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Least-squares linear estimation of signals from observations with Markovian delays

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ABSTRACT

The least-squares linear estimation of signals from randomly delayed measurements is addressed when the delay is modeled by a homogeneous Markov chain. To estimate the signal, recursive filtering and fixed-point smoothing algorithms are derived, using an innovation approach, assuming that the covariance functions of the processes involved in the observation equation are known. Recursive formulas for filtering and fixed-point smoothing error covariance matrices are obtained to measure the goodness of the proposed estimators.

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1. Introduction

Traditionally, the estimation problem of signals has been addressed assuming that the measurement data are transmitted over perfect communications channels. In practice, unfortunately, this is not the case; for example, in wireless communication networks an unavoidable problem is the existence of errors during the transmission, which can lead to delays in the arrival of the measurements. These delays may be deterministic (see e.g. [1,2]), but in most practical cases, such as mobile communications or exploration seismology, the delay is random and can be modeled by a stochastic process.

The practical application of systems affected by delays has been the main motivation to many recent studies of control and estimation problems. With respect to the problem of signal estimation in randomly delayed systems, different approaches have been considered, differing in the delay model and estimation criteria adopted. A common formulation for modeling the random delay is to consider it as a sequence of Bernoulli random variables that takes values of 0 or 1 depending on whether the real observation is received on time or otherwise. In this context, Yaz and Ray [3] studied linear unbiased state estimation for dynamic systems with a one-step sensor delay and presented full and reduced order estimators by reformulating the state estimation problem as one of the parameter designs within the filtering problem. More recently, Wang et al. [4] addressed the robust filtering problem with variance constraints. Wen et al. [5] derived new filtering algorithms, both full and reduced order, for a stochastic dynamic system with a random one-step sensor delay. The filtering problem for systems with uncertain observations, packet dropouts and a random sensor delay through the same framework using the stochastic H_2-norm for systems with stochastic parameters has been studied in [6]. In all the aforementioned papers, the state-space model is known. However, in many practical situations the only information available to estimate the signal are the covariance functions of the processes involved in the observation model. Under this assumption, recursive filtering and smoothing algorithms have also been derived assuming the random delay to be modeled by independent Bernoulli random variables (see e.g. [7,8]).

For randomly delayed models, the signal estimation problem has been usually addressed assuming that the delay is modeled by independent random variables. However, in real communication systems, current time delays are usually

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correlated with previous ones, and then a reasonable way to model the dependence on the delays is to consider the random delay as a homogeneous Markov chain. Thus, state estimation algorithms for a hidden Markov model have been presented in [9], while a H_∞ filtering was obtained in [10]. In both the papers, the delay steps were known on-line via time-stamped data. Assuming an unknown Markov delay, Han and Zhang [11] solved the optimal estimation problem by considering that the single Markov delayed measurement can be rewritten as an equivalent measurement with multiple constant delays; the estimator is then derived on the basis of geometric arguments in the Hilbert space. Song et al. [12] investigated the H_∞ filtering problem for a class of network systems with a random delay modeled by a Markov chain. When the delay is modeled by Bernoulli random variables, the signal estimation problem has been investigated assuming some kind of correlation on the Bernoulli random variables (see for example [13,14]). However, to the best of our knowledge, the problem of signal estimation in systems where Bernoulli random variables are a Markov chain has not yet been studied.

In this paper, assuming no signal equation is available and that the delay is modeled by a homogeneous discrete-time Markov chain to capture the dependence between delays, we study the least-squares linear estimation problem of a signal based on randomly delayed measurements. This signal estimation problem is addressed assuming that the covariance functions of the processes involved in the observation equation are known and that the covariance function of the signal is expressed in a semi-degenerated kernel form. The proposed recursive filtering and fixed-point smoothing algorithms are obtained using an innovation approach which, as it is known, enables straightforward derivation of the estimation algorithms. Recently, the innovation based methods are applied to estimation and filtering as well as system identification (e.g. [15–22]). Finally, a numerical simulation example is included to illustrate the feasibility of the proposed algorithms for estimating a signal from randomly delayed observations.

2. Problem formulation

2.1. Observation model

Consider the measurement of an $n \times 1$ signal, x_k, described by the equation

$$z_k = x_k + v_k, \quad k \geq 0, \tag{1}$$

where $\{v_k,\ k \geq 0\}$ is a white measurement noise. It is assumed that the measurement of the signal, z_k, is transmitted to a processing unit through an unreliable network, where some data may be delayed by one sampling time during the transmission; if so, the last available measurement is processed. According to [3], this situation can be modeled by the following equation

$$y_k = (1 - \xi_k)z_k + \xi_k z_{k-1}, \quad k \geq 1, \tag{2}$$

where $\{\xi_k,\ k \geq 1\}$ are Bernoulli random variables that model the random delay. If $\xi_k = 1$, then the measurement is delayed by one sample period; otherwise the measurement is up-to-date.

In order to address the least-squares linear estimation problem of the signal, x_k, from the observations given by (2), the following hypotheses are made regarding the signal and noise processes:

(i) The signal, $\{x_k,\ k \geq 0\}$, has a zero mean and its covariance function is

$$E[x_k x_s^T] = \alpha_k \beta_s^T,\ s \leq k$$

where α and β are known $n \times M$ matrix functions.

(ii) The measurement noise, $\{v_k,\ k \geq 0\}$, is white noise with a zero mean and $E[v_k v_k^T] = R_k$.

(iii) The sequence of Bernoulli variables $\{\xi_k,\ k \geq 1\}$ is a homogeneous Markov chain with $P[\xi_k = 1] = p_k$ and transition probability matrix $\mathbb{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$ with $p_{ij} = P[\xi_k = j/\xi_{k-1} = i],\ i, j \in \{0, 1\}$.

(iv) The signal, $\{x_k,\ k \geq 0\}$, and the measurement noises, $\{v_k,\ k \geq 0\}$ and $\{\xi_k,\ k \geq 1\}$, are mutually independent.

2.2. Least-squares linear estimation problem

Our aim is to determine the least-squares linear estimator $\widehat{x}_{k/L}$ of the signal, x_k, based on the information provided by the measurements $y_1, \ldots, y_L$, given by (2). Specifically, recursive algorithms for the filtering ($L = k$) and fixed-point smoothing (k fixed and $L > k$) problems will be derived.

As is well known, the estimator $\widehat{x}_{k/L}$ is the orthogonal projection of the vector x_k onto the linear space spanned by $y_1, \ldots, y_L$ and so the estimator is the only linear combination of $y_1, \ldots, y_L$ satisfying the orthogonality property $E[(x_k - \widehat{x}_{k/L})y_s^T] = 0,\ s = 1, \ldots, L$. Since the observations are generally nonorthogonal vectors, we use an innovation approach to address the estimation problem. This provides straightforward means of resolving the problem because the innovation process is white. This approach is based on an orthogonalization procedure by means of which the observation process is transformed into an equivalent one, the innovation process, defined as $\nu_k = y_k - \widehat{y}_{k|k-1},\ k \geq 1$, where $\widehat{y}_{k/k-1}$ denotes the linear estimator of y_k based on the observations $y_1, \ldots, y_{k-1}$. As the innovations and observations provide the same information and the innovation process is white, the linear estimator can be calculated as a linear combination of the innovations, as follows

$$\widehat{x}_{k/L} = \sum_{i=1}^{L} S_{k,i}\Pi_i^{-1}\nu_i, \tag{3}$$

where $S_{k,i} = E[x_k\nu_i^T]$ and Π_i is the covariance matrix of the innovation, ν_i.

Taking into account general expression (3) of the estimator, our first aim is to derive explicit formulas for the innovations as well as for their covariance matrices.

3. Innovation process

Theorem 1. *Under hypotheses* (i)–(iv) *set out in Section* 2*, the innovation process is given by*

$$\begin{aligned}\nu_k &= y_k - \Lambda_k Q_{k-1} - p_{01}(1-p_{k-1})R_{k-1}\Pi_{k-1}^{-1}\nu_{k-1}, \quad k \geq 2,\\ \nu_1 &= y_1,\end{aligned} \tag{4}$$

where the vectors Q_k *are recursively calculated as*

$$Q_k = Q_{k-1} + M_k\Pi_k^{-1}\nu_k, \quad k \geq 1; \qquad Q_0 = 0 \tag{5}$$

with

$$\begin{aligned}M_k &= \Gamma_k^T - m_{k-1}\Lambda_k^T - p_{01}(1-p_{k-1})M_{k-1}\Pi_{k-1}^{-1}R_{k-1}, \quad k \geq 2,\\ M_1 &= \Gamma_1^T,\end{aligned} \tag{6}$$

$$m_k = m_{k-1} + M_k\Pi_k^{-1}M_k^T, \quad k \geq 1; \qquad m_0 = 0, \tag{7}$$

and the matrices Λ_k *and* Γ_k *are given by*

$$\Lambda_k = (\alpha_k, \alpha_{k-1})(\mathbb{P}^k \otimes I_M)^T, \qquad \Gamma_k = ((1-p_k)\beta_k, p_k\beta_{k-1})(\mathbb{P}^{-k} \otimes I_M), \tag{8}$$

where $\otimes$ *denotes the Kronecker product and* I_M *is the* $M \times M$*-dimensional identity matrix.*

The innovation covariance matrix is given by

$$\begin{aligned}\Pi_k \;=\; & (1-p_k)[\alpha_k\beta_k^T + R_k] + p_k[\alpha_{k-1}\beta_{k-1}^T + R_{k-1}] - \Lambda_k m_{k-1}\Lambda_k^T - p_{01}(1-p_{k-1})\Lambda_k M_{k-1}\Pi_{k-1}^{-1}R_{k-1}\\ & - p_{01}(1-p_{k-1})R_{k-1}\Pi_{k-1}^{-1}M_{k-1}^T\Lambda_k^T - p_{01}^2(1-p_{k-1})^2R_{k-1}\Pi_{k-1}^{-1}R_{k-1}, \quad k \geq 2,\\ \Pi_1 = \; & (1-p_1)[\alpha_1\beta_1^T + R_1] + p_1[\alpha_0\beta_0^T + R_0].\end{aligned} \tag{9}$$

Proof. In order to obtain the explicit formula (4) for the innovations, $\nu_k = y_k - \widehat{y}_{k/k-1}$, it is necessary to determine $\widehat{y}_{k/k-1}$, the one-stage predictor of observation y_k,which is given by

$$\widehat{y}_{k/k-1} = \sum_{i=1}^{k-1} T_{k,i}\Pi_i^{-1}\nu_i, \quad k \geq 2; \qquad \widehat{y}_{1/0} = 0, \tag{10}$$

where $T_{k,i} = E[y_k\nu_i^T],\ i \leq k-1$.

First, we obtain an expression for the coefficients $T_{k,i} = E[y_ky_i^T] - E[y_k\widehat{y}_{i/i-1}^T]$. Using (2) and taking into account that $\{\xi_k,\ k \geq 1\}$ is a homogeneous Markov chain, we have

$$E[y_ky_i^T] = p_{00}^{(k-i)}(1-p_i)E[z_kz_i^T] + p_{10}^{(k-i)}p_iE[z_kz_{i-1}^T] + p_{01}^{(k-i)}(1-p_i)E[z_{k-1}z_i^T] + p_{11}^{(k-i)}p_iE[z_{k-1}z_{i-1}^T], \quad i \leq k-1,$$

where $p_{sj}^{(k-i)}$ denotes the $(k-i)$-step transition probability from state s to j, with $s, j \in \{0, 1\}$. Using (1) and the hypotheses of the model, we have

$$\begin{aligned}E[y_ky_i^T] = \; & p_{00}^{(k-i)}(1-p_i)\alpha_k\beta_i^T + p_{10}^{(k-i)}p_i\alpha_k\beta_{i-1}^T + p_{01}^{(k-i)}(1-p_i)\alpha_{k-1}\beta_i^T\\ & + p_{11}^{(k-i)}p_i\alpha_{k-1}\beta_{i-1}^T + p_{01}(1-p_{k-1})R_{k-1}\delta_{i,k-1}, \quad i \leq k-1,\end{aligned}$$

which, equivalently, can be expressed in matrix form as

$$E[y_ky_i^T] = (\alpha_k, \alpha_{k-1})(\mathbb{P}^{(k-i)} \otimes I_M)^T((1-p_i)\beta_i, p_i\beta_{i-1})^T + p_{01}(1-p_{k-1})R_{k-1}\delta_{i,k-1}, \quad i \leq k-1,$$

where $\mathbb{P}^{(k-i)}$ is the $(k-i)$-step transition probability matrix and is the δ the Kronecker delta function.

Substituting the above expectation in $T_{k,i} = E[y_ky_i^T] - E[y_k\widehat{y}_{i/i-1}]$, taking into account the properties of the transition probability matrix and those of the Kronecker product, $(\mathbb{P}^{(k-i)} \otimes I_M)^T = (\mathbb{P}^k \otimes I_M)^T(\mathbb{P}^{-i} \otimes I_M)^T$ and using (10) for predictor

$\widehat{y}_{i/i-1}$, the coefficients $T_{k,i}$ can be written as

$$T_{k,i} = \Lambda_k \Gamma_i^T - \sum_{j=1}^{i-1} T_{k,j}\Pi_j^{-1}T_{i,j}^T + p_{01}(1-p_{k-1})R_{k-1}\delta_{i,k-1}, \quad i \le k-1,$$

where Λ_k and Γ_i are given by (8).

Introducing a function M_i that satisfies

$$\begin{aligned} M_i &= \Gamma_i^T - \sum_{j=1}^{i-1} M_j \Pi_j^{-1} T_{i,j}^T, \quad i \ge 2, \\ M_1 &= \Gamma_1^T \end{aligned} \tag{11}$$

the coefficients $T_{k,i}$ are given by

$$T_{k,i} = \Lambda_k M_i + p_{01}(1-p_{k-1})R_{k-1}\delta_{i,k-1}, \quad i \le k-1. \tag{12}$$

Substituting the above expression of $T_{k,i}$ in (10) and defining $Q_k = \sum_{i=1}^{k} M_i \Pi_i^{-1}\nu_i$, for $k \ge 1$ and $Q_0 = 0$, we obtain

$$\begin{aligned} &\widehat{y}_{k/k-1} = \Lambda_k Q_{k-1} + p_{01}(1-p_{k-1})R_{k-1}\Pi_{k-1}^{-1}\nu_{k-1}, \quad k \ge 2, \\ &\widehat{y}_{1/0} = 0 \end{aligned} \tag{13}$$

and, therefore, expression (4) is obtained.

From the definition of Q_k, the recursive relation given by (5) is clear.

Now, performing $i = k$ in (11) and using expression (12) for $T_{k,j}$ for $j \le k-1$, we obtain

$$\begin{aligned} M_k &= \Gamma_k^T - \sum_{j=1}^{k-1} M_j \Pi_j^{-1} M_j^T \Lambda_k^T - p_{01}(1-p_{k-1})M_{k-1}\Pi_{k-1}^{-1}R_{k-1}, \quad k \ge 2, \\ M_1 &= \Gamma_1^T, \end{aligned}$$

and denoting

$$m_k = E[Q_k Q_k^T] = \sum_{j=1}^{k} M_j \Pi_j^{-1} M_j^T, \quad k \ge 1; \qquad m_0 = 0,$$

formula (6) for M_k is obtained. From the definition of m_k, the recursive relation (7) is immediate.

Finally, we obtain the covariance matrix of the innovation process. Since the estimation error is orthogonal to the estimator, $\Pi_k = E[y_k y_k^T] - E[\widehat{y}_{k/k-1}\widehat{y}_{k/k-1}^T]$; so, let us calculate both expectations. From the expression of observation Eq. (2) and taking into account the hypotheses of the model, it is clear that

$$E[y_k y_k^T] = (1-p_k)[\alpha_k \beta_k^T + R_k] + p_k[\alpha_{k-1}\beta_{k-1}^T + R_{k-1}], \quad k \ge 1.$$

In order to determine the expectation $E[\widehat{y}_{k/k-1}\widehat{y}_{k/k-1}^T]$, we use (13) for $\widehat{y}_{k/k-1}$ and the definition of $m_k = E[Q_k Q_k^T]$ obtaining

$$\begin{aligned} E[\widehat{y}_{k/k-1}\widehat{y}_{k/k-1}^T] &= \Lambda_k m_{k-1}\Lambda_k^T + p_{01}(1-p_{k-1})\Lambda_k M_{k-1}\Pi_{k-1}^{-1}R_{k-1} + p_{01}(1-p_{k-1})R_{k-1}\Pi_{k-1}^{-1}M_{k-1}^T\Lambda_k^T \\ &\quad + p_{01}^2(1-p_{k-1})^2 R_{k-1}\Pi_{k-1}^{-1}R_{k-1}, \quad k \ge 2, \\ E[\widehat{y}_{1/0}\widehat{y}_{1/0}^T] &= 0, \end{aligned}$$

where we make use of the fact that $M_k = E[O_k \nu_k^T]$. From these two expectations, the innovation covariance matrix, given by (9), is then obtained. □

Now, taking into account the general expression for the linear estimator (3) together with the results of Theorem 1, in the following section, recursive algorithms are derived for the filtering and fixed-point smoothing estimation problems.

4. Recursive estimation algorithms

4.1. Filtering algorithm

Under hypotheses (i)–(iv) set out in Section 2, the filter of the signal is obtained by

$$\widehat{x}_{k/k} = \alpha_k O_k, \quad k \ge 0, \tag{14}$$

where the vectors O_k are recursively calculated as

$$O_k = O_{k-1} + J_k \Pi_k^{-1}\nu_k, \quad k \ge 1; \qquad O_0 = 0 \tag{15}$$

with

$$J_k = (1-p_k)\beta_k^T + p_k\beta_{k-1}^T - r_{k-1}\Lambda_k^T - p_{01}(1-p_{k-1})J_{k-1}\Pi_{k-1}^{-1}R_{k-1}, \quad k \geq 2,$$
$$J_1 = (1-p_1)\beta_1^T + p_1\beta_0^T \tag{16}$$

and

$$r_k = r_{k-1} + J_k\Pi_k^{-1}M_k^T, \quad k \geq 1; \qquad r_0 = 0. \tag{17}$$

The innovation, $\nu_k,\ k \geq 1$, its covariance matrix, Π_k, and M_k are given in Theorem 1.

Proof. Taking into account the general expression of estimators (3) in terms of the innovations, to derive the linear filter of the signal, it is necessary to calculate the coefficients $S_{k,i} = E[x_k\nu_i^T]$ for $1 \leq i \leq k$. Using expression (4) for ν_i, (2) and the hypotheses of the model, we obtain

$$S_{k,i} = \alpha_k[(1-p_i)\beta_i^T + p_i\beta_{i-1}^T] - E[x_kQ_{i-1}^T]\Lambda_i^T - p_{01}(1-p_{i-1})E[x_k\nu_{i-1}^T]\Pi_{i-1}^{-1}R_{i-1}, \quad i \leq k,$$
$$S_{k,1} = \alpha_k[(1-p_1)\beta_1^T + p_1\beta_0^T],$$

and from the definition $Q_i = \sum_{j=1}^{i} M_i\Pi_j^{-1}\nu_j$,

$$S_{k,i} = \alpha_k[(1-p_i)\beta_i^T + p_i\beta_{i-1}^T] - \sum_{j=1}^{i-1} S_{k,j}\Pi_j^{-1}M_j^T\Lambda_i^T - p_{01}(1-p_{i-1})S_{k,i-1}\Pi_{i-1}^{-1}R_{i-1}.$$

So,

$$S_{k,i} = \alpha_k J_i, \quad 1 \leq i \leq k, \tag{18}$$

where J_i is a function satisfying

$$J_i = (1-p_i)\beta_i^T + p_i\beta_{i-1}^T - \sum_{j=1}^{i-1} J_j\Pi_j^{-1}M_j^T\Lambda_i^T - p_{01}(1-p_{i-1})J_{i-1}\Pi_{i-1}^{-1}R_{i-1}, \quad 1 < i \leq k,$$
$$J_1 = (1-p_1)\beta_1^T + p_1\beta_0^T. \tag{19}$$

Now, substituting (18) in the expression of the filter, $\widehat{x}_{k/k} = \sum_{i=1}^{k} S_{k,i}\Pi_i^{-1}\nu_i$, and defining $O_k = \sum_{i=1}^{k} J_i\Pi_i^{-1}\nu_i,\ k \geq 1$ and $O_0 = 0$, expression (14) for the filter is deduced.

Clearly, the recursive relation (15) is obtained from the definition of O_k.

Expression (16) for J_k is derived by performing $i = k$ in (19) and defining

$$r_k = E[O_kQ_k^T] = \sum_{j=1}^{k} J_j\Pi_j^{-1}M_j^T, \quad k \geq 1; \qquad r_0 = 0.$$

Finally, from the above definition of r_k, the recursive relation (17) is immediate. □

4.2. Filtering error covariance matrices

The performance of the filter can be measured by the estimation errors, $x_k - \widehat{x}_{k/k}$, and, more specifically, by the covariance matrices of these errors,

$$P_{k/k} = E[(x_k - \widehat{x}_{k/k})(x_k - \widehat{x}_{k/k})^T], \quad k \geq 0.$$

Since the estimation error is orthogonal to the estimator, taking into account hypothesis (i),

$$P_{k/k} = \alpha_k\beta_k^T - E[\widehat{x}_{k/k}\widehat{x}_{k/k}^T], \quad k \geq 0.$$

Now, using (14) for $\widehat{x}_{k/k}$, the filtering error covariance matrices are given by

$$P_{k/k} = \alpha_k[\beta_k^T - d_k\alpha_k^T], \quad k \geq 0,$$

where $d_k = E[O_kO_k^T]$. Using the recursive relation (15) for O_k and taking into account that O_{k-1} and ν_k are uncorrelated, the following recursive relation is obtained for d_k:

$$d_k = d_{k-1} + J_k\Pi_k^{-1}J_k^T, \quad k \geq 1; \qquad d_0 = 0.$$

In the following, we focus on the fixed-point smoothing problem; that is, our aim is to obtain the least-squares linear estimator of the signal x_k based on the observations $y_1, \ldots, y_L$, being $L > k$.

4.3. Fixed-point smoothing algorithm

Under hypotheses (i)–(iv) set out in Section 2, the fixed-point smoother of the signal x_k can be recursively obtained by

$$\widehat{x}_{k/L} = \widehat{x}_{k/L-1} + S_{k,L}\Pi_L^{-1}\nu_L, \quad L > k, \ k \geq 0 \tag{20}$$

with initial condition $\widehat{x}_{k/k}$, given by (14).

The smoothing gain, $S_{k,L}$, is given by

$$\begin{aligned} S_{k,L} &= \beta_k[(1-p_L)\alpha_L^T + p_L\alpha_{L-1}^T] - F_{k,L-1}\Lambda_L^T - p_{01}(1-p_{L-1})S_{k,L-1}\Pi_{L-1}^{-1}R_{L-1}, \quad L > k, \\ S_{0,1} &= \beta_0[(1-p_1)\alpha_1^T + p_1\alpha_0^T], \end{aligned} \tag{21}$$

where the matrices $F_{k,L}$ satisfy the following recursive formula

$$\begin{aligned} F_{k,L} &= F_{k,L-1} + S_{k,L}\Pi_L^{-1}M_L^T, \quad L > k, \\ F_{k,k} &= \alpha_k r_k. \end{aligned} \tag{22}$$

The innovation ν_L and the matrices M_L, Π_L are determined according to Theorem 1 and the matrix r_k is given by (17).

Proof. The recursive relation (20) is immediate from (3). Then, our aim is to determine the smoothing gain, $S_{k,L} = E[x_k\nu_L^T]$, $L > k$ by means of reasoning analogous to that used to obtain $S_{k,i}$, $i \leq k$ in the filtering algorithm, obtaining

$$\begin{aligned} S_{k,L} &= \beta_k[(1-p_L)\alpha_L^T + p_L\alpha_{L-1}^T] - \sum_{j=1}^{L-1} S_{k,j}\Pi_j^{-1}M_j^T\Lambda_L^T - p_{01}(1-p_{L-1})S_{k,L-1}\Pi_{L-1}^{-1}R_{L-1}, \\ S_{0,1} &= \beta_0[(1-p_1)\alpha_1^T + p_1\alpha_0^T]. \end{aligned}$$

Then, by defining

$$F_{k,L} = \sum_{j=1}^{L} S_{k,j}\Pi_j^{-1}M_j^T,$$

expression (21) for the smoothing gain is derived.

From the above definition of $F_{k,L}$ the recursive relation (22) is immediate. □

4.4. Fixed-point error covariance matrices

As observed above, a measure of the goodness of the estimators is the estimation error covariance matrices; then, as in the study of the filter, we determine a recursive formula for the fixed-point smoothing error covariance matrices $P_{k/L} = E[x_kx_k^T] - E[\widehat{x}_{k/L}\widehat{x}_{k/L}^T]$. Using (20) and taking into account that $\widehat{x}_{k/L-1}$ is uncorrelated with ν_L, we obtain

$$P_{k/L} = E[x_kx_k^T] - E[\widehat{x}_{k/L-1}\widehat{x}_{k/L-1}^T] - S_{k,L}\Pi_L^{-1}S_{k,L}^T, \quad L > k$$

and the following recursive expression at L for the smoothing error covariance matrices is obtained

$$P_{k/L} = P_{k/L-1} - S_{k,L}\Pi_L^{-1}S_{k,L}^T, \quad L > k$$

with the initial condition of the filtering error covariance matrices, $P_{k/k}$.

5. Simulation example

In this section, a simulation example is given to illustrate the effectiveness of the proposed algorithms. This is implemented in a MATLAB program which, at each iteration, simulates the signal and the observed values and provides the filtering and smoothing estimates, as well as their corresponding error variances.

Consider a zero-mean scalar signal $\{x_k,\ k \geq 0\}$ with covariance function

$$E[x_kx_s] = 1.025641 \times 0.95^{k-s}, \quad s \leq k$$

which is expressed according to hypothesis (i) by taking

$$\alpha_k = 1.025641 \times 0.95^k \quad \text{and} \quad \beta_s = 0.95^{-s}.$$

The measurement of the signal, z_k, is given by (1) where the measurement noise, $\{v_k,\ k \geq 0\}$, is a zero-mean white noise with $R_k = 0.9$.

According to the theoretical model, it is assumed that the available measurements of the signal can be delayed by one sample period during the transmission; that is, the processed observations are modeled by

$$y_k = (1-\xi_k)z_k + \xi_k z_{k-1}, \quad k \geq 1,$$

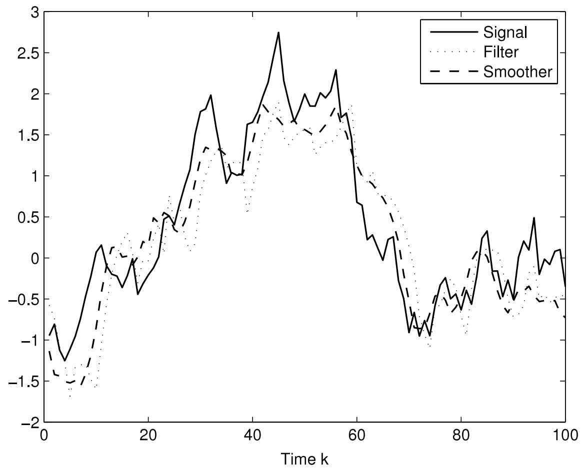

Fig. 1. Simulated signal, filtering and smoothing estimates.

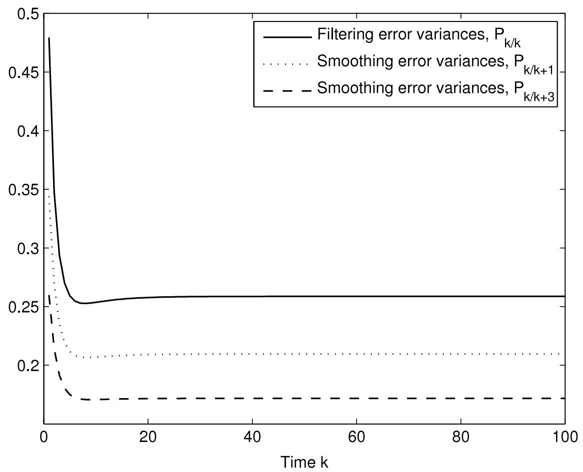

Fig. 2. Filtering and smoothing error variances.

where $\{\xi_k,\ k \geq 1\}$ are Bernoulli random variables. As in Section 2.1, it is assumed that this sequence is a homogeneous Markov chain with initial distribution $P[\xi_1 = 1] = 0$ (the first observation is not delayed), and that the transition probability matrix is given by $\mathbb{P} = \begin{pmatrix} 0.95 & 0.05 \\ 0.11 & 0.89 \end{pmatrix}$.

Moreover, the signal and noise processes are assumed to be mutually independent.

In order to realize the simulation process, the signal is assumed to be generated from the following first-order autoregressive model,

$$x_{k+1} = 0.95x_k + w_k$$

where $\{w_k,\ k \geq 0\}$ is a zero-mean white Gaussian noise with $E[w_k^2] = 0.1,\ \forall k$.

To analyze the performance of the algorithms proposed in Section 4, 100 iterations of each one were performed and the filtering and smoothing error variances were calculated.

Fig. 1 shows a simulated signal together with the filtering estimates, $\widehat{x}_{k/k}$, and fixed-point smoothing estimates, $\widehat{x}_{k/k+3}$. This figure reveals, as expected, that the smoothing estimates follow the evolution better than the filtering estimates, which means that the estimation performance is better as the number of available observations increases. Fig. 2 shows the filtering and fixed-point smoothing error variances, $P_{k/k}$, $P_{k/k+1}$ and $P_{k/k+3}$. Examination of this figure confirms the simulated results in Fig. 1, showing that the best estimations are obtained by the fixed-point smoothers since, as seen, the smoothing error variances, $P_{k/k+1}$ and $P_{k/k+3}$, are smaller than the filtering error variances, $P_{k/k}$. On the other hand, we also observe that the smoothing error variances decrease as the number of observations increase; that is, the best results are obtained for $\widehat{x}_{k/k+3}$.

Moreover, we have also calculated the filtering error variances assuming that the arrival of the signal (no delay or delay) is modeled by different Markov chains. Concretely, we assume the same initial distribution and the following transition

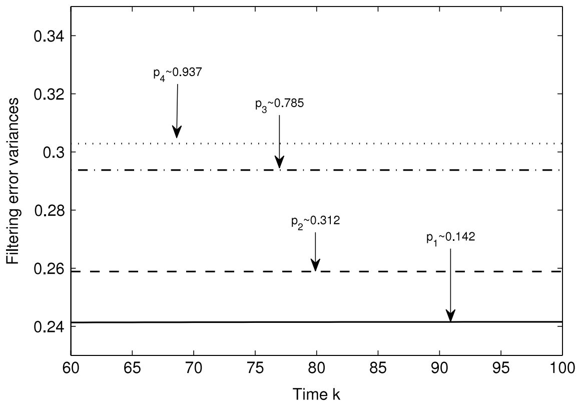

Fig. 3. Filtering error variances for different transition probability matrix.

probability matrices

$$\mathbb{P}_1 = \begin{pmatrix} 0.99 & 0.01 \\ 0.06 & 0.94 \end{pmatrix}, \qquad \mathbb{P}_2 = \begin{pmatrix} 0.95 & 0.05 \\ 0.11 & 0.89 \end{pmatrix},$$
$$\mathbb{P}_3 = \begin{pmatrix} 0.89 & 0.11 \\ 0.03 & 0.97 \end{pmatrix} \quad \text{and} \quad \mathbb{P}_4 = \begin{pmatrix} 0.85 & 0.15 \\ 0.01 & 0.99 \end{pmatrix}.$$

The properties of the Markov chains lead us to conclude that the delay probability converges to a constant value; in our case these values, for the different transition probability matrices considered, are $p_1 \simeq 0.142,\ p_2 \simeq 0.312,\ p_3 \simeq 0.785$ and $p_4 \simeq 0.937$, respectively. Fig. 3 shows the filtering error variances for these models, reflecting only the values of the filtering error variances for time $k \geq 60$ since from this iteration the delay probabilities are stabilized in all cases considered. Analysis of this figure reveals that as the limit probability of delay decreases, the filtering error variances become smaller and, consequently the performance of the estimator improves.

6. Conclusions

Recursive filtering and fixed-point smoothing algorithms to estimate the signal from randomly delayed observations are proposed. The variables modeling the random delays are assumed to be non-independent Bernoulli random variables, and the dependence is modeled by a homogeneous Markov chain. Assuming that the state-space model generating the signal is unknown, the algorithms are derived, using an innovation approach, under the assumption that the second-order moments of the signal and noises are known. In order to provide a measure of the goodness of the proposed estimators, recursive expressions to calculate the filtering and smoothing error covariance matrices are also derived.

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