}_{\mathrm{ext}}(v_E)\|_T \lesssim \gamma^{1/2} h_{E,T}^{1/2} \|v_E\|_E, \tag{23}$$

$$\|A^{1/2} \nabla(b_{E,\gamma} \mathrm{F}_{\mathrm{ext}}(v_E))\|_T \lesssim \gamma^{-1/2} h_{E,T}^{1/2} h_{\min,A,T}^{-1} \|v_E\|_E. \tag{24}$$

**Proof.** Scaling arguments yield

$$\|b_{E,\gamma} \mathrm{F}_{\mathrm{ext}}(v_E)\|_T = |T|^{1/2} |T_A|^{-1/2} \|\tilde{b}_{E,\gamma} \tilde{\mathrm{F}}_{\mathrm{ext}}(\tilde{v}_E)\|_{T_A},$$

where we write $\tilde{v}(\tilde{x}) = v(x)$. Now using Lemma 2 of [7] in $T_A$, we have

$$|T_A|^{-1/2}\|\tilde{b}_{E,\gamma}\tilde{\mathrm{F}}_{\mathrm{ext}}(\tilde{v}_E)\|_{T_A} \lesssim \gamma^{1/2}|E_A|^{-1/2}\|\tilde{v}_E\|_{E_A}.$$

Again scaling arguments lead to

$$|E_A|^{-1/2}\|\tilde{v}_E\|_{E_A} \lesssim |E|^{-1/2}\|v_E\|_E. \tag{25}$$

The three above estimates imply (23).

For the second estimate, scaling arguments yield

$$\|A^{1/2}\nabla(b_{E,\gamma}\mathrm{F}_{\mathrm{ext}}(v_E))\|_T = |T|^{1/2}|T_A|^{-1/2}\|\tilde{\nabla}(\tilde{b}_{E,\gamma}\tilde{\mathrm{F}}_{\mathrm{ext}}(\tilde{v}_E))\|_{T_A}.$$

Again Lemma 2 of [7] applied in $T_A$ leads to

$$\|A^{1/2}\nabla(b_{E,\gamma}\mathrm{F}_{\mathrm{ext}}(v_E))\|_T \lesssim |T|^{1/2}\gamma^{1/2}|E_A|^{-1/2}\|\tilde{v}_E\|_{E_A}\min\{\gamma h_{E_A,T_A}, h_{\min,T_A}\}^{-1}.$$

Using the estimate (25), we arrive at

$$\|A^{1/2}\nabla(b_{E,\gamma}\mathrm{F}_{\mathrm{ext}}(v_E))\|_T \lesssim \gamma^{1/2}h_{E,T}^{1/2}\|v_E\|_E\min\{\gamma h_{E_A,T_A}, h_{\min,T_A}\}^{-1}.$$

The estimate (24) will be proved if we can show that

$$\min\{\gamma h_{E_A,T_A}, h_{\min,T_A}\}^{-1} \lesssim \gamma^{-1}h_{\min,A,T}^{-1},$$

or equivalently

$$\min\{\gamma h_{E_A,T_A}, h_{\min,T_A}\} \gtrsim \gamma h_{\min,A,T}. \tag{26}$$

But it was proved in Lemma 3.1 of [25] that

$$h_{E_A,T_A} \gtrsim h_{\min,T_A}.$$

Since $\gamma \in (0, 1]$ we then have

$$\gamma h_{E_A,T_A} \gtrsim \gamma h_{\min,T_A} \geq \gamma h_{\min,A,T} \quad \text{and} \quad h_{\min,T_A} \geq \gamma h_{\min,A,T}.$$

This leads to (26). □

### 3.3. Anisotropic interpolation error estimates

In order to obtain an accurate discrete solution $u_h$, it is obviously helpful to align the elements of the mesh according to the anisotropy of the solution. It turns out that this intuitive alignment is also necessary to prove sharp upper error bounds. In particular the proof employs specific interpolation error estimates. These interpolation estimates hold for isotropic meshes, but do not hold for general anisotropic meshes; instead the mesh has to have the aforementioned anisotropic alignment with the function to be interpolated.

In order to quantify this alignment, we introduce a so-called alignment measure $m_1(v, A, T_h)$ which was originally introduced in [15] for the identity matrix $A$ and that we extend here to any matrix $A$.

**Definition 3.5** (*Alignment Measure*)**.** Let $v \in H^1(\Omega)$, and $\mathcal{T} = \{T_h\}$ be a family of triangulations of $\Omega$. Define the *alignment measure* $m_1 : H^1(\Omega) \times \mathcal{T} \mapsto \mathbb{R}$ by

$$m_1(v, A, T_h) := \left(\sum_{T\in T_h} h_{\min,A,T}^{-2}\|C_{A,T}^{\top}\nabla v\|_T^2\right)^{1/2} \Big/ \|A^{1/2}\nabla v\|. \tag{27}$$

One has $m_1(v, A, T_h) \gtrsim 1$ since

$$\|C_{A,T}^{\top}\nabla v\|_T = \|C_{T_A}^{\top}A^{1/2}\nabla v\|_T \gtrsim h_{\min,A,T}\|A^{1/2}\nabla v\|_T.$$

For arbitrary isotropic meshes one obtains that $m_1(v, Id, T_h) \sim 1$. The same is achieved for anisotropic meshes $T_h$ that are aligned with the anisotropic function $v$. Therefore the alignment measure is not an obstacle for reliable a posteriori error estimation. We refer to [15,26] for discussions concerning this alignment measure.

Now we recall the definition of the Clément interpolation operator that maps a function from $H^1_{\Gamma_D}(\Omega)$ into

$$V_{h,1} := \{v_h \in H^1_{\Gamma_D}(\Omega) : v_{h|T} \in \mathbb{P}^1, \forall T \in T_h\} \subset V_h.$$

For that purpose, let the basis function $\varphi_x \in V_{h,1}$ associated with the node $x$ be determined by the condition

$$\varphi_x(y) = \delta_{x,y} \quad \forall y \in N_h,$$

where $N_h$ is the set of nodes of the triangulation included into $\Omega$ and $\Gamma_N$. Then, the Clément interpolation operator will be defined via these basis functions:

**Definition 3.6** (*Clément Interpolation Operator*)**.** The *Clément interpolation operator* $\mathrm{I}_{\mathrm{Cl}} : H^1_{\Gamma_D}(\Omega) \to V_{h,1}$ is defined by

$$\mathrm{I}_{\mathrm{Cl}} v := \sum_{x \in N_h} \left( \frac{1}{|\omega_x|} \int_{\omega_x} v \right) \varphi_x,$$

with $\omega_x$ being the union of elements $T$ of $T_h$ having $x$ has vertex.

**Lemma 3.7** (*Global Interpolation Error Bounds*)**.** *For each edge/face $E$, let us set*

$$\beta_E = \max_{T \subset \omega_E} (h_{E,T} h^{-1}_{\min,A,T}). \tag{28}$$

*Let $v \in H^1_{\Gamma_D}(\Omega)$, then the following estimates hold:*

$$|||\mathrm{I}_{\mathrm{Cl}} v||| \lesssim m_1(v, A, T_h) |||v|||, \tag{29}$$

$$\sum_{T \in T_h} \alpha_T^{-2} \|v - \mathrm{I}_{\mathrm{Cl}} v\|_T^2 \lesssim m_1(v, A, T_h)^2 |||v|||^2, \tag{30}$$

$$\sum_{T \in T_h} \alpha_T^{-1} \sum_{\substack{E \in \partial T \setminus \Gamma_D: \\ \beta_E = h_{E,T} h^{-1}_{\min,A,T}}} \beta_E \|v - \mathrm{I}_{\mathrm{Cl}} v\|_E^2 \lesssim m_1(v, A, T_h)^2 |||v|||^2. \tag{31}$$

*Note that in (31), every edge/face $E \in E_h$ with $E \not\subset \Gamma_D$ appears in the double sum at least once.*

**Proof.** Lemma 3.1 of [14] says that

$$\|\mathrm{I}_{\mathrm{Cl}} v\| \lesssim \|v\|, \tag{32}$$

$$\|v - \mathrm{I}_{\mathrm{Cl}} v\|_T \lesssim \sum_{T' \subset \omega_T} \|C^{\top}_{A,T'} \nabla v\|_{T'}, \tag{33}$$

$$\|C^{\top}_{A,T} \nabla (v - \mathrm{I}_{\mathrm{Cl}} v)\|_T \lesssim \sum_{T' \subset \omega_T} \|C^{\top}_{A,T'} \nabla v\|_{T'}. \tag{34}$$

Multiplying the estimate (34) by $h^{-1}_{\min,A,T}$ and summing the squares for all $T$ yields

$$\sum_{T \in T_h} h^{-2}_{\min,A,T} \|C^{\top}_{A,T} \nabla (v - \mathrm{I}_{\mathrm{Cl}} v)\|_T^2 \lesssim \sum_{T \in T_h} h^{-2}_{\min,A,T} \|C^{\top}_{A,T} \nabla v\|_T^2.$$

By the definition of the alignment measure, we get

$$\sum_{T \in T_h} h^{-2}_{\min,A,T} \|C^{\top}_{A,T} \nabla (v - \mathrm{I}_{\mathrm{Cl}} v)\|_T^2 \lesssim m_1(v, A, T_h)^2 \|A^{1/2} \nabla v\|^2. \tag{35}$$

In the same manner by (34) and (33) we have

$$\sum_{T \in T_h} h^{-2}_{\min,A,T} \|C^{\top}_{A,T} \nabla \mathrm{I}_{\mathrm{Cl}} v\|_T^2 \lesssim m_1(v, A, T_h)^2 \|A^{1/2} \nabla v\|^2, \tag{36}$$

$$\sum_{T \in T_h} h^{-2}_{\min,A,T} \|(v - \mathrm{I}_{\mathrm{Cl}} v)\|_T^2 \lesssim m_1(v, A, T_h)^2 \|A^{1/2} \nabla v\|^2. \tag{37}$$

Now we remark that

$$\begin{aligned} \|A^{1/2} \nabla \mathrm{I}_{\mathrm{Cl}} v\|^2 &= \sum_{T \in T_h} \|A_T^{1/2} \nabla \mathrm{I}_{\mathrm{Cl}} v\|_T^2 \\ &\le \sum_{T \in T_h} \|C^{-1}_{T_A}\|^2 \|C^{\top}_{A,T} \nabla \mathrm{I}_{\mathrm{Cl}} v\|_T^2 \\ &\lesssim \sum_{T \in T_h} h^{-2}_{\min,A,T} \|C^{\top}_{A,T} \nabla \mathrm{I}_{\mathrm{Cl}} v\|_T^2. \end{aligned}$$

By the estimate (36), we conclude that

$$\|A^{1/2} \nabla \mathrm{I}_{\mathrm{Cl}} v\|^2 \lesssim m_1(v, A, T_h)^2 \|A^{1/2} \nabla v\|^2. \tag{38}$$

The estimates (32) and (38) prove (29).

Let us go on with the estimate (30):

$$\sum_{T\in\mathcal{T}_h} \alpha_T^{-2}\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T^2 \le \sum_{T\in\mathcal{T}_h} c_0\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T^2 + \sum_{T\in\mathcal{T}_h} h_{\min,A,T}^{-2}\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T^2.$$

By (32) and (37), we obtain (30).

For the last estimate, for any edge/face $E$ of $\mathcal{E}_h$, take any element $T \in \mathcal{T}_h$ such that $\beta_E = h_{E,T}h_{\min,A,T}^{-1}$. Now we use a standard trace inequality on $T$ to get

$$\|v-\mathrm{I}_{\mathrm{Cl}}v\|_E^2 \lesssim h_{E,T}^{-1}\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T(\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T + \|C_{A,T}^\top\nabla(v-\mathrm{I}_{\mathrm{Cl}}v)\|_T).$$

Multiplying this estimate by $\beta_E$ $(= h_{E,T}h_{\min,A,T}^{-1})$, we get

$$\beta_E\|v-\mathrm{I}_{\mathrm{Cl}}v\|_E^2 \lesssim h_{\min,A,T}^{-1}\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T(\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T + \|C_{A,T}^\top\nabla(v-\mathrm{I}_{\mathrm{Cl}}v)\|_T).$$

Multiplying this estimate by $\alpha_T^{-1}$ and summing on $E$ and then on $T$, we have obtained

$$\sum_{T\in\mathcal{T}_h}\alpha_T^{-1}\sum_{\substack{E\subset\partial T\setminus\Gamma_D:\\ \beta_E=h_{E,T}h_{\min,A,T}^{-1}}}\beta_E\|v-\mathrm{I}_{\mathrm{Cl}}v\|_E^2 \lesssim \sum_{T\in\mathcal{T}_h}\alpha_T^{-1}h_{\min,A,T}^{-1}\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T\cdot(\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T + \|C_{A,T}^\top\nabla(v-\mathrm{I}_{\mathrm{Cl}}v)\|_T).$$

Now using the discrete Cauchy–Schwarz inequality, we arrive at

$$\sum_{T\in\mathcal{T}_h}\alpha_T^{-1}\sum_{\substack{E\subset\partial T\setminus\Gamma_D:\\ \beta_E=h_{E,T}h_{\min,A,T}^{-1}}}\beta_E\|v-\mathrm{I}_{\mathrm{Cl}}v\|_E^2$$
$$\lesssim \left(\sum_{T\in\mathcal{T}_h}\alpha_T^{-2}\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T^2\right)^{1/2}\cdot\left(\sum_{T\in\mathcal{T}_h}h_{\min,A,T}^{-2}(\|v-\mathrm{I}_{\mathrm{Cl}}v\|_T^2 + \|C_{A,T}^\top\nabla(v-\mathrm{I}_{\mathrm{Cl}}v)\|_T^2)\right)^{1/2}.$$

We conclude thanks to (30), (34) and (37). □

**Remark 3.8.** If $A = \varepsilon Id$, then the estimate (31) implies the estimate (21) of [7], since

$$\beta_{E,T} = \sqrt{\varepsilon}\frac{h_{E,T}}{h_{\min,T}} \gtrsim \sqrt{\varepsilon}.$$

## 4. Error estimator

### 4.1. Definition of the error estimator

We investigate a residual error estimator. The exact element residual is defined by

$$R_T := f - Au_h \quad \text{on } T.$$

Similarly the exact edge/face residual is

$$R_E = \begin{cases} [\![A\nabla u_h\cdot n_E]\!]_E & \text{on } E\in\mathcal{E}_h^{\mathrm{int}},\\ g - A\nabla u_h\cdot n & \text{on } E\in\mathcal{E}_h^{\mathrm{ext}}\cap\Gamma_N,\\ 0 & \text{on } E\in\mathcal{E}_h^{\mathrm{ext}}\cap\Gamma_D.\end{cases}$$

As usual, these exact residuals are replaced by some finite-dimensional approximation $r_T\in\mathbb{P}^{k_0}(T)$ and $r_E\in\mathbb{P}^{k_1}(E)$ called approximate element residuals.

Now for further uses for any edge/face $E$, we set

$$\alpha_E = \alpha_T,$$

for one element $T\subset\omega_E$ such that $\beta_E = h_{E,T}h_{\min,A,T}^{-1}$. Note that from the definition of $\beta_E$, we have

$$h_{E,T}h_{\min,A,T}^{-1} \ge h_{E,T'}h_{\min,A,T'}^{-1}, \quad \forall T'\subset\omega_E,$$

and since the assumption (18) implies that

$$h_{E,T}\sim h_{E,T'}, \quad \forall T'\subset\omega_E,$$

we deduce that

$$h_{\min,A,T}^{-1} \gtrsim h_{\min,A,T'}^{-1}, \quad \forall T'\subset\omega_E,$$

and consequently

$$\alpha_E = \alpha_T \lesssim \alpha_{T'}, \quad \forall T'\subset\omega_E. \tag{39}$$

**Definition 4.1** (*Residual Error Estimator*)**.** The local and global residual error estimators are defined by

$$\eta_T^2 := \alpha_T^2 \|r_T\|_T^2 + \sum_{E\in\partial T\setminus\Gamma_D} \alpha_E \beta_E^{-1} \|r_E\|_E^2, \qquad \eta^2 := \sum_{T\in\mathcal{T}_h} \eta_T^2.$$

The local and global approximation terms are defined by

$$\zeta_T^2 := \alpha_T^2 \sum_{T'\subset\omega_T} \|R_{T'} - r_{T'}\|_{T'}^2 + \sum_{E\in\partial T\setminus\Gamma_D} \alpha_E \beta_E^{-1} \|R_E - r_E\|_E^2, \qquad \zeta^2 := \sum_{T\in\mathcal{T}_h} \zeta_T^2.$$

### 4.2. Upper error bound

**Theorem 4.2.** *Assume that $\delta_T$ satisfies (10). Let $u$ be a solution of (3) and $u_h$ a solution of (9). Then the error is bounded as follows:*

$$|||u-u_h||| \lesssim m_1(u-u_h, A, \mathcal{T}_h)(\eta+\zeta). \tag{40}$$

**Proof.** By (6) we have

$$|||u-u_h|||^2 \le B(u-u_h, u-u_h) \le B(u-u_h, v-\mathrm{I}_{\mathrm{Cl}}v) + B(u-u_h, \mathrm{I}_{\mathrm{Cl}}v), \tag{41}$$

where for shortness we write $v = u - u_h$.

For the first term, element-wise integration by parts yields

$$B(u-u_h, v-\mathrm{I}_{\mathrm{Cl}}v) = \sum_{T\in\mathcal{T}_h} (R_T, v-\mathrm{I}_{\mathrm{Cl}}v)_T + \sum_{E\in\mathcal{E}_h} (R_E, v-\mathrm{I}_{\mathrm{Cl}}v)_E.$$

By the continuous and by the discrete Cauchy–Schwarz inequality and the use of Lemma 3.7, we arrive at

$$B(u-u_h, v-\mathrm{I}_{\mathrm{Cl}}v) \lesssim m_1(v, A, \mathcal{T}_h)(\eta+\zeta)|||v|||. \tag{42}$$

For the second term of the right-hand side of (41), we first estimate $\|A^{1/2}\nabla \mathrm{I}_{\mathrm{Cl}}v\|_T$. Indeed we first write

$$\begin{aligned}
\|A^{1/2}\nabla \mathrm{I}_{\mathrm{Cl}}v\|_T &= \|C_{T_A}^{-\top} C_{T_A}^{\top} A^{1/2}\nabla \mathrm{I}_{\mathrm{Cl}}v\|_T \\
&\lesssim h_{\min,A,T}^{-1} \|C_{A,T}^{\top} A^{1/2}\nabla \mathrm{I}_{\mathrm{Cl}}v\|_T \\
&\lesssim h_{\min,A,T}^{-1} \|\mathrm{I}_{\mathrm{Cl}}v\|_T,
\end{aligned}$$

this last estimate coming from the inverse inequality on $\hat{T}_{T_A}$ and scaling arguments. This finally implies that

$$\|A^{1/2}\nabla \mathrm{I}_{\mathrm{Cl}}v\|_T \lesssim h_{\min,A,T}^{-1} c_0^{-1/2} |||\mathrm{I}_{\mathrm{Cl}}v|||_T.$$

On the other hand, we trivially have

$$\|A^{1/2}\nabla \mathrm{I}_{\mathrm{Cl}}v\|_T \lesssim |||\mathrm{I}_{\mathrm{Cl}}v|||_T = h_{\min,A,T}^{-1} h_{\min,A,T} |||\mathrm{I}_{\mathrm{Cl}}v|||_T,$$

and, by the definition of $\alpha_T$, we have obtained

$$\|A^{1/2}\nabla \mathrm{I}_{\mathrm{Cl}}v\|_T \lesssim h_{\min,A,T}^{-1} \alpha_T |||\mathrm{I}_{\mathrm{Cl}}v|||_T. \tag{43}$$

Now using (3) and (9), we get

$$B(u-u_h, \mathrm{I}_{\mathrm{Cl}}v) = -\sum_{T\in\mathcal{T}_h} \delta_T (R_T, \mathbf{b}\cdot\nabla \mathrm{I}_{\mathrm{Cl}}v)_T,$$

and by the Cauchy–Schwarz inequality we obtain

$$B(u-u_h, \mathrm{I}_{\mathrm{Cl}}v) \le \sum_{T\in\mathcal{T}_h} \delta_T \|R_T\|_T \|A^{-1/2}\mathbf{b}\|_{\infty,T} \|A^{1/2}\nabla \mathrm{I}_{\mathrm{Cl}}v\|_T.$$

Using (43), we obtain

$$B(u-u_h, \mathrm{I}_{\mathrm{Cl}}v) \lesssim \sum_{T\in\mathcal{T}_h} \delta_T \|R_T\|_T \|A^{-1/2}\mathbf{b}\|_{\infty,T} h_{\min,A,T}^{-1} \alpha_T |||\mathrm{I}_{\mathrm{Cl}}v|||_T,$$

and by the assumption on $\delta_T$, we arrive at

$$B(u-u_h, \mathrm{I}_{\mathrm{Cl}}v) \lesssim \sum_{T\in\mathcal{T}_h} \alpha_T \|R_T\|_T |||\mathrm{I}_{\mathrm{Cl}}v|||_T.$$

The discrete Cauchy–Schwarz inequality and the estimate (29) lead to

$$B(u-u_h, \mathrm{I}_{\mathrm{Cl}}v) \lesssim m_1(v, A, \mathcal{T}_h)(\eta+\zeta)|||v|||.$$

This estimate and (42) in the identity (41) lead to the conclusion. □

Now we estimate the dual norm of the convective derivative.

**Theorem 4.3.** *Assume that $\delta_T$ satisfies (10). Let $u$ be a solution of (3) and $u_h$ a solution of (9). Let $v_1$ be any function in $H^1_{\Gamma_D}(\Omega)$ such that (see (5))*

$$|||v_1||| \le 1 \quad \textit{and} \quad |||\mathbf{b}\cdot\nabla(u-u_h)|||_* = \int_\Omega \mathbf{b}\cdot\nabla(u-u_h)v_1. \tag{44}$$

*Then the error is bounded as follows:*

$$|||\mathbf{b}\cdot\nabla(u-u_h)|||_* \lesssim \kappa|||u-u_h||| + m_1(v_1, A, \mathcal{T}_h)(\eta+\zeta). \tag{45}$$

**Proof.** According to (44) and the definition of $B$, we have

$$|||\mathbf{b}\cdot\nabla(u-u_h)|||_* = \int_\Omega \mathbf{b}\cdot\nabla(u-u_h)v_1 = B(u-u_h, v_1) - \int_\Omega (A\nabla(u-u_h)\cdot\nabla v_1 + c(u-u_h)v_1).$$

Consequently, applying the Cauchy–Schwarz inequality we obtain (see (7))

$$|||\mathbf{b}\cdot\nabla(u-u_h)|||_* \le B(u-u_h, v_1) + \kappa|||u-u_h|||.$$

We conclude by using the arguments of the previous proof with $v$ replaced by $v_1$. □

**Corollary 4.4.** *Under the assumptions of the previous theorem, we have the error bound*

$$|||u-u_h||| + |||\mathbf{b}\cdot\nabla(u-u_h)|||_* \lesssim (\kappa m_1(u-u_h, A, \mathcal{T}_h) + m_1(v_1, A, \mathcal{T}_h))(\eta+\zeta).$$

### 4.3. *Lower error bound*

**Theorem 4.5.** *The following global lower error bound holds:*

$$\eta \lesssim \kappa|||u-u_h||| + |||\mathbf{b}\cdot\nabla(u-u_h)|||_* + \zeta. \tag{46}$$

**Proof.** As already mentioned element-wise integration by parts yields

$$B(u-u_h, w) = \sum_{T\in\mathcal{T}_h}(R_T, w)_T + \sum_{E\in\mathcal{E}_h}(R_E, w)_E, \quad \forall w \in H^1_{\Gamma_D}(\Omega). \tag{47}$$

*Element residual*. For a fixed element $T$ define $w_T = r_T b_T$ which belongs to $H^1_{\Gamma_D}(\Omega)$. From the definition of $R_T$ and using (47) with $w = \sum_{T\in\mathcal{T}_h}\alpha_T^2 w_T$ we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\alpha_T^2\int_T r_T w_T &= \sum_{T\in\mathcal{T}_h}\alpha_T^2\int_T (r_T-R_T)w_T + \sum_{T\in\mathcal{T}_h}\alpha_T^2\int_T R_T w_T\\ &= \sum_{T\in\mathcal{T}_h}\alpha_T^2\int_T (r_T-R_T)w_T + B(u-u_h, w).\end{aligned}$$

Using the equivalence (20) and the estimate (7) we obtain

$$\sum_{T\in\mathcal{T}_h}\alpha_T^2\|r_T\|_T^2 \lesssim \left(\sum_{T\in\mathcal{T}_h}\alpha_T^2\|r_T-R_T\|_T^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}\alpha_T^2\|w_T\|_T^2\right)^{\frac12} + (\kappa|||u-u_h||| + |||\mathbf{b}\cdot\nabla(u-u_h)|||_*)|||w|||.$$

By the definition of $w$, we have

$$|||w|||^2 = \sum_{T\in\mathcal{T}_h}\alpha_T^4|||w_T|||_T^2,$$

and by the inverse inequality (22) we get

$$|||w|||^2 \lesssim \sum_{T\in\mathcal{T}_h}\alpha_T^2\|r_T\|_T^2.$$

Similarly by the inverse inequality (20), we have

$$\sum_{T\in\mathcal{T}_h}\alpha_T^2\|w_T\|_T^2 \lesssim \sum_{T\in\mathcal{T}_h}\alpha_T^2\|r_T\|_T^2.$$

This last three estimates yield

$$\left(\sum_{T\in\mathcal{T}_h}\alpha_T^2\|r_T\|_T^2\right)^{\frac{1}{2}} \lesssim \zeta + \kappa|||u-u_h||| + |||\mathbf{b}\cdot\nabla(u-u_h)|||_*. \tag{48}$$

*Edge/face residual*. Fix an arbitrary edge/face $E \in \mathcal{E}_h \setminus \Gamma_D$. We apply (47) with $w = w_E$, where

$$w_E := \mathrm{F}_{\mathrm{ext}}(r_E) b_{E,\gamma_{E,T}} \quad \text{on } T_{E,\gamma_E} \subset T \subset \omega_E,$$

where $T_{E,\gamma_{E,T}}$ is the squeezed element associated with $T$ defined with the parameter $\gamma_{E,T} \in (0, 1]$ that will be fixed later on. This yields

$$\begin{aligned}(r_E, w_E)_E &= (r_E - R_E, w_E)_E + (R_E, w_E)_E\\ &= (r_E - R_E, w_E)_E + B(u-u_h, w_E) - \sum_{T\subset\omega_E}(R_T, w_E)_T.\end{aligned}$$

Multiplying this identity by $\alpha_E\beta_E^{-1}$, and setting

$$w = \sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}w_E,$$

we arrive at

$$\begin{aligned}\sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}(r_E, w_E)_E &= \sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}(r_E - R_E, w_E)_E + \sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}\sum_{T\subset\omega_E}(r_T - R_T, w_E)_T\\ &\quad - \sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}\sum_{T\subset\omega_E}(r_T, w_E)_T + B(u-u_h, w).\end{aligned}$$

Using the equivalence (21) and the estimate (7) we obtain

$$\begin{aligned}\sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}\|r_E\|_E^2 &\lesssim \sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}(r_E - R_E, w_E)_E + \sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}\sum_{T\subset\omega_E}(r_T - R_T, w_E)_T\\ &\quad - \sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}\sum_{T\subset\omega_E}(r_T, w_E)_T + (\kappa|||u-u_h||| + |||\mathbf{b}\cdot\nabla(u-u_h)|||_*)|||w|||.\end{aligned}$$

Using the Cauchy–Schwarz inequality, (23), (24), (39) and (48) we deduce that

$$\begin{aligned}\sum_{E\in\mathcal{E}_h}\alpha_E\beta_E^{-1}\|r_E\|_E^2 &\lesssim \sum_{E\in\mathcal{E}_h}\sum_{T\subset\omega_E}\zeta_T\alpha_E^{1/2}\beta_E^{-1/2}\|r_E\|_E + \sum_{E\in\mathcal{E}_h}\sum_{T\subset\omega_E}\zeta_T\alpha_E^{1/2}\beta_E^{-1}\gamma_{E,T}^{1/2}h_{E,T}^{1/2}\|r_E\|_E\\ &\quad + (\zeta + \kappa|||u-u_h||| + |||\mathbf{b}\cdot\nabla(u-u_h)|||_*)\left(\sum_{E\in\mathcal{E}_h}\sum_{T\subset\omega_E}\beta_E^{-2}\gamma_{E,T}h_{E,T}\|r_E\|_E^2\right)^{1/2}\\ &\quad + (\kappa|||u-u_h||| + |||\mathbf{b}\cdot\nabla(u-u_h)|||_*)|||w|||.\end{aligned}$$

Similarly by the definition of $w$ and (23) and (24), we have

$$|||w|||^2 \lesssim \sum_{E\in\mathcal{E}_h}\alpha_E^2\beta_E^{-2}\sum_{T\subset\omega_E}(\kappa^2\gamma_{E,T}h_{E,T} + \gamma_{E,T}^{-1}h_{E,T}h_{\min,A,T}^{-2})\|r_E\|_E^2.$$

In view of these two estimates we need that

$$\alpha_E^{1/2}\beta_E^{-1}\gamma_{E,T}^{1/2}h_{E,T}^{1/2} \lesssim \alpha_E^{1/2}\beta_E^{-1/2}, \quad \forall T\subset\omega_E, \tag{49}$$

$$\beta_E^{-2}\gamma_{E,T}h_{E,T} \lesssim \alpha_E\beta_E^{-1}, \quad \forall T\subset\omega_E, \tag{50}$$

$$\alpha_E^2\beta_E^{-2}\gamma_{E,T}h_{E,T} \lesssim \alpha_E\beta_E^{-1}, \quad \forall T\subset\omega_E, \tag{51}$$

$$\alpha_E^2\beta_E^{-2}\gamma_{E,T}^{-1}h_{E,T}h_{\min,A,T}^{-2} \lesssim \alpha_E\beta_E^{-1}, \quad \forall T\subset\omega_E, \tag{52}$$

since these conditions imply that

$$\left(\sum_{E\in E_h} \alpha_E \beta_E^{-1} \|r_E\|_E^2\right)^{\frac{1}{2}} \lesssim \zeta + \kappa |||u-u_h||| + |||\mathbf{b}\cdot\nabla(u-u_h)|||_*. \tag{53}$$

The conclusion then follows from the estimates (48) and (53) if we can show that conditions (49)–(52) hold with an appropriate choice of $\gamma_{E,T}$.

First since $\gamma_{E,T} \le 1$ and $\alpha_E \lesssim 1$, condition (50) implies (49) and (51). Hence it remains to check (50) and (52). The first one is clearly equivalent to

$$\gamma_{E,T} h_{E,T} \alpha_E^{-1} \beta_E^{-1} \lesssim 1, \tag{54}$$

and since $\beta_E \gtrsim h_{E,T} h_{\min,A,T}^{-1}$, for all $T \subset \omega_E$, the estimate (54) holds if

$$\gamma_{E,T} \lesssim \alpha_E h_{E,T}^{-1} h_{E,T} h_{\min,A,T}^{-1} = \alpha_E h_{\min,A,T}^{-1} \tag{55}$$

is satisfied.

On the other hand (52) holds if and only if

$$\alpha_E \beta_E^{-1} \gamma_{E,T}^{-1} h_{E,T} h_{\min,A,T}^{-2} \lesssim 1.$$

Again by the definition of $\beta_E$, this estimate holds if

$$\alpha_E h_{\min,A,T}^{-1} \lesssim \gamma_{E,T}. \tag{56}$$

To satisfy these two conditions (55) and (56), we take

$$\gamma_{E,T} = \min\{\alpha_E h_{\min,A,T}^{-1}, 1\}.$$

Obviously this right-hand side is $\le 1$ and condition (55) holds. Second to check that condition (56) is satisfied, we distinguish two cases:

1. If $\alpha_E h_{\min,A,T}^{-1} \le 1$, then $\gamma_{E,T} = \alpha_E h_{\min,A,T}^{-1}$ and (56) is trivially satisfied.
2. If $\alpha_E h_{\min,A,T}^{-1} > 1$, then $\gamma_{E,T} = 1$, but by the property $\alpha_E \lesssim \alpha_T$, for all $T \subset \omega_E$ (see (39)), we have

$$\alpha_E h_{\min,A,T}^{-1} \lesssim \alpha_T h_{\min,A,T}^{-1} \lesssim 1,$$

which again yields (56). □

**Remark 4.6.** In comparison with the case $A = \varepsilon Id$ treated in [27,24,7] for anisotropic meshes and in [28,10] for isotropic meshes, we have obtained as in [12] for isotropic meshes, a robust lower bound due to the use of the dual norm of the convective derivative.

## 5. Numerical results

The aim is to test the behavior of the estimated error in the relationship with the true error. Therefore we use a test example with a known exact solution. We consider the problem

$$-\mathrm{div}\,(A\nabla u) + \mathbf{b}\cdot\nabla u = f \quad \text{in } \Omega = (0,1)^2, \qquad u = g \quad \text{on } \Gamma,$$

with

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$\varepsilon = 10^{-k}$, $k = 4, 8$, and choose the data

$$f := 10y(1-y)(-\varepsilon e^{-x} - e^{-x}) + 20(e^{-x} - e^{-1+\frac{x-1}{\varepsilon}}),$$

$$g := \begin{cases} 10y(1-y)(1-e^{-1-1/\varepsilon}) & \text{if } x = 0, \\ 0 & \text{else.} \end{cases}$$

This results in the solution

$$u = 10y(1-y)(e^{-x} - e^{-1+\frac{x-1}{\varepsilon}}),$$

which is illustrated in Fig. 3. Note that both the data and the solution are $O(1)$ in the $L^2(\Omega)$- and $L^\infty(\Omega)$-norms uniformly in $\varepsilon$. The solution contains a typical boundary layer of that problem.

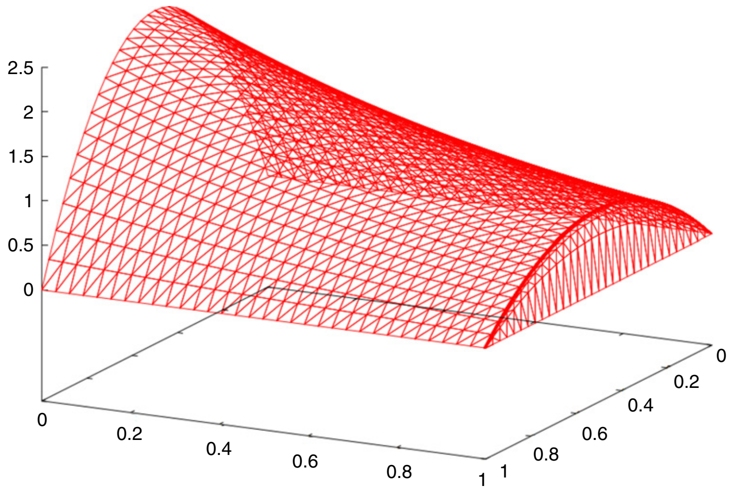


**Fig. 3.** The solution $u$ for $\varepsilon = 10^{-4}$.

**Table 1**
Computation of errors in the maximum norm and norm $|[\cdot]|$ for $\varepsilon = 10^{-4}$.

| $N$ | $\|e\|_{L^\infty(\Omega)}$ | Rate | $|[e]|$ | Rate |
|---|---|---|---|---|
| 153 | 1.62E−01 | | 6.86E−01 | |
| 561 | 7.12E−02 | 1.27 | 3.16E−01 | 1.19 |
| 2145 | 2.57E−02 | 1.52 | 1.53E−01 | 1.08 |
| 8385 | 7.89E−03 | 1.73 | 7.61E−02 | 1.03 |
| 33,153 | 2.20E−03 | 1.86 | 3.80E−02 | 1.01 |
| 131,841 | 5.81E−04 | 1.93 | 1.90E−02 | 1.00 |
| 525,825 | 1.49E−04 | 1.96 | 9.48E−03 | 1.00 |

**Table 2**
Computation of errors in the maximum norm and norm $|[\cdot]|$ for $\varepsilon = 10^{-8}$.

| $N$ | $\|e\|_{L^\infty(\Omega)}$ | Rate | $|[e]|$ | Rate |
|---|---|---|---|---|
| 153 | 3.42E−01 | | 5.35E+00 | |
| 561 | 1.64E−01 | 1.14 | 1.78E+00 | 1.69 |
| 2145 | 6.92E−02 | 1.29 | 5.19E−01 | 1.84 |
| 8385 | 2.50E−02 | 1.49 | 1.60E−01 | 1.73 |
| 33,153 | 7.68E−03 | 1.72 | 5.97E−02 | 1.43 |
| 131,841 | 2.14E−03 | 1.85 | 2.67E−02 | 1.17 |
| 525,825 | 5.65E−04 | 1.92 | 1.29E−02 | 1.05 |

The mesh is piecewise uniform with an anisotropic part in the boundary strip $\Omega_L = (1 - 2\varepsilon|\ln\varepsilon|, 1) \times (0, 1)$. Both $\Omega_L$ and $\Omega_0 = \Omega \setminus \Omega_L$ are subdivided into $2^k \times 2^k$, $k = 3, \dots, 9$, congruent rectangles which are afterwards split into two triangles each. In this way the aspect ratio of the elements is about $\varepsilon^{-1}$ in $\Omega_L$ and about unity in $\Omega_0$.

The problem is discretized with the SUPG scheme (9) where

$$\delta_T = \begin{cases} \sqrt{\varepsilon} h_{T,\min}^2 & \text{in the boundary layer,} \\ h_{T,\min}^2 & \text{elsewhere} \end{cases}$$

is chosen. The error is computed in various norms and also estimated with the method investigated above, see Definition 4.1. Tables 1 and 2 display the error in norms which are typically investigated in an a priori error analysis, the maximum norm and “SUPG norm” defined by $|[v]|^2 = B(v, v) + \sum_{T \in T_h} \delta_T \|\mathbf{b} \cdot \nabla v\|_{L^2(T)}^2$. The convergence rates are computed with respect to the mesh size $h = 2^{-k}$ which seems more convenient than the relationship to the number $N = (2 \cdot 2^k + 1)(2^k + 1)$, $k = 3, \dots, 9$. The error behaviour shows that the meshes are appropriately chosen.

Tables 3 and 4 show the error estimator $\eta$ as well as the error in the norms $|||e|||$, see (2), $|||\mathbf{b} \cdot \nabla e|||_*$, see (4), and the efficiency index

$$I_{\text{eff}} := \frac{\eta}{|||e||| + |||\mathbf{b} \cdot \nabla e|||_*}.$$

It can be seen well, that the effectivity index converges for $h \to 0$ to some limit of about 6 independent of $\varepsilon$. This experiment illustrates the efficiency and reliability of our estimator.

**Table 3**
Behaviour of the error estimator for $\varepsilon = 10^{-4}$.

| $N$ | $\eta$ | Rate | $\|\|\|e\|\|\|$ | Rate | $\|\|\mathbf{b}\cdot\nabla e\|\|_*$ | Rate | $I_{\text{eff}}$ |
|---|---|---|---|---|---|---|---|
| 153 | 3.10E+00 | | 5.86E−01 | | 7.73E−02 | | 4.68 |
| 561 | 1.63E+00 | 0.988 | 3.00E−01 | 1.03 | 2.44E−02 | 1.77 | 5.03 |
| 2145 | 8.46E−01 | 0.981 | 1.51E−01 | 1.02 | 6.85E−03 | 1.89 | 5.35 |
| 8385 | 4.33E−01 | 0.983 | 7.58E−02 | 1.01 | 1.86E−03 | 1.91 | 5.58 |
| 33,153 | 2.20E−01 | 0.988 | 3.79E−02 | 1.01 | 4.95E−04 | 1.93 | 5.72 |
| 131,841 | 1.11E−01 | 0.993 | 1.90E−02 | 1.00 | 1.28E−04 | 1.96 | 5.79 |
| 525,825 | 5.55E−02 | 0.996 | 9.48E−03 | 1.00 | 3.23E−05 | 1.99 | 5.84 |

**Table 4**
Behaviour of the error estimator for $\varepsilon = 10^{-8}$.

| $N$ | $\eta$ | Rate | $\|\|\|e\|\|\|$ | Rate | $\|\|\mathbf{b}\cdot\nabla e\|\|_*$ | Rate | $I_{\text{eff}}$ |
|---|---|---|---|---|---|---|---|
| 153 | 3.69E+00 | | 6.65E−01 | | 2.19E−01 | | 4.17 |
| 561 | 1.96E+00 | 0.973 | 3.78E−01 | 0.869 | 8.50E−02 | 1.46 | 4.23 |
| 2145 | 1.04E+00 | 0.939 | 2.00E−01 | 0.952 | 2.52E−02 | 1.81 | 4.64 |
| 8385 | 5.52E−01 | 0.935 | 1.02E−01 | 0.993 | 6.71E−03 | 1.94 | 5.10 |
| 33,153 | 2.87E−01 | 0.950 | 5.10E−02 | 1.00 | 1.74E−03 | 1.97 | 5.45 |
| 131,841 | 1.47E−01 | 0.969 | 2.55E−02 | 1.00 | 4.46E−04 | 1.97 | 5.67 |
| 525,825 | 7.46E−02 | 0.982 | 1.28E−02 | 1.00 | 1.14E−04 | 1.97 | 5.80 |

**Remark 5.1.** The error in the dual norm $|||\phi|||_*$ is approximately computed here by

$$|||\phi|||_* = \sup_{v\in H^1_{\Gamma_D}(\Omega)\setminus\{0\}} \frac{\int_\Omega \phi v}{|||v|||} \approx \sup_{v_h\in V_h\setminus\{0\}} \frac{\int_\Omega \phi_h v_h}{|||v_h|||}$$

where $\phi_h$ is an approximation of $\phi$ in a finite dimensional space $W_h$, here the space of piecewise constants. This expression can be easily computed: Let $\underline{\phi}$ and $\underline{v}$ be the vectors of the coefficients of $\phi_h$ and $v_h$ in some bases of $W_h$ and $V_h$, respectively. With appropriate mass and stiffness matrices $M$ and $K$ we can write $\int_\Omega \phi_h v_h = \underline{\phi}^T M \underline{v}$ and $|||v_h|||^2 = \underline{v}^T K \underline{v}$. By using the Cholesky decomposition $K = LL^T$ and the substitutions $\underline{w} = L^T \underline{v}$ and $\underline{\psi} = L^{-1} M^T \underline{\phi}$ we can reformulate

$$\begin{aligned}\sup_{v_h\in V_h\setminus\{0\}} \frac{\left(\int_\Omega \phi_h v_h\right)^2}{|||v_h|||^2} &= \sup_{\underline{v}\in\mathbb{R}^n\setminus\{0\}} \frac{(\underline{\phi}^T M\underline{v})^2}{\underline{v}^T K\underline{v}} = \sup_{\underline{w}\in\mathbb{R}^n\setminus\{0\}} \frac{(\underline{\phi}^T M L^{-T}\underline{w})^2}{\underline{w}^T\underline{w}} \\ &= \sup_{\underline{w}\in\mathbb{R}^n\setminus\{0\}} \frac{(\underline{\psi}^T\underline{w})^2}{\underline{w}^T\underline{w}} = \sup_{\underline{w}\in\mathbb{R}^n\setminus\{0\}} \frac{\underline{w}^T\underline{\psi}\,\underline{\psi}^T\underline{w}}{\underline{w}^T\underline{w}} = \underline{\psi}^T\underline{\psi}\end{aligned}$$

where we used for the computation of the Rayleigh quotient that the only non-zero eigenvalue of the matrix $\underline{\psi}\,\underline{\psi}^T$ is $\underline{\psi}^T\underline{\psi}$. By substituting back we obtain

$$\sup_{v_h\in V_h\setminus\{0\}} \frac{\int_\Omega \phi_h v_h}{|||v_h|||} = (\underline{\phi}^T M K^{-1} M^T \underline{\phi})^{1/2}.$$

## 6. Conclusions

We have proposed and rigorously analyzed a new a posteriori error estimate for the finite element approximation of anisotropic diffusion–convection–reaction equations with anisotropic finite elements. We have shown that this estimate is reliable and efficient. Numerical experiments confirm our theoretical predictions.

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